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How should acoustics adapt to meet future demands?

A detailed tutorial for evaluating in-duct net acoustic power transmission in a circular duct with an attached cylindrical Helmholtz resonator using transfer matrix method

Sarabjeet Singh (1), Colin H. Hansen (2) and Carl Q. Howard (2)

(1) Vipac Engineers and Scientists Ltd., PO Box 2419, Kent Town, SA 5071, Australia

(2) School of Mechanical Engineering, The University of Adelaide, SA 5000, Australia

ABSTRACT

The transfer matrix method has been extensively explored by many researchers for analysing acoustic duct systems. However, there does not exist a comprehensive source that elaborates the use of transfer matrix method for evaluating the performance of acoustic filters. The current approach describes a detailed step-by-step method for evaluating in-duct net acoustic power transmission for a harmonic plane wave travelling in a circular duct with an attached cylindrical Helmholtz resonator using the transfer matrix method. The net acoustic power transmission is evaluated using two different methods: (1) the product of acoustic pressure and acoustic volume velocity at the duct exit, and (2) estimates of the in-duct sound field using the two-microphone technique.

INTRODUCTION

The transfer matrix method (also known as transmission matrix or four-pole parameter presentation) has been widely accepted as a tool for analysing complex systems due to its computational efficiency and flexibility. Many previous researchers (Munjal 1987, Pierce 1989, Snowdon 1971, and To and Doige 1978) have used the transfer matrix method for analysing acoustic duct systems. Although many research papers, in addition to standard acoustic text books (Beranek and Ver 1992, Kinsler *et al.* 1992, and Pierce 1989), provide a brief overview of the transfer matrix method and its associated uses, they do not incorporate detailed analyses of a particular system.

This paper describes in detail a step-by-step method for approximating the downstream net acoustic power transmission for a harmonic plane wave travelling inside a circular duct to which a cylindrical Helmholtz resonator (HR) is attached. It begins with a description of the transfer matrix of a uniform circular duct which includes all the steps and pertinent calculations in order to estimate the in-duct net acoustic power transmission. Then, the building of the transfer matrix of a coupled duct-HR system is described and the net acoustic power transmission inside the duct downstream of the HR is approximated.

The net acoustic power transmission inside the duct was estimated by using the estimates of acoustic pressure and acoustic volume velocity at the terminating end of the duct. However, in addition to the above stated method, net acoustic

power transmission inside the duct was also estimated using the two-microphone method developed by Chung and Blaser (Chung and Blaser 1980) and extended by Åbom (Åbom 1989). The purpose of estimating the acoustic power transmission using the latter method is to incorporate additional detailed calculations using the transfer matrix method. Furthermore, the latter method provides a means to verify the estimations of the former method.

UNIFORM CIRCULAR DUCT

Figure 1 shows a schematic of a circular duct of (physical) length l , radius a , and cross-sectional area S .

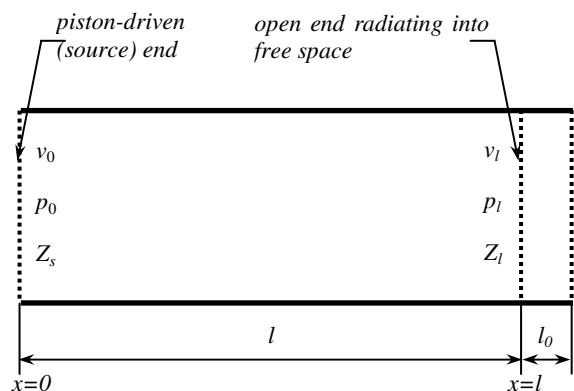


Figure 1. A schematic of a uniform circular duct modelled as being driven by a constant amplitude piston source mounted at one (left) end and open at the other (right).

Also shown at the end of the duct is the end-correction factor, l_0 , which will be discussed later. The left end of the duct, referred to as a source end, was modelled as being driven by a constant amplitude piston with a unit acoustic volume velocity, and the right end of the duct, referred to as a duct exit, was modelled as an open end radiating into free space.

Plane Wave Assumption

As the transfer matrix method works on a principle of plane wave propagation inside the duct, it is important to estimate the *cut-on* frequency, which is defined as the frequency below which only plane waves propagate inside the duct. The *cut-on* frequency is given by:

$$f_c = \frac{1.8412c}{2\pi a} \quad (1)$$

where, c is the speed of sound in air (343 m/s at 20°C), and a is the radius of the duct.

For the current paper, the dimensions of the circular duct were taken as: diameter = 0.1555 m and length = 3 m. The *cut-on* frequency for the above dimensioned duct is 1292 Hz. Therefore, the plane wave mode is the only mode which propagates below 1292 Hz.

Radiation Impedance

For the purpose of modelling the end of the duct as open and radiating into free space, the calculation of radiation impedance is required. It is approximated by assuming a circular piston located in an infinite baffle, and is the ratio of the force exerted by the piston on the acoustic field to the velocity of the piston. Radiation impedance is given by (Beranek and Ver 1992, Bies and Hansen 2003, Pierce 1989, Kinsler *et al.* 1982, Munjal 1987):

$$Z_r = R + jX \quad (2)$$

The real part of radiation impedance, R , is termed as *radiation reactance* and represents the energy radiated away from the open end in the form of sound waves. The imaginary part, X , is termed as *radiation admittance* and represents the mass loading of the fluid (air) just outside the open end.

It is well known that the theoretical expression for the impedance of an unflanged open duct, with plane waves propagating inside it, is given by (Beranek and Ver 1992, Bies and Hansen 2003, Pierce 1989, Kinsler *et al.* 1982, Munjal 1987):

$$Z_l = \frac{\rho c}{S} \left[\frac{(ka)^2}{4} + j(0.6)ka \right] \quad (3)$$

where, Z_l is the radiation impedance, ρ is the density of the fluid medium, c is the speed of sound, k is the wave number, and a is the radius of the duct.

End-Correction

The term end-correction refers to an additional length to the actual length of a tube or an orifice that accounts for the entrained mass of the fluid that vibrates outside the tube's or orifice's opening. The expression for the end-correction of an unflanged open end of a duct radiating into free space can be found in standard acoustical text books (Beranek and Ver 1992, Bies and Hansen 2003, Kinsler *et al.* 1992, and Munjal 1987). It is given by:

$$l_0 = 0.6a \quad (4)$$

where, a is the radius of the duct.

Transfer Matrix of a Circular Duct

The transfer matrix of a circular duct of uniform cross-sectional area, S , and length, l , is given by (Munjal 1987):

$$\begin{bmatrix} p_r \\ q_r \end{bmatrix} = \begin{bmatrix} \cos(\hat{k}l) & jY_r \sin(\hat{k}l) \\ jY_r \sin(\hat{k}l) & \cos(\hat{k}l) \end{bmatrix} \begin{bmatrix} p_{r-1} \\ q_{r-1} \end{bmatrix} \quad (5)$$

The subscript r implies that the duct is defined/denoted as an element r . The quantity $Y_r = \frac{c}{S}$ is the characteristic impedance,

and $\hat{k} = k(1 + i\eta)$ is the complex wave number. Complex wave numbers are introduced in order to conveniently incorporate damping in the system, even though this formulation implies hysteretic damping when in fact the system is better described as viscously damped. The quantity η is the loss factor, which is equivalent to twice the critical damping ratio for a viscous system.

The quantities, p_r , p_{r-1} and q_r , q_{r-1} represent the acoustic pressures and acoustic mass velocities at the extreme ends of the duct, respectively (input and output sides). These quantities can be r.m.s values, amplitudes or instantaneous, provided they are all the same type.

Rewriting equation (5) by relating acoustic volume velocity and acoustic pressure instead of acoustic mass velocity and acoustic pressure, respectively gives:

$$\begin{bmatrix} v_r \\ p_r \end{bmatrix} = \begin{bmatrix} \cos(\hat{k}l) & j\frac{S}{\rho c} \sin(\hat{k}l) \\ j\frac{\rho c}{S} \sin(\hat{k}l) & \cos(\hat{k}l) \end{bmatrix} \begin{bmatrix} v_{r-1} \\ p_{r-1} \end{bmatrix} \quad (6)$$

where, v_r and v_{r-1} represent acoustic volume velocities at the input and output sides of the duct (element r), respectively.

The above shown transition in equation (6) affected the definition of the elements of the transfer matrix compared to those defined by Munjal (1987) (in equation (5)). A pivotal reason for implementing this transition is to enable a quick comparison between theoretical predictions and experimental results. Although no experimental results are discussed in this paper, in most of the relevant experimental investigations reported in the literature, a loudspeaker backed by a small air-tight cavity, as an acoustic source, is attached to one end of the duct. And, generally, sound pressure level measurements inside the duct are normalised by the acoustic volume velocity of the loudspeaker. Therefore, from a practical point of view, a relation between acoustic volume velocity and acoustic pressure is much preferred over a relation between acoustic mass velocity and acoustic pressure. Although acoustic mass velocity is directly proportional to acoustic particle velocity which, in turn, is directly proportional to acoustic volume velocity, the transfer matrix equations shown in this paper eliminate the need to derive a relationship between acoustic volume velocity and pressure.

As indicated in figure 1, v_0 and p_0 are acoustic volume velocity and pressure, respectively, at the input side of the duct. Similarly, v_l and p_l have the same definition at the output side. Relating these state variables gives the resultant transfer matrix of the duct, which is described below:

$$\begin{bmatrix} v_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} \cos(\hat{k}l_{\text{eff}}) & j\frac{S}{\rho c}\sin(\hat{k}l_{\text{eff}}) \\ j\frac{\rho c}{S}\sin(\hat{k}l_{\text{eff}}) & \cos(\hat{k}l_{\text{eff}}) \end{bmatrix} \begin{bmatrix} v_l \\ p_l \end{bmatrix} \quad (7)$$

A very important point which needs to be considered is the end-correction factor of the unflanged open end of the duct. While estimating the elements of the transfer matrix, the end-correction factor, l_0 , must be added to the physical length of the duct. Therefore, in equation (7), l_{eff} represents the effective length of the duct, which corresponds to the sum of physical duct length, l , and the end-correction factor, l_0 .

In order to make the evaluation of equation (7) easier for relevant future calculations, the elements of the above described transfer matrix were assigned alphabetical characters as shown below:

$$\begin{bmatrix} v_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} A_l & B_l \\ C_l & D_l \end{bmatrix} \begin{bmatrix} v_l \\ p_l \end{bmatrix} \quad (8)$$

where, $A_l = D_l = \cos(\hat{k}(l+l_0))$, $B_l = j\frac{S}{\rho c}\sin(\hat{k}(l+l_0))$,

and $C_l = j\frac{\rho c}{S}\sin(\hat{k}(l+l_0))$.

As described earlier, the in-duct net acoustic power transmission is calculated using two different methods, (1) the product of acoustic pressure and velocity at the duct exit, and (2) the two-microphone method. These methods are described in the following text.

METHOD 1: PRODUCT OF ACOUSTIC PRESSURE AND VOLUME VELOCITY AT THE DUCT EXIT

The equations shown below describe the procedure to estimate the acoustic pressure and volume velocity at the duct exit. Estimation of these two variables will in turn facilitate the estimation of in-duct net acoustic power transmission.

From equation (8), we get

$$v_0 = A_l v_l + B_l p_l \quad (9)$$

By using $Z_l = \frac{p_l}{v_l}$, equation (9) can be written as:

$$v_0 = A_l \frac{p_l}{Z_l} + B_l p_l \quad (10)$$

Solving equation (10) for p_l with respect to v_0 gives

$$\frac{p_l}{v_0} = \frac{Z_l}{A_l + B_l Z_l} \quad (11)$$

Equation (11) gives the estimate of the ratio of acoustic pressure p_l at the duct exit to the input volume velocity v_0 . Z_l is the radiation impedance of an unflanged open end of the duct and can be calculated using equation (3).

Similarly, solving equation (9) for v_l with respect to v_0 , which incorporates the use of equation (11), gives:

$$\frac{v_l}{v_0} = \frac{1 - \frac{B_l Z_l}{A_l + B_l Z_l}}{A_l} \quad (12)$$

Equation (12) gives the measure of the ratio of acoustic volume velocity v_l at the duct exit to the input volume velocity v_0 .

If v_0 is set to unity (1 m³/s), then p_l and v_l would be

$$p_l = \frac{Z_l}{A_l + B_l Z_l} \quad (13)$$

$$v_l = \frac{1 - \frac{B_l Z_l}{A_l + B_l Z_l}}{A_l} \quad (14)$$

Equations (13) and (14) represent the estimates of acoustic pressure and acoustic volume velocity, respectively, at the duct exit; the duct being driven by a constant amplitude piston at the source end with a unit volume velocity.

In-Duct Net Acoustic Power Transmission

The net acoustic power flux associated with a harmonic plane wave, which takes into account the pertinent incident and reflected plane waves, travelling inside a duct is given by (Fahy, 1995):

$$W_{\text{net}} = \frac{1}{2} \text{Re}(p \times v^*) \quad (15)$$

where, p and v represent the complex amplitudes of pressure and volume velocity, respectively, and the $*$ indicates the complex conjugate. This expression can be written as:

$$W_{\text{net}} = \frac{1}{2} P V \cos(\phi) \quad (16)$$

where, P and V are the scalar amplitudes that are the moduli of the complex amplitudes of the pressure and volume velocity, respectively shown in equation (15), and ϕ is the phase angle between them. Equation (16) represents the net acoustic power transmission inside the duct as the product of acoustic pressure and volume velocity amplitudes.

Substituting the values of equations (13) and (14) in equation (16) provides the estimate of net acoustic power transmission in the duct. Assuming that the analysis in the preceding sections used amplitude quantities, equation (16) calculates the estimate of the net acoustic power transmission in the duct when the fluid present inside the duct is assumed to be driven by a constant amplitude harmonic piston source with a unit volume velocity.

Figure 2 shows a plot of the net acoustic power transmission in the uniform circular duct of diameter equals 0.1555 m and length equals 3 m.

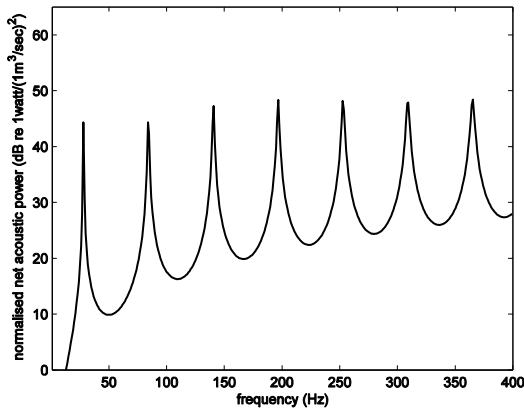


Figure 2. Net acoustic power transmission as a function of harmonic frequency in a uniform circular duct driven by a constant amplitude harmonic piston source (of unit acoustic volume velocity) mounted at one (left) end and open at the other (right).

Although the above discussed method showed the equations and procedure required to estimate the in-duct net acoustic power transmission, the use of the transfer matrix method is better illustrated in the second analysis method to follow.

METHOD 2: TWO-MICROPHONE TECHNIQUE

The two-microphone technique was developed by Chung and Blaser (Chung and Blaser 1980) and requires an estimate of acoustic pressure at two different locations inside a duct. The following paragraphs detail the procedure for estimating the acoustic pressure at different locations inside a duct.

Acoustic Pressure and Velocity at Arbitrary Locations Inside a Duct

Figure 3 shows a schematic of a uniform duct of length l depicting the end-correction, l_0 , at the open end of the duct.

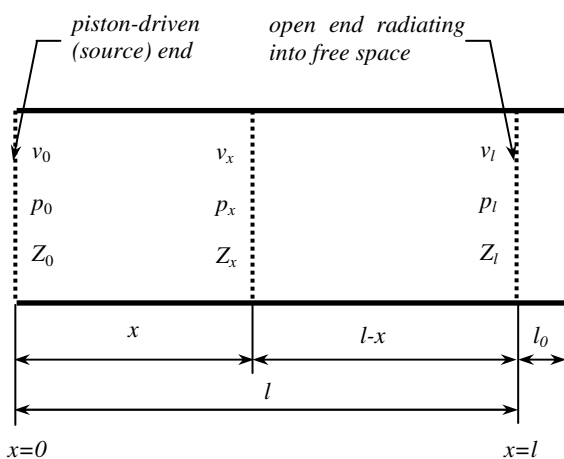


Figure 3. A schematic of a uniform duct depicting location x along the duct where acoustic pressure and acoustic volume velocity need to be estimated.

Suppose that acoustic pressure and acoustic volume velocity at location x need to be estimated, and are denoted by p_x and v_x , respectively. As evident from equations (11) and (12), the estimation of acoustic pressure and volume velocity at a particular location requires the corresponding value of acoustic impedance at that particular location. Therefore, estimation of acoustic pressure and volume velocity at location x requires estimating the acoustic impedance at location x , which is detailed below.

Acoustic Impedance at Arbitrary Location Inside a Duct

From standard acoustic text books (Beranek and Ver 1992, Bies and Hansen 2003, Kinsler *et al.* 1992, and Pierce 1989), the specific acoustic impedance at any location inside a duct can be expressed as

$$Z_x = \frac{\rho c}{S} \frac{a_+ \exp^{-jkx} + a_- \exp^{+jkx}}{a_+ \exp^{-jkx} - a_- \exp^{+jkx}} \quad (17)$$

where, a_+ and a_- are the modal amplitudes of incident and reflected acoustic waves, respectively, the - and + signs represent the propagation of acoustic wave in +ve and -ve x directions, respectively, and x is the location along the duct.

Using equation (17), the values of acoustic impedance at locations $x = 0$ and $x = l$ are given by:

$$Z_0 = \frac{\rho c}{S} \frac{a_+ + a_-}{a_+ - a_-} \quad (18)$$

$$Z_l = \frac{\rho c}{S} \frac{a_+ \exp^{-jkl} + a_- \exp^{+jkl}}{a_+ \exp^{-jkl} - a_- \exp^{+jkl}} \quad (19)$$

Eliminating constants a_+ and a_- facilitates a relationship between Z_0 and Z_l , which is given by:

$$Z_0 = \frac{Z_l + j \frac{\rho c}{S} \tan(kl)}{1 + j \frac{S}{\rho c} Z_l \tan(kl)} \quad (20)$$

If the impedance at a duct termination is known, the value of the acoustic impedance at any point inside the duct can be calculated using equation (20). In order to distinguish between the source and radiation impedance, equation (20) was rewritten with a slight modification in terms of nomenclature. Denoting the acoustic impedance at $x=0$ by Z_s (source impedance) and at $x = l$ by Z_r (radiation impedance), gives:

$$Z_s = \frac{Z_r + j \frac{\rho c}{S} \tan(kl)}{1 + j \frac{S}{\rho c} Z_r \tan(kl)} \quad (21)$$

Finally, the value of acoustic impedance at location x looking into the duct from the source end ($x = 0$) can be given by (utilising equation (21)):

$$Z_x = \frac{Z_s - j \frac{\rho c}{S} \tan(kx)}{1 - j \frac{S}{\rho c} Z_s \tan(kx)} \quad (22)$$

The alternate expression for calculating the value of Z_x , by using the value of Z_r instead of Z_s , is given by:

$$Z_x = \frac{Z_r + j \frac{\rho c}{S} \tan(k(l-x))}{1 + j \frac{S}{\rho c} Z_r \tan(k(l-x))} \quad (23)$$

Equation (23) shows the acoustic impedance at location x looking into the duct from the open end ($x = l$).

End-Corrections

Before using equations (22) and (23) for calculating the value of Z_x , the end-correction of an unflanged open end of a duct must be accounted.

After incorporating the value of the end-correction in equations (22) and (23), Z_x is expressed as:

$$Z_x = \frac{Z_s - j \frac{\rho c}{S} \tan(k(x-l_0))}{1 - j \frac{S}{\rho c} Z_s \tan(k(x-l_0))} \quad (24)$$

$$Z_x = \frac{Z_r + j \frac{\rho c}{S} \tan(k(l-x+l_0))}{1 + j \frac{S}{\rho c} Z_r \tan(k(l-x+l_0))} \quad (25)$$

Once the value of acoustic impedance at a particular location in the duct is known, the acoustic pressure and velocity can be calculated using equations (11) and (12), respectively. The acoustic pressure and volume velocity at location x with respect to unit volume velocity, v_0 , are given by:

$$p_x = \frac{Z_x}{A_x + B_x Z_x} \quad (26)$$

$$v_x = \frac{1 - \frac{B_x Z_x}{A_x + B_x Z_x}}{A_x} \quad (27)$$

where, $A_x = \cos(\hat{k}x)$ and $B_x = j \frac{S}{\rho c} \sin(\hat{k}x)$.

Knowledge of the acoustic pressure at two appropriately spaced different locations inside a duct estimates the in-duct net acoustic power transmission by using the decomposition of sound field which is discussed below.

Decomposition of the Sound Field

Decomposition is a technique for determining the amplitudes of the acoustic waves propagating each way inside a duct. It facilitates the separation of modes into incident and reflected parts. Once the acoustic power associated with the incident and reflected waves is determined, the net acoustic power transmission is simply their difference. Åbom (Åbom 1989) extended the two-microphone technique developed by Chung and Blaser (Chung and Blaser 1980) and documented the scheme for in-duct modal decomposition. However, in this paper we are not concerned with higher order modes and we are only decomposing the sound field into right-travelling and left-travelling plane waves.

Generally, for a uniform straight duct with rigid walls and an arbitrary cross-sectional shape carrying an axial mean flow, the sound field in the duct can be expressed as:

$$\hat{p} = \sum_{n=0}^{\infty} [\hat{a}_+ \exp^{-ik_{+n}x} + \hat{a}_- \exp^{+ik_{-n}x}] \Psi_n \quad (28)$$

where, \hat{p} is the acoustic pressure, \hat{a}_+, \hat{a}_- are the modal amplitudes of acoustic pressure associated with the incident (+ve x direction) and reflected waves (-ve x direction), respectively, $\hat{k}_{+n}, \hat{k}_{-n}$ are the axial wave numbers in the positive and

negative directions, respectively, n is the modal number, and Ψ_n is the eigenfunction for mode n .

For the case of a uniform circular duct with rigid-walls and carrying no axial flow, with only plane waves propagating inside it, equation (28) can be written as:

$$\hat{p} = \hat{a}_+ \exp^{-ikx} + \hat{a}_- \exp^{+ikx} \quad (29)$$

The terms \hat{a}_+ and \hat{a}_- can be calculated by either estimating or experimentally measuring the acoustic pressure at two locations in a duct and using the two-microphone technique. Expressing the results of such a measurement in a matrix formulation yields (Åbom 1989):

$$\hat{\mathbf{p}} = \mathbf{M}\hat{\mathbf{a}} \quad (30)$$

where, $\hat{\mathbf{p}}$ is a $[2 \times 1]$ column vector containing the estimates or measures of the acoustic pressures at two different locations, \mathbf{M} is a $[2 \times 2]$ modal matrix containing the propagation terms, and $\hat{\mathbf{a}}$ is a $[2 \times 1]$ column vector containing the unknown modal amplitudes.

Writing vector $\hat{\mathbf{p}}$ in terms of transfer function between the two pressure measuring or estimating locations yields:

$$\hat{\mathbf{p}} = \hat{p}_r \mathbf{H} \quad (31)$$

where, \hat{p}_r is the acoustic pressure at one of the two pressure measuring or estimating locations, and \mathbf{H} is the $[2 \times 1]$ column vector containing the transfer function between the two locations.

Using equation (31), equation (30) can be written as:

$$\hat{p}_r \mathbf{H} = \mathbf{M}\hat{\mathbf{a}} \quad (32)$$

Solving equation (32) for $\hat{\mathbf{a}}$ gives:

$$\hat{\mathbf{a}} = \hat{p}_r \mathbf{M}^{-1} \mathbf{H} \quad (33)$$

$$\begin{bmatrix} \hat{a}_+ \\ \hat{a}_- \end{bmatrix} = \hat{p}_r \begin{bmatrix} 1 & 1 \\ \exp^{-iks} & \exp^{+iks} \end{bmatrix} \begin{bmatrix} 1 \\ H_{12} \end{bmatrix} \quad (34)$$

For this analysis, \hat{p}_r was taken as the acoustic pressure estimate at the location closer to the source end, H_{12} represents the pressure transfer function between two pressure estimates, and s is the axial distance between the two pressure estimating locations. Solving equation (34), for the estimates of \hat{a}_+ and \hat{a}_- , yields:

$$\hat{a}_+ = \frac{\hat{p}_r (\exp^{+iks} - H_{12})}{\exp^{+iks} - \exp^{-iks}} \quad (35)$$

$$\hat{a}_- = \frac{\hat{p}_r (H_{12} - \exp^{-iks})}{\exp^{+iks} - \exp^{-iks}} \quad (36)$$

The acoustic power associated with the incident and reflected waves is expressed as:

$$W_+ = \frac{S|\hat{a}_+|^2}{2\rho c} \quad (37)$$

$$W_- = \frac{S|\hat{a}_-|^2}{2\rho c} \quad (38)$$

The in-duct net acoustic power transmission in x direction is estimated as the difference between W_+ and W_- , and is expressed as:

$$W_{net} = W_+ - W_- = \frac{S}{2\rho c} \left[|\hat{a}_+|^2 - |\hat{a}_-|^2 \right] \quad (39)$$

In-Duct Net Acoustic Power Transmission

Assuming that the analysis in the preceding sections used amplitude quantities, equation (39) calculates the estimate of the net acoustic power transmission in the duct when the fluid present in the duct is assumed to be driven by a constant amplitude harmonic piston source with a unit volume velocity.

The net acoustic power estimates from the two methods described above exactly match with one another.

COUPLED DUCT – HR SYSTEM

Mounting a resonator on a duct reduces the noise transmission at a particular frequency which is a characteristic of the resonator's geometry. Comparison of the acoustic power transmission inside the duct without and with the resonator can be used to evaluate its performance.

In order to develop the complete transfer matrix of the duct-HR system, it was discretised into three elements: (1) *element 0* - section of the duct upstream of the HR, (2) *element 1* - section of the duct downstream of the HR, and (3) *element HR* - the Helmholtz resonator. Figure 4 shows a schematic of a HR mounted on to a duct of length l at location x . The figure also illustrates the three elements described above, along with the neck-cavity and neck-duct interfaces.

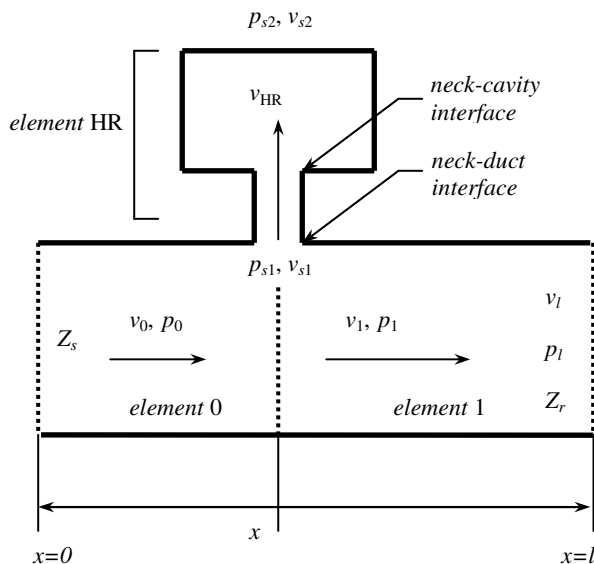


Figure 4. A schematic of a coupled duct-HR system depicting its division into three elements along with neck-duct and neck-cavity interfaces, and relevant notations.

End-Corrections of a HR's Neck

A key issue related to the theoretical analysis of a duct-HR system is the incorporation of two end-correction factors for the neck of the HR; (1) the neck-cavity interface end-correction factor, and (2) the neck-duct interface end correction factor, both of which must be added to the physical neck length to get the effective neck length.

The expression for the neck-cavity interface end-correction factor closely approximates the geometry of a cylindrical piston radiating in a tube. It is given by (Selamat and Ji 2000):

$$\delta_e = 0.82r \left(1 - 1.33 \frac{r}{R} \right) \quad (40)$$

where, r denotes the radius of the neck of a HR, and R denotes the radius of the cavity of a HR.

Another equation which has been widely used in the literature for the neck-cavity interface end-correction factor was presented by Ingard (Ingard 1953). His equation slightly differs from equation (40) and was for an orifice in an anechoically terminated duct. However, equation (40) is considered to be better than the equation presented by Ingard and has been used in this current paper.

On the other hand, the neck-duct interface end-correction factor has not been modelled analytically due to the difficulty in interpreting the non-planar sound field in the region of the duct adjacent to the neck opening. However, Ji (2005) derived two-curve fitting expressions for the circular neck-duct interface end-correction factor based on the ratio of neck and duct diameter using Boundary Element Analysis. The corresponding expressions are given by:

$$\begin{aligned} \delta_i &= r \left\{ 0.8216 - 0.0644 \left(\frac{r}{a} \right) - 0.694 \left(\frac{r}{a} \right)^2 \right\}; & \frac{r}{a} \leq 0.4 \\ &= r \left\{ 0.9326 - 0.6196 \left(\frac{r}{a} \right) \right\}; & \frac{r}{a} > 0.4 \end{aligned} \quad (41)$$

where, r denotes the radius of the neck of a HR, and a denotes the radius of a duct to which a HR is attached.

Transfer Matrix of a HR

Figure 4 shows a uniform duct with a HR attached as a side branch to one its walls at location x , along with the relevant notations required to develop the complete transfer matrix of the duct-HR system. The complex amplitude of the volume velocity entering the duct, before the HR's location, is denoted by v_0 . Similarly, it is v_1 in the section after the resonator's location and v_{HR} flowing into the HR. From the continuity condition, the volume velocity and pressure are locally conserved at junction x . Therefore,

$$v_0(x) = v_{HR}(x) + v_1(x) \quad (42)$$

$$p_0(x) = p_{HR}(x) = p_1(x) \quad (43)$$

Putting equations (42) and (43) in a matrix form yields:

$$\begin{bmatrix} v_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} v_{HR} + v_1 \\ p_l \end{bmatrix} \quad (44)$$

$$\begin{bmatrix} v_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{v_{HR}}{p_l} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ p_l \end{bmatrix} \quad (45)$$

From equation (43), $p_1(x) = p_{HR}(x)$ and also, $\frac{p_{HR}}{v_{HR}} = Z_{HR}$.

Here, Z_{HR} is the acoustic impedance just outside the opening of the resonator, which can be referred to as the *input point impedance*. Therefore, equation (45) can be re-written as:

$$\begin{bmatrix} v_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{Z_{HR}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_l \\ p_l \end{bmatrix} \quad (46)$$

Finally, the matrix $\begin{bmatrix} 1 & \frac{1}{Z_{HR}} \\ 0 & 1 \end{bmatrix}$ represents the transfer matrix of the Helmholtz resonator.

Input Point Impedance of a HR

The impedance at the opening of the HR, i.e. Z_{HR} , can be estimated by applying the transfer matrix equation to the neck and cavity of the HR.

The neck section of the resonator is denoted by s_1 having cross-sectional area A_n and length l_1 , and the cavity section is denoted by s_2 having cross-sectional area A_c and length l_2 . The length l_1 represents the effective length of the neck, which includes two end-corrections at each side of the neck, which were described in the previous section below figure 4 (equations (40) and (41)).

Relating the acoustic volume velocity and pressure at the opening of the neck v_{s1}, p_{s1} (see figure 4) and at the end of the cavity v_{s2}, p_{s2} (see figure 4), the transfer matrix equation for the Helmholtz resonator is expressed as:

$$\begin{bmatrix} v_{s1} \\ p_{s1} \end{bmatrix} = \begin{bmatrix} 2 \times 2 & \\ \text{transfer matrix} & \\ \text{of the neck} & \end{bmatrix} \begin{bmatrix} 2 \times 2 & \\ \text{transfer matrix} & \\ \text{of the cavity} & \end{bmatrix} \begin{bmatrix} v_{s2} \\ p_{s2} \end{bmatrix} \quad (47)$$

or

$$\begin{bmatrix} v_{s1} \\ p_{s1} \end{bmatrix} = \begin{bmatrix} A_{s1} & B_{s1} \\ C_{s1} & D_{s1} \end{bmatrix} \begin{bmatrix} A_{s2} & B_{s2} \\ C_{s2} & D_{s2} \end{bmatrix} \begin{bmatrix} v_{s2} \\ p_{s2} \end{bmatrix} \quad (48)$$

where, $A_{s1} = D_{s1} = \cos(\hat{k}l_1)$, $B_{s1} = j \frac{S}{\rho c} \sin(\hat{k}l_1)$, and

$C_{s1} = j \frac{\rho c}{S} \sin(\hat{k}l_1)$. Similarly, $A_{s2} = D_{s2} = \cos(\hat{k}l_2)$,

$B_{s2} = j \frac{S}{\rho c} \sin(\hat{k}l_2)$, and $C_{s2} = j \frac{\rho c}{S} \sin(\hat{k}l_2)$.

Let us denote the resultant elements of equation (48) as:

$$\begin{bmatrix} v_{s1} \\ p_{s1} \end{bmatrix} = \begin{bmatrix} A_s & B_s \\ C_s & D_s \end{bmatrix} \begin{bmatrix} v_{s2} \\ p_{s2} \end{bmatrix} \quad (49)$$

As the end of the resonator (cavity) is blocked (rigidly terminated), there will be no flow at the end of the resonator, hence, $v_{s2} = 0$. Substituting $v_{s2} = 0$ in equation (49), the acoustic pressure and volume velocity at the opening of the neck becomes

$$v_{s1} = B_s p_{s2} \quad (50)$$

$$p_{s1} = D_s p_{s2} \quad (51)$$

Dividing equation (51) by (50), gives the estimate of the input point impedance of the HR, as:

$$\frac{p_{s1}}{v_{s1}} = Z_{HR} = \frac{D_s}{B_s} \quad (52)$$

Complete Transfer Matrix of a Duct-HR System

Referring to figure 4, for the convenience of presenting the complete transfer matrix of the duct-HR system, let us denote: (1) *element 0*, the first section of the duct upstream of the HR, by *subscript 0*, and (2) *element 1*, the second section of the duct downstream of the HR, by *subscript 1*. The complete transfer matrix method of the duct-HR system shown in figure 4 can be expressed as:

$$\begin{bmatrix} v_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} 2 \times 2 \\ \text{transfer matrix} \\ \text{of element 0} \end{bmatrix} \begin{bmatrix} 2 \times 2 \\ \text{transfer matrix} \\ \text{of element HR} \end{bmatrix} \begin{bmatrix} 2 \times 2 \\ \text{transfer matrix} \\ \text{of element 1} \end{bmatrix} \begin{bmatrix} v_l \\ p_l \end{bmatrix} \quad (53)$$

$$\begin{bmatrix} v_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{Z_{HR}} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} v_l \\ p_l \end{bmatrix} \quad (54)$$

Let the resultant of matrix multiplication of equation (54) be denoted by:

$$\begin{bmatrix} v_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_l \\ p_l \end{bmatrix} \quad (55)$$

In-Duct Net Acoustic Power Transmission Downstream of the HR

The dimensions of the duct were kept same as were considered in the analysis of the uniform circular duct (diameter of 0.1555 m and length of 3 m). The dimensions of the cylindrical Helmholtz resonator were: cavity diameter = 0.131 m, cavity length = 0.070 m, neck diameter = 0.525 m, and physical neck length = 0.093 m. The HR was mounted at a distance of 0.5 m from the source end of the duct.

Method 1: Product of the acoustic pressure and volume velocity at the duct exit.

The transfer matrices of three elements of the duct-HR system shown in figure 4, *element 0*, *element 1*, and *element HR*, were written using equation (54) as per the dimensions of the duct-HR system described in the immediate preceding paragraph.

The acoustic pressure and acoustic volume velocity at the duct exit (open end radiating into free space), $x = l$, were estimated in the similar way as described in equations (11) and (12), and are given by:

$$\frac{p_l}{v_0} = \frac{Z_l}{A + B Z_l} \quad (56)$$

$$\frac{v_l}{v_0} = \frac{1 - \frac{B Z_l}{A + B Z_l}}{A} \quad (57)$$

where, p_l and v_l represent the acoustic pressure and volume velocity at the duct exit, v_0 denotes the input volume velocity, Z_l is the radiation impedance at the unflanged open end of the duct (also denoted by Z_r), and A , B , C and D are the elements of the resultant complete transfer matrix (equation (55)).

Multiplying the estimates of acoustic pressure (equation (56)) and acoustic volume velocity (equation (57)) at the duct exit as per equation (16) results in the net acoustic power transmission inside the duct downstream of the HR.

Method 2: Two-Microphone Decomposition of Sound Field

The net acoustic power transmission inside the duct downstream of the HR was also estimated by using the two-microphone in-duct decomposition method and exactly the same results as for Method 1 were obtained.

Figure 5 shows a plot of the net acoustic power transmission in the duct downstream of the HR. For comparison purposes, this figure also includes the plot of the net acoustic power transmission in the duct without the resonator.

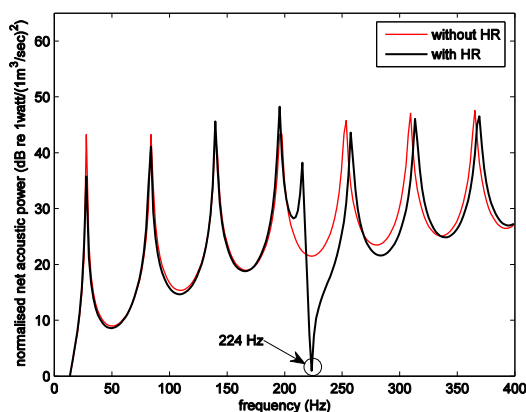


Figure 5. Net acoustic power transmission in a uniform circular duct with a HR mounted on to it at a distance of 0.5 m from the source end; the duct being driven by a constant amplitude harmonic piston source mounted at one (left) end and open at the other (right).

DISCUSSION

The effect of mounting the HR on to the duct, as evident from Figure 5, minimised the net acoustic power transmission by 20 dB at 224 Hz. Generally, a frequency at which the maximum reduction of in-duct net acoustic power transmission downstream of a HR occurs is considered to be the resonance frequency of the HR, which is 224 Hz in the current analysis. However, when a HR is mounted on to a duct, a coupled system is created whose resonance frequency is different to that of the HR as a stand-alone device. Therefore, the frequency at which the maximum reduction of in-duct net acoustic power transmission downstream of a HR occurs does not correspond to the resonance frequency of the HR (Singh *et al.* 2006 and Singh 2006). The acoustic performance obtained using the transfer matrix method for the circular duct-cylindrical HR system should only be considered as approximate. This is because of the duct-HR system related limitations of the transfer matrix method, which are discussed below.

DUCT-HR SYSTEM RELATED LIMITATIONS OF THE TRANSFER MATRIX METHOD

The key issue in the development of the duct-HR system transfer matrix is the incorporation of the two end-correction

factors of the neck of the HR in addition to the actual dimensions of the duct-HR system. The neck-cavity interface end-correction factor is well documented. However, there exists some uncertainty concerning the end-correction factor at the neck-duct interface, due to the difficulty in analytically modelling the complex sound field in the regions of the duct adjacent to the neck opening. However, empirical equations have been formulated by Ji (2005). Even though Ji found that his empirical expressions agreed with his experimental results, in another study (Singh *et al.* 2006 and Singh 2006), it was observed that the experimental performance of a duct mounted HR did not match the corresponding theoretical estimations based on the empirical end-correction factor reported by Ji.

Singh (2006) developed and solved a three-dimensional finite element model of the above described duct-HR system using the software package ANSYS. Unlike the theoretical analysis of a duct-HR system in which the end-correction factors have to be calculated, ANSYS automatically determines and incorporates the end-correction factors during its solution phase, and therefore, ANSYS results are considered more reliable than the theoretical calculations which differ slightly from the ANSYS results. In order to further identify the validity of the ANSYS results, experiments were conducted, and the experimentally measured performance of the HR mounted on to a duct was found to agree with the ANSYS results.

The difficulty in accurately estimating the neck duct interface end correction factor compromises the ability of the transfer matrix method to provide an accurate performance estimation of the duct mounted HR.

Also, as of the magnitude of the damping that occurs in a practical duct-HR system cannot be accurately modelled, the transfer matrix method with damping excluded can result in unrealistically high estimates of acoustic power reduction. However, in this paper, damping has been accounted for in the transfer matrix method by using the value of loss factor, η , equal to 0.004.

CONCLUSION

A detailed method for evaluating the acoustic performance of a cylindrical Helmholtz resonator mounted on to a uniform circular duct using the transfer matrix method has been presented. The in-duct net acoustic power transmission was estimated using two different methods: (1) product of the acoustic pressure and acoustic volume velocity at the open-ended duct exit, and (2) in-duct decomposition of sound field. The net acoustic power results from both methods exactly matched each other.

The transfer matrix method for the described duct-HR system application only provides a rough estimate of the acoustic performance of the duct mounted HR in terms of the frequency corresponding to the maximum net acoustic power reduction and the amount of net acoustic power reduction. The limitations corresponding to the duct-HR system are also described.

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