Enhancements to a tool for underwater acoustic sensitivity analysis, and relative importance of uncertainties in environmental parameters

Douglas R. Sweet

Maritime Operations Division, Defence Science and Technology Organisation, Edinburgh, Australia

ABSTRACT

Current sonar prediction models for an active sonar scenario give outputs such as probability of detection. However, the inputs to the models can be very uncertain, and it would be desirable to present measures of uncertainty in the output, and of sensitivity of the output to uncertainty in each of the inputs. In a previous paper, a tool was developed to predict uncertainty and sensitivity in acoustic transmission loss (TL) to uncertainty in sound speed profile (SSP) and bottom loss (BL). For the Malta plateau environment, it was found that SSP uncertainty was more important than BL uncertainty at some ranges, and vice-versa at other ranges. In this paper, an option has been provided to evaluate the sensitivity measure directly and more accurately than the (albeit faster) Monte-Carlo method used in the previous paper. The tool has been extended to compute probability of detection (Pd) and its uncertainty and sensitivity. For the Malta plateau environment and for some input parameters, it was found that SSP uncertainty was the most important parameter in Pd sensitivity at all ranges greater than 3.2 km, whereas for TL, BL sensitivity was most important at some ranges greater than 3.2 km. An explanation for these results is suggested.

INTRODUCTION

Current sonar performance prediction models give outputs such as “detection range of the day” versus azimuth, and probability of detection versus range. However, the inputs to a sonar performance prediction model, such as sound speed profile (SSP) and bottom loss (BL) can be very uncertain due to incomplete measurements, or spatial or temporal fluctuations. It would be desirable to present to the operator some measure of prediction uncertainty, and of sensitivity to uncertainties in each of the inputs. In the sensitivity literature, the analysis of output uncertainty due to all the input parameters is uncertainty analysis (UA); and the analysis of output uncertainty due to each input parameter (or the “importance” of each input parameter) is sensitivity analysis (SA). The main steps of UA and SA are:

1. generate a Monte-Carlo ensemble of uncertain inputs according to a specified multivariate distribution
2. propagate the ensemble of inputs through the input-output model to yield an ensemble of outputs
3. estimate the uncertainty in the output from its ensemble
4. estimate the sensitivity of the output to uncertainty in each individual input.

In a previous paper (Sweet and Jones 2008), an existing UA/SA modelling package – Algorithms for Stochastic Sensitivity Analysis (ASSA, by Jansen 2005) – was adapted to develop a tool to use uncertainty models of SSP and BL to predict uncertainty and sensitivity in acoustic transmission loss (TL). The uncertainty and sensitivity measures were the total variance (VTOT) and the “top marginal variance” (TMV) respectively of TL – the latter is the expected reduction in variance which would occur if the input of interest were known perfectly. We refer to this tool as the Underwater Acoustic Sensitivity Tool, version 1 (UAST v.1).

Results (Sweet and Jones 2008) were obtained for the Malta Plateau environment (Dosso et al 2007), and it was found that SSP uncertainty was more important than BL uncertainty at some ranges, and vice-versa at other ranges.

This paper describes two main extensions to UAST v.1, and discusses results obtained from them. The first provides an option to evaluate the expression for TMV directly, and more accurately; UAST v.1 used a Monte-Carlo method. Secondly, the tool has been extended to compute probability of detection (Pd) for an active sonar scenario, and its uncertainty and sensitivity. We refer to the updated version of the UAST as UAST v.2.

Pd results were obtained for the Malta Plateau for assumed signal and noise levels, and it was found that for certain values of noise level, SSP was more important than BL at all ranges greater than 3.2 km, whereas for TL, BL sensitivity was most important at some ranges greater than 3.2 km. An explanation of these results is suggested.

The paper is organized as follows. In the next section, the ASSA package and UAST v.1 are reviewed. Some uncertainty and sensitivity results for TL obtained using UAST v.1 are then reviewed and discussed. In the following two sections, the two extensions to UAST v.1 are described: direct evaluation of TMV, and computation of active sonar Pd and its uncertainty and sensitivity. Pd results are then presented and compared with the TL results, and an explanation offered for the reversal of importance of input parameters between the TL and Pd plots at some ranges. Finally, some conclusions are drawn and proposals for future work are given.
ASSA AND UAST V.1

ASSA

ASSA – Algorithms for Stochastic Sensitivity Analysis (Jansen 2005) – is a library of C-language algorithms which can be used to perform UA and SA. They fall into four groups, corresponding to the four main steps of UA and SA listed above in the Introduction:

- input generators;
- input-output models;
- uncertainty analysis; and
- sensitivity analysis.

Input Generators

The input generators provide a set of multidimensional samples (one dimension per input) whose distributions and correlations are as specified by the user. ASSA provides two basic generators – uniform and multinormal. For other distributions, ASSA allows the user to carry out two steps. Here, $m$ is the number of inputs.

- Draw a set of $m$-dimensional samples uniformly from the $m$-dimensional hypercube, with the desired correlation matrix
- Transform these samples to variables with the desired distribution

Input-Output Models

Only two very simple models, suitable for software development, are available (Jansen 2005). The user is expected to integrate their own model into their analyses, which includes computation of the output ensemble, say $y$, from the input ensemble.

Uncertainty Analysis

Variance based methods are used for both uncertainty and sensitivity analysis. ASSA uses the total output variance due to all input parameters

$$ VTOT = \text{Var}(y). $$ (1)

Sensitivity Analysis

ASSA uses two measures of sensitivity to input parameters, the top marginal variance (TMV) and the bottom marginal variance (Jansen et al. 1994). The former is the most useful, and we review that here.

The TMV due to a group $S$ of inputs is the expected variance reduction that would occur if one knew $S$ perfectly. For example, suppose there are two input parameters $a_1$ and $a_2$ and one wishes to find the TMV of the output $y$ due to $a_1$. Let the limits of $a_1$ be $a_{1\text{min}}$ and $a_{1\text{max}}$. Fix $a_2 = a_{2\text{max}}$, compute $y$ and its (reduced) variance $\text{Var}_r(y)$ for an ensemble of values of $a_2$. Increment $a_1$ a little and repeat, yielding $\text{Var}_r(y)$. Continue until $a_1 = a_{1\text{min}}$, yielding $\text{Var}_r(y)$. Average all the $\text{Var}_r(y)$, yielding the average reduced variance, say $\text{Var}_{\text{red}}(y)$. Then the average reduction, or the TMV, is $\text{Var}(y) - \text{Var}_{\text{red}}(y)$.

It can be shown (Saltelli et al 2004) that the TMV due to $S$, denoted by $\text{TMV}_S$, is given by

$$ \text{TMV}_S = \text{Var}[E(y|S)] $$ (2)

where E denotes expectation. ASSA’s measure of sensitivity is actually the ratio $\text{TMV}_S / VTOT$, the relative TMV.

ASSA provides two methods of computing TMV and BMV, a regression-based method, which fits a linear model to the output data, and a regression-free method, which makes no assumptions about the form of the model. We only consider the latter here.

Computation of $\text{TMV}_S / VTOT$ using the resampling method

Equation (2) computes $\text{TMV}_S$ exactly. ASSA does not compute (2) directly, but uses Saltelli et al’s (2000) resampling-based method, a Monte-Carlo method which provides an estimate of $\text{TMV}_S$ in fewer operations. They use the following procedure:

Algorithm 1 (Jansen 2005)

1. Construct two independent ensembles of inputs $X_1$ and $X_2$. $X_1$ and $X_2$ can be represented as matrices with $N$ rows and $m$ columns, where $N$ is the number of samples in each ensemble and $m$ is the number of inputs.
2. Let the column indices of the inputs in $S$ be $\{i_1, \ldots, i_p\}$. Note that $p \leq m$. Set columns $i_1, \ldots, i_p$ of $X_2$ equal to columns $i_1, \ldots, i_p$ respectively of $X_1$.
3. Using the chosen model, compute two output ensembles $y_1$ and $y_2$ from $X_1$ and $X_2$, respectively.
4. Compute the two sample variances $v_1$ and $v_2$ from $y_1$ and $y_2$ respectively. Estimate $VTOT = \text{Var}(y)$ by the geometric mean of $v_1$ and $v_2$, that is $VTOT \approx \sqrt{v_1v_2}$.
5. Let $\text{corr}(y_1, y_2)$ be the Pearson correlation coefficient of $y_1$ and $y_2$ (Parratt 1961). It can be shown (Saltelli et al. 2000) that $\text{corr}(y_1, y_2)$ is an estimate of the relative TMV:

$$ \text{TMV}_S / VTOT \equiv \text{corr}(y_1, y_2). $$ (3)

UAST v.1

UAST v.1 adapts the ASSA methodology to the estimation of uncertainty and sensitivity of transmission loss ($TL$) in the underwater acoustic channel to uncertainty in sound speed profile (SSP) and bottom loss (BL). $TL$ is an important parameter in the estimation of sonar performance. It is defined by

$$ TL = -20 \log_{10}(p / p_1) $$ (4)

where $p_1$ and $p$ are the rms sound pressure at 1m from the transmitter and at the receiver respectively. We review how the four main steps of UASA are implemented in UAST v.1.

Input Generators

The two main parameters that affect $TL$ are $SSP$ and $BL$. We neglect the effect of surface absorption loss in this paper.

SSP uncertainty model

In UAST v.1, the SSP uncertainty model of Dosso et al (2007) is used – a schematic diagram is given in Figure 1. The SSP uncertainty (indicated by the dotted lines) is taken to represent variability due to surface heating/cooling and wind mixing, with the effects decaying exponentially with depth over the top 30m, as shown in Figure 1. This variability is represented in terms of the standard deviation of the surface sound speed, which is assumed to have a Gaussian distribu-
tion. Mathematically, the variation of the perturbation with depth can be expressed as

$$\Delta c_z = \Delta c_0 \exp(-\gamma z)$$  \hspace{1cm} (5)

where \(z\) is the depth, \(\Delta c_z\) is the perturbation at depth \(z\), \(\Delta c_0\) is the perturbation at the surface, and \(\gamma\) is the rate of exponential decay. So the perturbed SSP is given by

$$c_z = \tilde{c}_z + \Delta c_z$$  \hspace{1cm} (6)

where \(\tilde{c}_z\) is the unperturbed SSP. Note from (5) and (6) that the perturbed SSP can be expressed in terms of a single uncertain parameter, viz. \(\Delta c_0\). So an ensemble of uncertain SSPs can be generated from a Gaussian ensemble of \(\Delta c_0\)’s using (5) and (6).

$$c_0 = 1520 \pm 4 \text{ m/s}$$

Figure 1. Source: (Dosso et al 2007)

**Figure 1.** Sound speed profile, bottom parameters and associated uncertainties for the Malta Plateau environment

Key points in SSP (depth, sound speed):

A: (0, 1516); B: (0, 1520); C: (0, 1524)

D: (30, 1515); E: (64, 1511); F: (131, 1512)

**BL uncertainty model**

The bottom can be characterized by BL versus grazing angle (β) curves. Jones et al (2008) have shown that for many bottom types, these curves may be characterized by a single parameter, viz the slope, \(g\), of the BL versus β curve at low grazing angles. They show that the magnitude of BL can be approximated by

$$BL \cong g \beta$$  \hspace{1cm} (7)

and that the reflection phase shift, \(\varphi\), can be approximated by

$$\varphi \cong -\pi + \pi g \beta / 6, \quad \beta \leq 6/g, \quad 0 \quad \text{otherwise.}$$ \hspace{1cm} (8a)\hspace{1cm} (8b)

So if the distribution of \(g\) is known, an ensemble of uncertain complex bottom reflection coefficients \((\text{BRCs})\) can be generated using (7) and (8).

In UAST v.1, the distribution of \(g\) is estimated using Dosso et al’s (2007) uncertainty model for 11 geoaoustic parameters such as compressional sound speed, layer thickness, etc. (see Figure 1). The uncertainties in these parameters are modelled in Dosso et al. (2007) by Gaussian distributions, with unrealistic samples (such as negative thicknesses) being excluded.

In UAST v.1, a large set of samples of these parameters (varying all parameters) is generated, assuming that the perturbations are independent. For each sample of the 11 parameters, the BL versus \(\beta\) curve is calculated using a multi-layer reflection coefficient formula (Bartel, D.W., private communication 2007). The bottom loss slope \(g\) is then estimated by fitting a straight line to curve at small \(\beta\), specifically the interval from 0° to 10°. The distribution of the \(g\)’s is then calculated, and is shown in Figure 2.

Figure 2. Source: (Sweet and Jones 2008)

**Figure 2.** Probability density function (PDF) of bottom loss gradients derived from bottom parameters and uncertainties of Figure 1

Note that this PDF is highly skewed towards the lower bottom loss slopes. The unperturbed gradient is also indicated on Figure 2. Note that the uncertainty model of Figure 1 generates more samples of \(g\) below the unperturbed value than above.

**Input-output Model**

In UAST v.1, the inputs are the SSP, the complex BRC curve, the absorption coefficient and the source and receiver positions. The latter three inputs are assumed known. The output is \(TL\), which can be computed using a variety of models. Porter’s (2008) Acoustics Toolbox contains a number of codes, including BELLHOP (ray-tracing model) and KRAKENC (modal model). UAST v.1 uses BELLHOP because of its speed of execution and the fact that it gives accurate results at the chosen frequency of 1200 Hz.

The procedure for generating an ensemble of outputs for UA and SA is thus:

1. Generate ensembles of uncertain SSPs and BRCs as described above.
2. Input these ensembles (together with known parameters) to BELLHOP to yield an ensemble of TLs.

**Uncertainty and Sensitivity Analyses**

We estimate \(VTOT\) as the measure of TL uncertainty, and \(TMV_{SSP}/VTOT\) and \(TMV_{BL}/VTOT\) as the measures of TL
sensitivity to uncertainties in SSP and BL respectively. These are estimated using Algorithm 1. Some details of the application of Algorithm 1 to TL are as follows:

Step 1. The solid and dashed curves show the relative changes at these ranges.

Step 2. The set is either or . For sensitivity to , set the ’s in to those in . For sensitivity to , set the ’s in to those in .

Step 3. For each pair , the solid and dashed curves show the relative changes only slowly with position at . If is to “shift” the pattern, so at these ranges, TL would be expected to be relatively insensitive to SSP changes.

The UA results for this environment are shown in Figure 3, sub-plot (a). The solid curve shows the mean ensemble incoherent TL versus range. Note the peaks at 4km and 8km. The upper dashed curves show plots of mean TL=VTOT, i.e. mean TL± one standard deviation. The standard deviations vary between 4 dB and 8.5 dB.

Figure 4 shows a range-depth plot of unperturbed TL. For a slice through the plot at 15m (the receiver depth), there are peaks at ranges 4 km and 8km, and these correspond to the peaks in TL (Figure 3(a)). Figure 4 also shows that TL changes rapidly with position at depth 15m and ranges 500m, 4km and 8km. One effect of perturbing the SSP is to “shift” the TL pattern, so at these ranges, TL would be expected to be sensitive to SSP changes.

Figure 4 shows that TL changes only slowly with position at depth 15m and ranges 2km and 6.5km, hence TL would be expected to be relatively insensitive to SSP changes. Conversely TL would be expected to be relatively more sensitive to BL changes at these ranges.

EXAMPLE UNCERTAINTY/SENSITIVITY RESULTS

We review the results presented in (Sweet and Jones 2008) for the environment of the Malta Plateau (Dosso et al 2007). Figure 1 illustrates the environment, its parameters and associated uncertainties (standard deviations of Gaussian distributions). The source and receiver depths are both 15m.

The UA results for this environment are shown in Figure 3, sub-plot (a). The solid curve shows the mean ensemble incoherent TL versus range. Note the peaks at 4km and 8km. The upper dashed curves show plots of mean TL=VTOT, i.e. mean TL± one standard deviation. The standard deviations vary between 4 dB and 8.5 dB.

Step 4. Compute and using BELLHOP. These correspond to and in Algorithm 1.

Step 5. If , then , otherwise . For sensitivity to uncertainties respectively – we denote these by TMVSSP/VTOT and TMVBL/VTOT using (7) and (8). This generates two ensembles of SSPs and BSs. Generate corresponding ensembles of , say , using BELLHOP. These correspond to and in Algorithm 1.

The SA results for this environment are shown in Figure 3, sub-plot (b). The solid and dashed curves show the relative changes in the ensembles of and . For one input-output model (Simakov 2009), about 50,000 samples were required for convergence with two uncertain parameters, and the number of samples required increases rapidly as the number of parameters increases.

We recall the exact expression for due to a set of parameters in equation (2).

We consider the case where there are only two uncertain parameters, say and . Let their probability density functions be and respectively, and assume that and are independent. We evaluate (2) for . For any value of we have

For a set of parameters ,

Evaluation of TMV by Algorithm 1 is fast, but does not provide the exact value. It yields an approximation that asymptotically converges to the exact value as the number of samples in the ensembles of and .

Discussion of results

Figure 4 shows a range-depth plot of unperturbed TL. For a slice through the plot at 15m (the receiver depth), there are peaks at ranges 4 km and 8km, and these correspond to the peaks in TL (Figure 3(a)).
\[ TMV(a_1) = \int_{-\infty}^{\infty} P_1(a_1) \left( c(a_1) - \bar{c} \right)^2 da_1 \]  

(11)

The exact expression for the total variance is

\[ VTOT = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_1(a_1) P_2(a_2) \left( y(a_1, a_2) - \bar{y} \right)^2 da_1 da_2 \]  

(12)

and the relative TMV is

\[ RTMV(a_1) = \frac{TMV(a_1)}{VTOT} . \]  

(13)

Eqs. (9)-13) can be used to evaluate \( VTOT \) and \( RTMV(a_1) \) directly. For our underwater acoustic UA/SA, \( a_1 = \Delta c_0 \), \( P_1 \) is Gaussian, \( a_2 = g \), \( P_2 \) is the distribution of \( g \) as estimated above (Figure 2), and \( y = TL \). \( RTMV(a_2) \) can be evaluated similarly (by interchanging \( a_1 \) and \( a_2 \), and \( P_1 \) and \( P_2 \), in (9)-(13). The actual numerical steps in evaluating (9)-(13) are as follows:

- Define a grid of values of \( a_1 \) and \( a_2 \) covering the intervals of support of \( P_1 \) and \( P_2 \) respectively, or if either interval is infinite, the interval in which the probability is more than some small threshold.

- For each value of \( a_1 \), evaluate (9) numerically, yielding \( c(a_1) \)

- Evaluate (10) numerically, yielding \( \bar{c} = \bar{y} \)

- Evaluate (11) numerically, yielding TMV\( (a_1) \)

- For each value of \( a_2 \), evaluate

\[ r(a_2) = \int_{-\infty}^{\infty} P_1(a_1) \left( y(a_1, a_2) - \bar{y} \right)^2 da_1 \]  

numerically

- Evaluate \( VTOT = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_2(a_2) r(a_2) da_2 \) numerically

- \( RTMV(a_1) = \frac{TMV(a_1)}{VTOT} \).

Figure 5 shows TL, its uncertainty range (TL ± one standard deviation) and its top marginal variances versus range. The curves in Figures 3 and 5 are very similar, but not the same. Parts (a) of both figures are virtually indistinguishable by eye, except at range 500m, where a singularity in integral (12) at very high SNR caused a “not-a-number” to be generated in the direct TMV computation. In parts (b) of the figures (the TMVs) the peaks and troughs of the curves are in about the same places. The differences in TMVs between Figures 3 and 5 generally varied from zero to 10%. The differences can be attributed to the approximation used in the resampling method (Algorithm 1).

**PROBABILITY OF DETECTION**

An important measure of sonar performance is probability of detection. As well as TL it depends on several other parameters such as source level, noise level, receiver directivity index and target strength - the ratio of the echo intensity 1 m from the target to the incident intensity.

The probability of detection \( P_d \) of a sonar system depends on two parameters – the signal-to-noise ratio (SNR) at the input of the detection system and the value of a decision threshold, say \( T_d \); if the system output is greater than or equal to \( T_d \) it is decided that the target is present, and absent otherwise.

The SNR (in decibels) can be computed by

\[ SNR = SL + DI - 2 \cdot TL + TS - NL \]  

(14)

where \( SL \) is the source level, \( DI \) is the receiver directivity index, \( TL \) is the transmission loss, \( TS \) is the target strength and \( NL \) is the noise level. All these quantities are specified in decibels, either as ratios or relative to reference values.

In operating a sonar, \( T_d \) is commonly set to give a specified probability of false alarm \( P_{fa} \). Expressions relating \( P_d \) to \( P_{fa} \) and SNR have been derived in numerous textbooks – see, for example Burdic (1991). For given values of \( SL \), \( DI \), \( TS \), \( NL \) and \( P_{fa} \), \( P_d \) is a monotonic function of \( TL \), that is, \( P_d \) decreases as \( TL \) increases. Denote this monotonic function by

\[ P_d = F_{Pd}(TL) \]  

(15)

To perform UA and SA on the output parameter \( P_d \), we carry out the same steps as for TL, except that in the computation of the output ensemble, we first compute each \( TL \), then apply (15).

UA and SA for probability of detection were carried out for the same source and receiver positions, and the same environment as for TL above. Typical values of \( SL \), \( DI \), \( TS \), \( NL \) and \( P_{fa} \) were used in (14).

The results are shown in Figure 6. The solid curve in sub-plot (a) shows mean \( P_d \) versus range. The upper dashed curve shows \( P_d + one standard deviation (s.d.), with any values greater than 1 (due to the skewness of the distribution of \( P_d \)) clipped to 1. The lower dashed curve shows \( P_d - one s.d., with negative values (due to \( P_d \) skewness) clipped to zero.

The solid and dashed curves in sub-plot (b) show the relative TMVs of \( P_d \) due to uncertainty in SSP and BL respectively. The TMVs were computed using the direct evaluation
method. Note that \( P_d \) is more sensitive to SSP than BL at all ranges greater than 3200m. Recall that from Figure 5(b), TL is more sensitive to BL at some ranges greater than 3200m, viz. 5500m to 7300m and 9400m to 10000m. In other words, the order of sensitivities was reversed at these ranges.

The following explanation of the insensitivity of \( P_d \) to BL at longer ranges is offered by Adrian Jones (private communication, 2009). The observation that the \( P_d \) at longer ranges is sensitive to the sound speed profile, and not sensitive to the seafloor, is due to a combination of the fact that relevant \( P_d \) values at these ranges are low, and the fact that the SSP variability scales from mostly downward refracting to a few cases of upward refraction, or near to it. This probably means that the only detections one gets at longer ranges are due to the near-upward refracting scenarios for which the seafloor is not a major factor. That is, it is the SSP variation that gives rise to the detections.

If the ensemble of SSP were all downward refracting, one would expect \( P_d \) to be more strongly influenced by the seafloor, if the seafloor varied from absorptive to reflective.

So one would expect that the relative sensitivities of \( P_d \) to uncertainty in SSP and uncertainty in seafloor reflectivity are a strong function of (i) the nature of the “baseline” (unperturbed) SSP and of the assumed distribution for the SSPs due to uncertainty, and (ii) the nature of the “baseline” seafloor reflectivity and the assumed distribution for this due to the uncertainty. Another factor is likely to be whether the \( P_d \) is either (1) all high, (2) all low, (3) in between. If \( P_d \) is either all high or all low, the sensitivity to parameter uncertainty will be due to the conditions which give rise to the outliers.

The uncertainty and sensitivity in \( P_d \) versus range were compared with those of transmission loss (TL) versus range for the Malta Plateau environment. It was found that at some ranges (including ranges greater than 3200m), TL was more sensitive to bottom loss uncertainty than sound speed profile (SSP) uncertainty; however, for the parameters used, \( P_d \) was more sensitive to SSP uncertainty at all ranges greater than 3200m. An explanation of these results due to Jones (2009) is given.

In future work, the UAST will be enhanced to allow (i) more than two uncertain parameters; (ii) Latin hypercube sampling, which covers the sample hyperspace more thoroughly, and allows a reduction in the number of samples; (iii) modelling of SSP uncertainty by a set of independently varying orthogonal functions (iv) modelling of uncertainties in several bottom parameters, (v) using other acoustic models for TL and (vi) modelling of uncertainties in other parameters in \( P_d \), such as target strength, noise level uncertainty, positional uncertainty, etc. The tool will be used on a large number of scenarios to glean a better understanding of the relative importance of uncertainties in inputs in various environments.

**REFERENCES**


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**CONCLUSIONS AND FUTURE WORK**

This paper presents two enhancements to an underwater acoustic sensitivity tool (UAST v.1). These are (i) direct evaluation of sensitivity, as measured by top marginal variance, which is more accurate than the resampling method, but slower; and (ii) evaluation of uncertainty and sensitivity in probability of detection (\( P_d \)), an important measure of sonar performance.

![Figure 6.](image-url)