Analytical Modelling of Single and Two-Stage Vibration Isolation Systems

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ABSTRACT

The structural vibration induced by machinery often needs to be assessed during the concept design phase in order to determine the configuration of machinery that minimises transmitted vibration, in addition to satisfying other design constraints. However, detailed information about machinery and the foundation's structure may not be available, and in these cases simplified analytical models can be used to assess potential designs. This paper derives the equations of motion for generalised two-stage and two-stage rafted vibration isolation systems using a matrix methodology. The supported mass and intermediate masses are assumed to be rigid and supported by isolators with arbitrary locations and orientations. The use of the methodology is illustrated for three common isolator configurations: single stage isolation; two-stage isolation; and two-stage rafted isolation, and the characteristics of each are discussed.

INTRODUCTION

Isolation of machinery vibration is required in many situations including industrial applications, commercial buildings, and ships. Specification of vibration isolators can be as simple as selecting the static deflection of an isolator based on the required level of vibration isolation and the forcing frequency. In this case, the parameters are derived from the dynamics of a uni-axial single degree-of-freedom (DOF) system (Beranek and Vér, 1992; Harris, 1995). In some applications a high degree of vibration isolation is required and in these cases more detailed modelling is required. One approach may be to apply finite element modelling techniques to assess the dynamics of machinery coupled to a supporting structure. Another method is to develop an approximate analytical model of the system. This approach may be more appropriate than finite element modelling in cases where detailed information on the machinery and supporting foundation is not known, for example at the concept design stage. Analytical modelling can provide greater insight into coupling of degrees-of-freedom (DOFs), the relationship between different parameters, as well as their effect on the performance of a vibration isolation system.

Equations describing the rigid-body dynamics of a supported mass are available in a number of sources (Beranek and Vér, 1992; Harris, 1995; Mead, 2000; Smollen, 1966). Manipulating the equations describing the motions of a supported mass in six degrees of freedom, including coupled rotation and translational stiffness is cumbersome, and becomes more tedious when modelling two-stage vibration isolations systems. Smollen (1966) presented a general matrix method for the design and analysis of single-stage vibration-isolation systems that overcomes some of the difficulties encountered when dealing with equations in terms of scalar variables. In this paper the generalised matrix method of Smollen is extended to general two-stage and two-stage rafted vibration isolation systems. The equations of motion are used to calculate the performance of representative single-stage, two-stage, and two-stage rafted vibration isolation systems and the characteristics of each type of system are briefly discussed.

MATRIX EQUATIONS FOR A SINGLE-STAGE VIBRATION ISOLATION SYSTEM

A general single stage vibration isolation system is shown in Figure 1. It is parameterised by the mass and inertial properties of a supported mass; the translation and rotational stiffness of one or more vibration isolators; and the location and orientation of the vibration isolator(s) with respect to a set of coordinate axes located at the centre of gravity (CoG) of the supported mass. The damping of the vibration isolators can also be included as either viscous damping in parallel with the stiffness associated with each isolator, or as a structural damping factor implemented as a complex stiffness. Damping terms will not be included in the derivations that follow for the sake of brevity.

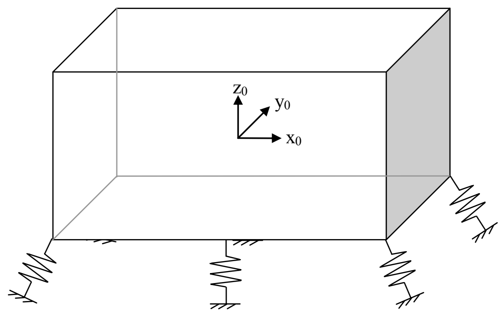

Figure 1. Single stage isolation system: a rigid mass is supported on one or more vibration isolators with arbitrary location and orientation.

The matrix method developed by Smollen involves transforming the stiffness (and damping) associated with each vibration isolator from its own local coordinate system to a coordinate system located at the CoG of the supported mass.

Figure 2 shows a rigid body of mass m_0 supported by a vibration isolator with arbitrary location and orientation. A set of Cartesian coordinate axes (x_0, y_0, z_0) is located at the CoG of the supported mass (see Figure 2, 3a).

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Each isolator has an associated set of principal elastic axes p_i, q_i, r_i (see Figure 3b). The principal elastic axes are defined such that a force along an axis results in deformation along that axis only, and no rotation. Similarly, a couple about any principal elastic axis results in no rotation about any other axis and no translation (Harris, 1995). Each isolator has translational stiffness $\boldsymbol{K}_{pi}$ and rotational stiffness $\boldsymbol{K}_{\lambda i}$ defined along the principal elastic axes; i.e. the matrices are diagonal. The origin of the principal elastic axes is located at the point of action of the isolator on the supported mass. A vector $\boldsymbol{r}_{0i} = \{r_{0x}, r_{0y}, r_{0z}\}_i^{\mathrm{T}}$ locates the point of action of isolator i with respect to the CoG of the supported mass (Figure 2). Another set of axes (x_i, y_i, z_i) is located at the point of action of each isolator and is parallel with the axes centred on the supported mass CoG (see Figure 3c).

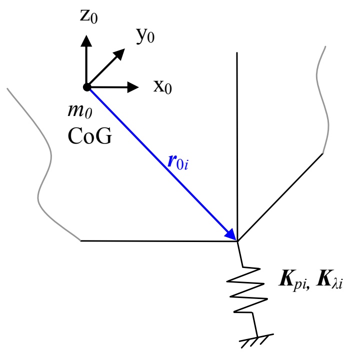


Figure 2. Mass supported by an arbitrarily located single-stage vibration isolator.

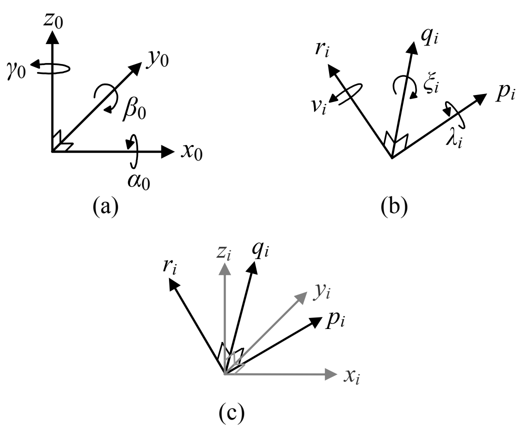


Figure 3. (a) Coordinate axes located at the CoG of the supported mass; (b) principal elastic axes associated with isolator i located at the point of action of the isolator on the supported mass; (c) principal elastic axes and local coordinate axes associated with isolator i (rotations are not shown for clarity).

Displacements $\boldsymbol{x}_i = \{x_i, y_i, z_i\}^{\mathrm{T}}$ at isolator i are related to displacements $\boldsymbol{p}_i = \{p_i, q_i, r_i\}^{\mathrm{T}}$ by $\boldsymbol{x}_i = \boldsymbol{A}_i\, \boldsymbol{p}_i$, where the transformation matrix $\boldsymbol{A}_i$ is given by

$$\boldsymbol{A}_i = \begin{bmatrix} a_{xp} & a_{xq} & a_{xr} \\ a_{yp} & a_{yq} & a_{yr} \\ a_{zp} & a_{zq} & a_{zr} \end{bmatrix}_i . \qquad (1)$$

The elements of $\boldsymbol{A}_i$ are the direction cosines between the positive axes listed in the subscripts; e.g. a_{xp} is the direction cosine between the positive x_i and p_i axes. Note that $\boldsymbol{A}_i$ is an orthogonal transformation matrix and therefore satisfies the relation $\boldsymbol{A}_i^{-1} = \boldsymbol{A}_i^{\mathrm{T}}$ (Kreyszig, 1993). In addition, for an arbitrary rotation, $\boldsymbol{A}_i$ can be obtained from the product of transformation matrices corresponding to rotations around individual axes.

The transformation matrix $\boldsymbol{A}_i$ also relates forces, rotations, and moments between the two sets of coordinates. That is, $\boldsymbol{f}_{xi} = \boldsymbol{A}_i \boldsymbol{f}_{pi}$; $\boldsymbol{\alpha}_{xi} = \boldsymbol{A}_i\, \boldsymbol{\lambda}_{pi}$; $\boldsymbol{h}_{xi} = \boldsymbol{A}_i\, \boldsymbol{h}_{pi}$, where $\boldsymbol{f}_{xi} = \{f_{xi}, f_{yi}, f_{zi}\}^{\mathrm{T}}$ are forces; $\boldsymbol{\alpha}_{xi} = \{\alpha_i, \beta_i, \gamma_i\}^{\mathrm{T}}$ are rotations; and $\boldsymbol{h}_{xi} = \{h_{xi}, h_{yi}, h_{zi}\}^{\mathrm{T}}$ are moments in x_i, y_i, z_i coordinates. Similarly $\boldsymbol{f}_{pi}$, $\boldsymbol{\lambda}_{pi}$, $\boldsymbol{h}_{pi}$, are forces, rotations and moments in the p_i, q_i, r_i coordinates.

The equations of motion for a supported mass can be written in terms of mass and stiffness matrices, where the stiffness matrices are initially defined for the action of a single isolator. The effect of multiple isolators is handled by summing the stiffness sub-matrices for each isolator. Results given by Smollen (1966) are repeated here without derivation and with changes in notation where appropriate. The reader is directed to the original reference for further details on the derivation.

Equations of motion for a supported mass are

$$\left[\begin{array}{ccc|ccc} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{xx} & -I_{xy} & -I_{xz} \\ 0 & 0 & 0 & -I_{yx} & I_{yy} & -I_{yz} \\ 0 & 0 & 0 & -I_{zx} & -I_{zy} & I_{zz} \end{array}\right] \begin{Bmatrix} \ddot{x}_0 \\ \ddot{y}_0 \\ \ddot{z}_0 \\ \hline \ddot{\alpha}_0 \\ \ddot{\beta}_0 \\ \ddot{\gamma}_0 \end{Bmatrix} + \left[\begin{array}{ccc|ccc} k_{xx} & k_{xy} & k_{xz} & k_{x\alpha} & k_{x\beta} & k_{x\gamma} \\ k_{yx} & k_{yy} & k_{yz} & k_{y\alpha} & k_{y\beta} & k_{y\gamma} \\ k_{zx} & k_{zy} & k_{zz} & k_{z\alpha} & k_{z\beta} & k_{z\gamma} \\ \hline k_{\alpha x} & k_{\alpha y} & k_{\alpha z} & k_{\alpha\alpha} & k_{\alpha\beta} & k_{\alpha\gamma} \\ k_{\beta x} & k_{\beta y} & k_{\beta z} & k_{\beta\alpha} & k_{\beta\beta} & k_{\beta\gamma} \\ k_{\gamma x} & k_{\gamma y} & k_{\gamma z} & k_{\gamma\alpha} & k_{\gamma\beta} & k_{\gamma\gamma} \end{array}\right] \begin{Bmatrix} x_0 \\ y_0 \\ z_0 \\ \hline \alpha_0 \\ \beta_0 \\ \gamma_0 \end{Bmatrix} = \boldsymbol{0} \qquad (2)$$

or in terms of partitioned matrices

$$\left[\begin{array}{c|c} \boldsymbol{m}_0 & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{I}_0 \end{array}\right] \begin{Bmatrix} \ddot{\boldsymbol{x}}_0 \\ \hline \ddot{\boldsymbol{\alpha}}_0 \end{Bmatrix} + \left[\begin{array}{c|c} \boldsymbol{K}_{xx} & \boldsymbol{K}_{x\alpha} \\ \hline \boldsymbol{K}_{\alpha x} & \boldsymbol{K}_{\alpha\alpha} \end{array}\right] \begin{Bmatrix} \boldsymbol{x}_0 \\ \hline \boldsymbol{\alpha}_0 \end{Bmatrix} = \boldsymbol{0} \qquad (3)$$

where

$$\boldsymbol{K}_{xx} = \sum_{i=1}^{n} \boldsymbol{K}_{xi} \qquad (4)$$

$$\boldsymbol{K}_{x\alpha} = \sum_{i=1}^{n} \boldsymbol{K}_{xi}\, \boldsymbol{R}_{0i}^{\mathrm{T}} = \boldsymbol{K}_{\alpha x}^{\mathrm{T}} = \left[\sum_{i=1}^{n} \boldsymbol{R}_{0i}\, \boldsymbol{K}_{xi}\right]^{\mathrm{T}} \qquad (5)$$

$$\boldsymbol{K}_{\alpha\alpha} = \sum_{i=1}^{n} \boldsymbol{R}_{0i}\, \boldsymbol{K}_{xi}\, \boldsymbol{R}_{0i}^{\mathrm{T}} + \boldsymbol{A}_i\, \boldsymbol{K}_{\alpha i}\, \boldsymbol{A}_i^{\mathrm{T}} \qquad (6)$$

$$\boldsymbol{K}_{xi} = \boldsymbol{A}_i\, \boldsymbol{K}_{pi}\, \boldsymbol{A}_i^{\mathrm{T}} \qquad (7)$$

$$\boldsymbol{K}_{\alpha i} = \boldsymbol{A}_i\, \boldsymbol{K}_{\lambda i}\, \boldsymbol{A}_i \qquad (8)$$

$$\boldsymbol{R}_{0i} = \begin{bmatrix} 0 & -r_{0z} & r_{0y} \\ r_{0z} & 0 & -r_{0x} \\ -r_{0y} & r_{0x} & 0 \end{bmatrix}_i \qquad (9)$$

and n is the number of vibration isolators.

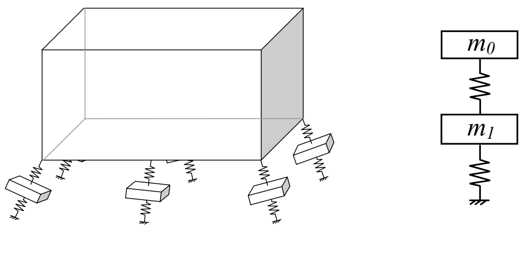
The matrix equations allow the coupling between the DOFs to be examined, as the level of coupling is given by off-diagonal terms of the mass and stiffness matrices. The off-diagonal terms of the mass matrix are due to non-symmetry of the supported mass and little can be done to change this in a real situation. The off-diagonal terms in the stiffness matrix are due to orientation of the isolators which give rise to off-diagonal terms in $\boldsymbol{K}_{xi}$ and $\boldsymbol{K}_{\alpha i}$ (see equations (7) and (8)), and also due to the location of each isolator with respect to the CoG of the supported mass (equations (5) and (6)).

Full decoupling of the DOFs in equation (2) is only possible if the whole set of isolators is symmetrical about the CoG of the supported mass (this eliminates $\boldsymbol{K}_{x\alpha}$ and $\boldsymbol{K}_{\alpha x}$ terms), and if principal elastic axes that result from the combined action of all isolators are aligned with the principal inertial axes of the supported mass. This is unlikely in real situations; however, partial decoupling of DOFs is possible and can be optimised for different applications. Mead (2000) gives an example of de-coupling rotational and translational degrees of freedom.

Single-stage vibration isolation systems are common as they are the simplest and provide good performance. Alternative configurations include two-stage vibration isolation systems where each isolator is an assembly of two springs separated by an intermediate mass (see Figure 4) and two-stage rafted systems where the supported mass is resiliently mounted on a platform, which in turn is resiliently supported on the foundation (see Figure 6). The matrix equations for these two cases are derived in the following sections.

MATRIX EQUATIONS FOR A TWO-STAGE VIBRATION ISOLATION SYSTEM

A two-stage vibration isolation system is shown in Figure 4a. The intermediate masses introduce extra DOFs into the system, and in general improve the high-frequency characteristics of the isolator when compared to a single stage vibration isolator. This characteristic is often derived from the dynamics of a uni-axial system (Figure 4b) (Beranek and Vér, 1992) and leads to the rule-of-thumb that the slope of the transmissibility is -40 dB/decade at high frequencies for single stage isolators and -80 dB/decade for dual-stage isolators (Mead, 2000). In this case high frequencies are greater than 1.5 times the highest natural frequency of the system. This will be further illustrated by the example at the end of the paper.



(a) (b)

Figure 4. (a) General two-stage vibration isolation system; (b) uniaxial two-stage vibration isolation system.

The equations of motion of a general two-stage vibration isolation system can be derived using matrix techniques in a similar way to that discussed for single-stage vibration isolation systems. Figure 5 shows a mass m_0 supported by a two-stage vibration isolator at a position $\boldsymbol{r}_{0i} = \{r_{0x}, r_{0y}, r_{0z}\}_i^{\mathrm{T}}$ with respect to a coordinate system at the CoG of the supported mass. The two-stage isolator consists of two collinear springs separated by the intermediate mass. The upper spring has translational and rotational stiffness $\boldsymbol{K}_{pui}$, $\boldsymbol{K}_{\lambda ui}$, respectively, and the lower spring stiffness is represented similarly by exchanging the subscript u for l. The intermediate masses are of mass m_i and have inertial properties $\boldsymbol{I}_{pi}$.

The upper and lower springs are assumed to act along the principal inertial axes of the intermediate masses and these are also assumed to be the principal elastic axes of the isolator assembly. This is a reasonable assumption as in practice the intermediate mass could be a rectangular prism; the upper and lower springs may be made up of multiple resilient elements arranged symmetrically about the top and bottom faces of the intermediate mass. The height of the intermediate mass could be made small so as to minimise coupling between lateral forces and rotational displacements and vice-versa.

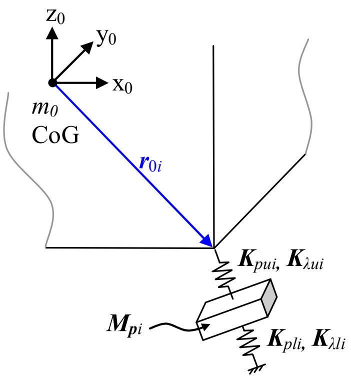


Figure 5. Mass supported by an arbitrarily located two-stage vibration isolator.

The local coordinate systems on the supported mass at the point of action of the isolator assembly are as described for the single stage system (see Figure 3). In addition, the motion of the intermediate mass ($\boldsymbol{x}_i$; $\boldsymbol{\alpha}_i$) is described with respect to coordinate systems centred on its CoG.

The isolator assembly has arbitrary orientation, and the stiffness of the upper and lower springs and the inertia of the intermediate mass are transformed from the principal elastic axes to axes parallel with x_0, y_0, z_0

$$\boldsymbol{K}_{xu\,i} = \boldsymbol{A}_i\,\boldsymbol{K}_{pu\,i}\,\boldsymbol{A}_i^{\mathrm{T}} \tag{10}$$

$$\boldsymbol{K}_{\alpha u\,i} = \boldsymbol{A}_i\,\boldsymbol{K}_{\lambda u\,i}\,\boldsymbol{A}_i^{\mathrm{T}} \tag{11}$$

$$\boldsymbol{K}_{xl\,i} = \boldsymbol{A}_i\,\boldsymbol{K}_{pl\,i}\,\boldsymbol{A}_i^{\mathrm{T}} \tag{12}$$

$$\boldsymbol{K}_{\alpha l\,i} = \boldsymbol{A}_i\,\boldsymbol{K}_{\lambda l\,i}\,\boldsymbol{A}_i^{\mathrm{T}} \tag{13}$$

$$\boldsymbol{I}_{xi} = \boldsymbol{A}_i\,\boldsymbol{I}_{pi}\,\boldsymbol{A}_i^{\mathrm{T}} \tag{14}$$

Forces Applied to Supported Mass

The forces applied to the supported mass by a single isolator are given by

$$\sum \boldsymbol{f}_x = \boldsymbol{m}_0\ddot{\boldsymbol{x}}_0 = -\boldsymbol{K}_{xui}(\boldsymbol{x}_0 - \boldsymbol{x}_1) - \boldsymbol{K}_{x\alpha ui}\boldsymbol{\alpha}_0 \tag{15}$$

therefore

$$\boldsymbol{m}_0\ddot{\boldsymbol{x}}_0 + \boldsymbol{K}_{xui}\boldsymbol{x}_0 - \boldsymbol{K}_{xui}\boldsymbol{x}_1 + \boldsymbol{K}_{x\alpha ui}\boldsymbol{\alpha}_0 = \boldsymbol{0} \tag{16}$$

The last term in equation (16) is the coupled translational-rotational stiffness $\boldsymbol{K}_{x\alpha ui}$, which relates forces at the isolator to a rotation of the supported mass. Translation at the isolator due to rotation of the supported mass is given by

$$\boldsymbol{x}_i = \boldsymbol{R}_{0i}^{\mathrm{T}}\boldsymbol{\alpha}_0 \tag{17}$$

where $\boldsymbol{R}_{0i}$ is defined in equation (9). The forces at the isolator resulting from a rotation of the supported mass are

$$\boldsymbol{f}_{xi} = \boldsymbol{K}_{xui}\boldsymbol{x}_i = \boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}}\boldsymbol{\alpha}_0 \tag{18}$$

Therefore the translational-rotational stiffness is given by

$$\boldsymbol{K}_{x\alpha ui} = \boldsymbol{K}_{xui}\,\boldsymbol{R}_{0i}^{\mathrm{T}} \tag{19}$$

Moments Applied to Supported Mass

Moments ($\boldsymbol{h}_{x0}$) applied to the supported mass result from rotations of the supported mass relative to rotations of the intermediate mass; i.e. $-\boldsymbol{K}_{\alpha ui}(\boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_i)$. Forces at the isolator location also produce moments about the CoG of the supported mass, and these forces are due to translations *and* rotations of the supported mass

$$\boldsymbol{f}_{xi} = -\boldsymbol{K}_{xui}(\boldsymbol{x}_0 - \boldsymbol{x}_i) - \boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}}\boldsymbol{\alpha}_0 \tag{20}$$

Summing the moments about the CoG of the supported mass

$$\sum \boldsymbol{h}_{x0} = \boldsymbol{I}_0\ddot{\boldsymbol{\alpha}} = -\boldsymbol{K}_{\alpha ui}(\boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1) + \boldsymbol{R}_{0i}\boldsymbol{f}_{xi}\,, \tag{21}$$

and substituting equation (20) into equation (21) gives

$$\begin{aligned}\sum \boldsymbol{h}_{x0} &= \boldsymbol{I}_0\ddot{\boldsymbol{\alpha}} \\ &= -\boldsymbol{K}_{\alpha ui}(\boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1) + \boldsymbol{R}_{0i}\left(-\boldsymbol{K}_{xui}(\boldsymbol{x}_0 - \boldsymbol{x}_i) - \boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}}\boldsymbol{\alpha}_0\right)\end{aligned} \tag{22}$$

Further manipulation yields

$$\boldsymbol{I}_0\ddot{\boldsymbol{\alpha}} + \left(\boldsymbol{K}_{\alpha ui} + \boldsymbol{R}_{0i}\boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}}\right)\boldsymbol{\alpha}_0 - \boldsymbol{K}_{\alpha ui}\boldsymbol{\alpha}_1 + \boldsymbol{K}_{\alpha xui}\boldsymbol{x}_0 - \boldsymbol{K}_{\alpha xui}\boldsymbol{x}_i = \boldsymbol{0} \tag{23}$$

where

$$\boldsymbol{K}_{\alpha xui} = \boldsymbol{R}_{0i}\boldsymbol{K}_{xui} = \left[\boldsymbol{K}_{x\alpha ui}\right]^{\mathrm{T}} = \left[\boldsymbol{K}_{xui}\,\boldsymbol{R}_i^{\mathrm{T}}\right]^{\mathrm{T}} \tag{24}$$

is the rotational-translational stiffness. Note that $\boldsymbol{K}_{xui}$ (and $\boldsymbol{K}_{\alpha ui}$) are symmetric.

Forces Applied to the Intermediate Mass

The forces applied to the intermediate mass are given by

$$\sum \boldsymbol{f}_x = \boldsymbol{m}_1\ddot{\boldsymbol{x}}_1 = -\boldsymbol{K}_{xui}\left(\boldsymbol{x}_1 - \boldsymbol{x}_0 - \boldsymbol{R}_{0i}^{\mathrm{T}}\boldsymbol{\alpha}_0\right) - \boldsymbol{K}_{xli}\boldsymbol{x}_1 \tag{25}$$

therefore

$$\boldsymbol{m}_1\ddot{\boldsymbol{x}}_1 + \left(\boldsymbol{K}_{xui} + \boldsymbol{K}_{xli}\right)\boldsymbol{x}_1 - \boldsymbol{K}_{xui}\boldsymbol{x}_0 - \boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}}\boldsymbol{\alpha}_0 = \boldsymbol{0} \tag{26}$$

and substituting equation (19) into equation (26) gives

$$\boldsymbol{m}_1\ddot{\boldsymbol{x}}_1 + \left(\boldsymbol{K}_{xui} + \boldsymbol{K}_{xli}\right)\boldsymbol{x}_1 - \boldsymbol{K}_{xui}\boldsymbol{x}_0 - \boldsymbol{K}_{x\alpha ui}\boldsymbol{\alpha}_0 = \boldsymbol{0} \tag{27}$$

Moments Applied to the Intermediate Mass

The moments applied to the intermediate mass are given by

$$\sum \boldsymbol{h}_{xi} = \boldsymbol{I}_i\ddot{\boldsymbol{\alpha}}_i = -\boldsymbol{K}_{\alpha ui}(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_0) - \boldsymbol{K}_{\alpha li}\,\boldsymbol{\alpha}_1 \tag{28}$$

therefore

$$\boldsymbol{I}_i\ddot{\boldsymbol{\alpha}}_i - \boldsymbol{K}_{\alpha ui}\,\boldsymbol{\alpha}_0 + \left(\boldsymbol{K}_{\alpha ui} + \boldsymbol{K}_{\alpha li}\right)\boldsymbol{\alpha}_1 = \boldsymbol{0} \tag{29}$$

Matrix Equations

Equations (16), (23), (26), and (29) can be assembled into a matrix equation for n isolators given in equation (30), with elements of the stiffness matrix given by equations (31) – (34).

$$\boldsymbol{K}_{xxu} = \sum_{i=1}^{n} \boldsymbol{K}_{xui} \tag{31}$$

$$\boldsymbol{K}_{\alpha\alpha u} = \sum_{i=1}^{n} \boldsymbol{R}_{0i}\,\boldsymbol{K}_{xui}\,\boldsymbol{R}_{0i}^{\mathrm{T}} + \boldsymbol{K}_{\alpha ui} \tag{32}$$

$$\boldsymbol{K}_{x\alpha u} = \sum_{i=1}^{n} \boldsymbol{K}_{xui}\,\boldsymbol{R}_{0i}^{\mathrm{T}} \tag{33}$$

$$\boldsymbol{K}_{\alpha xu} = \sum_{i=1}^{n} \boldsymbol{R}_{0i}\,\boldsymbol{K}_{xui} \tag{34}$$

$\boldsymbol{R}_{0i}$ is given by equation (9).

MATRIX EQUATIONS FOR A TWO-STAGE RAFTED VIBRATION ISOLATION SYSTEM

A two-stage rafted vibration isolation system is shown in Figure 6. The supported mass is resiliently supported on top of the raft and the raft is resiliently supported on a rigid foundation. The raft is effectively an intermediate mass but is typically much more substantial than intermediate masses used in a two-stage vibration isolation system. A desirable consequence is that the natural frequencies associated with the motion of the raft are lower than those associated with intermediate masses in a two-stage system.

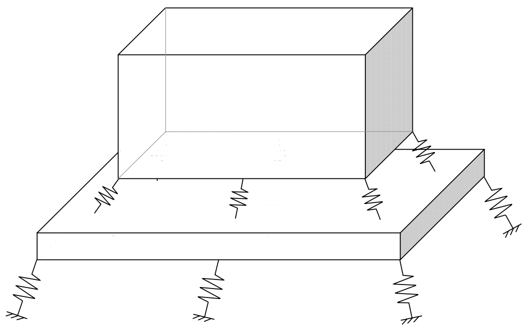


Figure 6. Two-stage rafted vibration isolation system.

Figure 7 shows a mass m_0 supported by isolator i with translational and rotational stiffness $\boldsymbol{K}_{pui}$ and $\boldsymbol{K}_{\lambda ui}$, respectively, which act along the principal elastic axes of the isolator. The isolator is coupled to the supported mass at a position $\boldsymbol{r}_{0i} = \{r_{0x}, r_{0y}, r_{0z}\}_i^{\mathrm{T}}$ with respect to a coordinate system at the CoG of the supported mass. Similarly, the isolator is coupled to the raft with mass m_1 at a position $\boldsymbol{r}_{1i} = \{r_{1x}, r_{1y}, r_{1z}\}_i^{\mathrm{T}}$ with respect to a coordinate system located at the CoG of the raft.

$$\begin{bmatrix} \boldsymbol{M}_0 & \boldsymbol{0} & & & & & \\ \boldsymbol{0} & \boldsymbol{I}_0 & & & \boldsymbol{0} & & \\ & & \boldsymbol{M}_1 & \boldsymbol{0} & & & \\ & & \boldsymbol{0} & \boldsymbol{I}_1 & & & \\ & & & & \ddots & & \\ & \boldsymbol{0} & & & & \boldsymbol{M}_i & \boldsymbol{0} \\ & & & & & \boldsymbol{0} & \boldsymbol{I}_i \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{x}}_0 \\ \ddot{\boldsymbol{\alpha}}_0 \\ \ddot{\boldsymbol{x}}_1 \\ \ddot{\boldsymbol{\alpha}}_1 \\ \vdots \\ \ddot{\boldsymbol{x}}_i \\ \ddot{\boldsymbol{\alpha}}_i \end{Bmatrix} + \begin{bmatrix} \boldsymbol{K}_{xxu} & \boldsymbol{K}_{x\alpha u} & -\boldsymbol{K}_{xxu1} & \boldsymbol{0} & \cdots & -\boldsymbol{K}_{xxui} & \boldsymbol{0} \\ \boldsymbol{K}_{\alpha xu} & \boldsymbol{K}_{\alpha\alpha u} & -\boldsymbol{K}_{\alpha xu1} & -\boldsymbol{K}_{\alpha\alpha u1} & \cdots & -\boldsymbol{K}_{\alpha xui} & -\boldsymbol{K}_{\alpha\alpha ui} \\ -\boldsymbol{K}_{xxu1} & -\boldsymbol{K}_{x\alpha u1} & \boldsymbol{K}_{xxu1} + \boldsymbol{K}_{xxl1} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{K}_{\alpha\alpha u1} & \boldsymbol{0} & \boldsymbol{K}_{\alpha\alpha u1} + \boldsymbol{K}_{\alpha\alpha l1} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\boldsymbol{K}_{xxui} & -\boldsymbol{K}_{x\alpha ui} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{K}_{xxui} + \boldsymbol{K}_{xxli} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{K}_{\alpha\alpha ui} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{K}_{\alpha\alpha ui} + \boldsymbol{K}_{\alpha\alpha li} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{\alpha}_0 \\ \boldsymbol{x}_1 \\ \boldsymbol{\alpha}_1 \\ \vdots \\ \boldsymbol{x}_i \\ \boldsymbol{\alpha}_i \end{Bmatrix} = \boldsymbol{0} \tag{30}$$

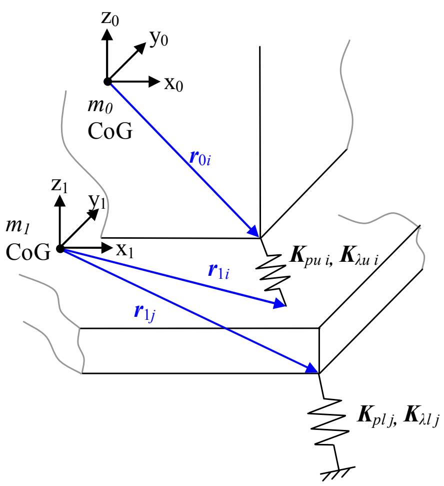


Figure 7. Supported mass and raft with arbitrarily located upper and lower isolators.

The raft is supported by isolator *j* with translational and rotational stiffness $\boldsymbol{K}_{plj}$ and $\boldsymbol{K}_{\lambda lj}$, respectively, which act along the principal elastic axes of the isolator. This isolator is coupled to the raft at a position $\boldsymbol{r}_{1j} = \{r_{1x}, r_{1y}, r_{1z}\}_j^{\mathrm{T}}$ with respect to a coordinate system at the CoG of the supported mass.

The isolator assembly has arbitrary orientation and the stiffness of the upper and lower springs are transformed from the principal elastic axes to coordinates parallel with x_0, y_0, z_0

$$\boldsymbol{K}_{xui} = \boldsymbol{A}_i \boldsymbol{K}_{pui} \boldsymbol{A}_i^{\mathbf{T}} \tag{35}$$

$$\boldsymbol{K}_{\alpha ui} = \boldsymbol{A}_i \boldsymbol{K}_{\lambda ui} \boldsymbol{A}_i^{\mathrm{T}} \tag{36}$$

$$\boldsymbol{K}_{xlj} = \boldsymbol{A}_j \boldsymbol{K}_{plj} \boldsymbol{A}_j^{\mathbf{T}} \tag{37}$$

$$\boldsymbol{K}_{\alpha lj} = \boldsymbol{A}_i \boldsymbol{K}_{\lambda lj} \boldsymbol{A}_j^{\mathbf{T}} \tag{38}$$

The coordinates x_1, y_1, z_1 are assumed to be parallel to x_0, y_0, z_0; the inertial properties of the supported mass and the raft are given by $\boldsymbol{I}_0$ and $\boldsymbol{I}_1$, respectively.

The equations of motion for the two-stage rafted system are derived in a similar way to that presented for the two-stage system.

Forces Applied to Supported Mass

The forces applied to the supported mass by the isolator *i* are given by

$$\sum \boldsymbol{f}_{x0} = \boldsymbol{m}_0 \ddot{\boldsymbol{x}}_0 = -\boldsymbol{K}_{xui}\left(\boldsymbol{x}_0 - \boldsymbol{x}_{1TOT}\right) - \boldsymbol{K}_{x\alpha ui}\boldsymbol{\alpha}_0 \tag{39}$$

therefore

$$\boldsymbol{m}_0 \ddot{\boldsymbol{x}}_0 + \boldsymbol{K}_{xui}\boldsymbol{x}_0 - \boldsymbol{K}_{xui}\boldsymbol{x}_{1TOT} + \boldsymbol{K}_{x\alpha ui}\boldsymbol{\alpha}_0 = \boldsymbol{0} \tag{40}$$

where $\boldsymbol{K}_{x\alpha ui}$ is given by equation (19) and

$$\boldsymbol{x}_{1TOT} = \boldsymbol{x}_1 + \boldsymbol{R}_{1i}^{\mathrm{T}}\boldsymbol{\alpha}_1 \tag{41}$$

$\boldsymbol{R}_{1i}$ is defined in an analogous way to $\boldsymbol{R}_{0i}$ (equation (9)) using $\boldsymbol{r}_{1i}$. Equation (41) represents translation of the raft at position $\boldsymbol{r}_{1i}$ due to translation *and* rotation of the raft. Substituting equation (41) into equation (40), and then using the expression for $\boldsymbol{K}_{x\alpha ui}$ in equation (19) gives

$$\boldsymbol{m}_0 \ddot{\boldsymbol{x}}_0 + \boldsymbol{K}_{xui}\boldsymbol{x}_0 + \boldsymbol{K}_{x\alpha ui}\boldsymbol{\alpha}_0 - \boldsymbol{K}_{xui}\boldsymbol{x}_1 - \boldsymbol{K}_{x\alpha ui}\boldsymbol{\alpha}_1 = \boldsymbol{0} \tag{42}$$

Moments Applied to Supported Mass

Moments ($\boldsymbol{h}_{x0}$) applied to the supported mass result from rotations of the supported mass relative to rotations of the raft; i.e. $-\boldsymbol{K}_{\alpha ui}(\boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1)$. Forces due to translation at the isolator location also produce moments about the CoG of the supported mass. Translation at the isolator location is due to translation *and* rotation of the supported mass

$$\boldsymbol{f}_{xi} = -\boldsymbol{K}_{xui}\left(\boldsymbol{x}_0 - \boldsymbol{x}_{1TOT}\right) - \boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}}\boldsymbol{\alpha}_0 . \tag{43}$$

Substituting equation (41) into equation (43) gives

$$\boldsymbol{f}_{xi} = -\boldsymbol{K}_{xui}\boldsymbol{x}_0 + \boldsymbol{K}_{xui}\boldsymbol{x}_1 + \boldsymbol{K}_{xui}\boldsymbol{R}_{1i}^{\mathrm{T}}\boldsymbol{\alpha}_1 - \boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}}\boldsymbol{\alpha}_0 \tag{44}$$

Summing the moments about the CoG of the supported mass

$$\sum \boldsymbol{h}_{x0} = \boldsymbol{I}_0 \ddot{\boldsymbol{\alpha}}_0 = -\boldsymbol{K}_{\alpha ui}\left(\boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1\right) + \boldsymbol{R}_{0i}\boldsymbol{f}_{xi} \tag{45}$$

and substituting equation (44) into equation (45) gives

$$\begin{aligned}\sum \boldsymbol{h}_{x0} &= \boldsymbol{I}_0 \ddot{\boldsymbol{\alpha}} = -\boldsymbol{K}_{\alpha ui}\left(\boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_1\right) \\ &\quad + \boldsymbol{R}_{0i}\left(-\boldsymbol{K}_{xui}\boldsymbol{x}_0 + \boldsymbol{K}_{xui}\boldsymbol{x}_1 + \boldsymbol{K}_{xui}\boldsymbol{R}_{1i}^{\mathrm{T}}\boldsymbol{\alpha}_1 - \boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}}\boldsymbol{\alpha}_0\right)\end{aligned} \tag{46}$$

Further manipulation yields

$$\begin{aligned}&\boldsymbol{I}_0 \ddot{\boldsymbol{\alpha}} + \left(\boldsymbol{K}_{\alpha ui} + \boldsymbol{R}_{0i}\boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}}\right)\boldsymbol{\alpha}_0 - \left(\boldsymbol{K}_{\alpha ui_1} + \boldsymbol{R}_{0i}\boldsymbol{K}_{xui}\boldsymbol{R}_{1i}^{\mathrm{T}}\right)\boldsymbol{\alpha}_1 \\ &\quad + \boldsymbol{R}_{0i}\boldsymbol{K}_{xui}\boldsymbol{x}_0 - \boldsymbol{R}_{0i}\boldsymbol{K}_{xui}\boldsymbol{x}_1 = \boldsymbol{0}\end{aligned} \tag{47}$$

Forces Applied to the Raft

The forces applied to the raft are given by

$$\begin{aligned}\sum \boldsymbol{f}_x &= \boldsymbol{m}_1 \ddot{\boldsymbol{x}}_1 \\ &= -\boldsymbol{K}_{xui}\left(\boldsymbol{x}_1 + \boldsymbol{R}_{1i}^{\mathrm{T}}\boldsymbol{\alpha}_1 - \boldsymbol{x}_0 - \boldsymbol{R}_{0i}^{\mathrm{T}}\boldsymbol{\alpha}_0\right) - \boldsymbol{K}_{xli}\left(\boldsymbol{x}_1 + \boldsymbol{R}_{1j}^{\mathrm{T}}\boldsymbol{\alpha}_1\right)\end{aligned} \tag{48}$$

which includes the effect of translation *and* rotation of both the supported mass and the raft. Rearranging equation (48) gives

$$\begin{aligned}&\boldsymbol{m}_1 \ddot{\boldsymbol{x}}_1 - \boldsymbol{K}_{xui}\boldsymbol{x}_0 - \boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}}\boldsymbol{\alpha}_0 + \left(\boldsymbol{K}_{xui} + \boldsymbol{K}_{xli}\right)\boldsymbol{x}_1 \\ &\quad + \left(\boldsymbol{K}_{xui}\boldsymbol{R}_{1i}^{\mathrm{T}} + \boldsymbol{K}_{xli}\boldsymbol{R}_{1j}^{\mathrm{T}}\right)\boldsymbol{\alpha}_1 = \boldsymbol{0}\end{aligned} \tag{49}$$

Moments Applied to the Raft

Moments applied to the raft are due to upper and lower rotational stiffness, and the rotations of the raft relative to the supported mass and the foundation; i.e. $-\boldsymbol{K}_{\alpha ui}(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_o) - \boldsymbol{K}_{\alpha lj}(\boldsymbol{\alpha}_1)$. In addition, moments about the CoG of the raft are caused by forces from the upper and lower isolators, and these forces are due to translations *and* rotations of the raft. Forces on the raft due to the upper isolator are given by

$$\boldsymbol{f}_{x1i} = -\boldsymbol{K}_{xui}\left(\boldsymbol{x}_1 - \boldsymbol{x}_{0TOT}\right) - \boldsymbol{K}_{xui}\boldsymbol{R}_{1i}^{\mathrm{T}}\boldsymbol{\alpha}_1 \tag{50}$$

Substituting $\boldsymbol{x}_{0TOT} = \boldsymbol{x}_0 + \boldsymbol{R}_{0i}^{\mathrm{T}}\boldsymbol{\alpha}_0$ into equation (50) gives

$$\boldsymbol{f}_{x1i} = \boldsymbol{K}_{xui}\boldsymbol{x}_0 + \boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}}\boldsymbol{\alpha}_0 - \boldsymbol{K}_{xui}\boldsymbol{x}_1 - \boldsymbol{K}_{xui}\boldsymbol{R}_{1i}^{\mathrm{T}}\boldsymbol{\alpha}_1 \tag{51}$$

Forces on the raft due to the lower isolator are given by

$$\boldsymbol{f}_{x1j} = -\boldsymbol{K}_{xlj}\boldsymbol{x}_1 - \boldsymbol{K}_{xlj}\boldsymbol{R}_{1j}^{\mathrm{T}}\boldsymbol{\alpha}_1 \tag{52}$$

where $\boldsymbol{R}_{1j}$ is defined in an analogous way to $\boldsymbol{R}_{0i}$ (equation (9)) using $\boldsymbol{r}_{1j}$. Summing the moments about the CoG of the raft

$$\sum \boldsymbol{h}_{x1} = \boldsymbol{I}_1\ddot{\boldsymbol{\alpha}}_1 = -\boldsymbol{K}_{\alpha ui}(\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_0) - \boldsymbol{K}_{\alpha li}\boldsymbol{\alpha}_1 + \boldsymbol{R}_{1i}\boldsymbol{f}_{x1i} + \boldsymbol{R}_{1j}\boldsymbol{f}_{x1j} \quad (53)$$

Substituting equations (51) and (52) into (53) and rearranging gives

$$\begin{aligned}\boldsymbol{I}_1\ddot{\boldsymbol{\alpha}}_1 - \boldsymbol{R}_{1i}\boldsymbol{K}_{xui}\boldsymbol{x}_0 - \left(\boldsymbol{R}_{1i}\boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}} + \boldsymbol{K}_{\alpha ui}\right)\boldsymbol{\alpha}_0 \\ + \left(\boldsymbol{R}_{1i}\boldsymbol{K}_{xui} + \boldsymbol{R}_{1j}\boldsymbol{K}_{xlj}\right)\boldsymbol{x}_1 \\ + \left(\boldsymbol{K}_{\alpha ui} + \boldsymbol{R}_{1i}\boldsymbol{K}_{xui}\boldsymbol{R}_{1i}^{\mathrm{T}} + \boldsymbol{K}_{\alpha li} + \boldsymbol{R}_{1j}\boldsymbol{K}_{xlj}\boldsymbol{R}_{j}^{\mathrm{T}}\right)\boldsymbol{\alpha}_1 = \boldsymbol{0}\end{aligned} \quad (54)$$

Matrix Equations

Equations (42), (47), (49), and (54) can be assembled into a matrix equation given by equation (55). The effect of n isolators above the raft and m isolators below the raft is taken into account by summing the stiffness sub-matrices for each isolator, as shown by equations (56) – (64).

$$\begin{bmatrix} \boldsymbol{M}_0 & \boldsymbol{0} & & \\ \boldsymbol{0} & \boldsymbol{I}_0 & & \\ & & \boldsymbol{M}_1 & \boldsymbol{0} \\ & & \boldsymbol{0} & \boldsymbol{I}_1 \end{bmatrix} \begin{Bmatrix} \ddot{\boldsymbol{x}}_0 \\ \ddot{\boldsymbol{\alpha}}_0 \\ \ddot{\boldsymbol{x}}_1 \\ \ddot{\boldsymbol{\alpha}}_1 \end{Bmatrix} + \begin{bmatrix} \boldsymbol{K}_{x0x0} & \boldsymbol{K}_{x0\alpha0} & -\boldsymbol{K}_{x0x1} & -\boldsymbol{K}_{x0\alpha1} \\ \boldsymbol{K}_{\alpha0x0} & \boldsymbol{K}_{\alpha0\alpha0} & -\boldsymbol{K}_{\alpha0x1} & -\boldsymbol{K}_{\alpha0\alpha1} \\ -\boldsymbol{K}_{x1x0} & -\boldsymbol{K}_{x1\alpha0} & \boldsymbol{K}_{x1x1} & -\boldsymbol{K}_{x1\alpha1} \\ -\boldsymbol{K}_{\alpha1x0} & -\boldsymbol{K}_{\alpha1\alpha0} & -\boldsymbol{K}_{\alpha1x1} & \boldsymbol{K}_{\alpha1\alpha1} \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_0 \\ \boldsymbol{\alpha}_0 \\ \boldsymbol{x}_1 \\ \boldsymbol{\alpha}_1 \end{Bmatrix} = \boldsymbol{0} \quad (55)$$

where

$$\boldsymbol{K}_{x0x0} = \boldsymbol{K}_{x0x1} = \boldsymbol{K}_{x1x0} = \sum_{i=1}^{n}\boldsymbol{K}_{xui} \quad (56)$$

$$\boldsymbol{K}_{x0\alpha0} = \sum_{i=1}^{n}\boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}} = \left[\boldsymbol{K}_{\alpha0x0}\right]^{\mathrm{T}} = \left[\sum_{i=1}^{n}\boldsymbol{R}_{0i}\boldsymbol{K}_{xui}\right]^{\mathrm{T}} \quad (57)$$

$$\boldsymbol{K}_{x0\alpha1} = \sum_{i=1}^{n}\boldsymbol{K}_{xui}\boldsymbol{R}_{1i}^{\mathrm{T}} = \left[\boldsymbol{K}_{\alpha1x0}\right]^{\mathrm{T}} = \left[\sum_{i=1}^{n}\boldsymbol{R}_{1i}\boldsymbol{K}_{xui}\right]^{\mathrm{T}} \quad (58)$$

$$\boldsymbol{K}_{\alpha0\alpha0} = \sum_{i=1}^{n}\boldsymbol{R}_{0i}\boldsymbol{K}_{xui}\,\boldsymbol{R}_{0i}^{\mathrm{T}} + \boldsymbol{K}_{\alpha ui} \quad (59)$$

$$\boldsymbol{K}_{\alpha0x1} = \sum_{i=1}^{n}\boldsymbol{R}_{0i}\boldsymbol{K}_{xui} = \left[\boldsymbol{K}_{x1\alpha0}\right]^{\mathrm{T}} = \left[\sum_{i=1}^{n}\boldsymbol{K}_{xui}\boldsymbol{R}_{0i}^{\mathrm{T}}\right]^{\mathrm{T}} \quad (60)$$

$$\begin{aligned}\boldsymbol{K}_{\alpha0\alpha1} &= \sum_{i=1}^{n}\boldsymbol{R}_{0i}\,\boldsymbol{K}_{xui}\,\boldsymbol{R}_{1i}^{\mathrm{T}} + \boldsymbol{K}_{\alpha ui} \\ &= \left[\boldsymbol{K}_{\alpha1\alpha0}\right]^{\mathrm{T}} = \left[\sum_{i=1}^{n}\boldsymbol{R}_{1i}\,\boldsymbol{K}_{xui}\,\boldsymbol{R}_{0i}^{\mathrm{T}} + \boldsymbol{K}_{\alpha ui}\right]^{\mathrm{T}}\end{aligned} \quad (61)$$

$$\boldsymbol{K}_{x1x1} = \sum_{i=1}^{n}\boldsymbol{K}_{xui} + \sum_{j=1}^{m}\boldsymbol{K}_{xlj} \quad (62)$$

$$\begin{aligned}\boldsymbol{K}_{x1\alpha1} &= \sum_{i=1}^{n}\boldsymbol{K}_{xui}\,\boldsymbol{R}_{1i}^{\mathrm{T}} + \sum_{j=1}^{m}\boldsymbol{K}_{xlj}\,\boldsymbol{R}_{1j}^{\mathrm{T}} \\ &= \left[\boldsymbol{K}_{\alpha1x1}\right]^{\mathrm{T}} = \left[\sum_{i=1}^{n}\boldsymbol{R}_{1i}\,\boldsymbol{K}_{xui} + \sum_{j=1}^{m}\boldsymbol{R}_{1j}\,\boldsymbol{K}_{xlj}\right]^{\mathrm{T}}\end{aligned} \quad (63)$$

$$\boldsymbol{K}_{\alpha1\alpha1} = \sum_{i=1}^{n}\boldsymbol{R}_{1i}\,\boldsymbol{K}_{xui}\,\boldsymbol{R}_{1i}^{\mathrm{T}} + \boldsymbol{K}_{\alpha ui} + \sum_{j=1}^{m}\boldsymbol{R}_{1j}\,\boldsymbol{K}_{xlj}\,\boldsymbol{R}_{1j}^{\mathrm{T}} + \boldsymbol{K}_{\alpha lj} \quad (64)$$

ANALYSIS OF VIBRATION ISOLATION SYSTEMS

Equations (3), (30), and (55) can be represented by

$$[\boldsymbol{M}]\ddot{\boldsymbol{X}} + [\boldsymbol{K}]\boldsymbol{X} = \boldsymbol{0} \quad (65)$$

Natural frequencies and mode shapes (eigenvectors) can be obtained by solving the generalised eigenvalue problem

$$[\boldsymbol{M}]^{-1}[\boldsymbol{K}]\boldsymbol{X} = \boldsymbol{\lambda}\boldsymbol{X} \quad (66)$$

The natural frequencies are given by $\omega_k = (\lambda_{kk})^{0.5}$; and λ_{kk} are the diagonal elements of $\boldsymbol{\lambda}$.

Forced vibration of the systems described by equations (3), (30), and (55) is given by

$$[\boldsymbol{M}]\ddot{\boldsymbol{X}} + [\boldsymbol{K}]\boldsymbol{X} = \boldsymbol{F} \quad (67)$$

The transfer function matrix relating the displacement response at each DOF to forces applied at each DOF is given by

$$\boldsymbol{H}(\omega) = \left[-[\boldsymbol{M}]\omega^2 + [\boldsymbol{K}]\right]^{-1} \quad (68)$$

Assuming a rigid foundation, the transmissibility of a vibration isolator is defined as the ratio of the force transmitted to the foundation and the force applied to the supported mass (Mead, 2000). The force applied to the foundation is calculated by first obtaining the displacements across isolators coupled to the foundation that result from the force applied to the supported mass. The displacements across the springs attached to the foundation are then multiplied by the corresponding stiffness to yield the force applied to the foundation.

The transmissibility between DOFs that are of interest will vary depending on the particular application and the level of coupling between different DOFs. This results from the location and stiffness of the isolators and the inertial properties of the supported mass (and any intermediate masses). For example, reciprocating engines may be excited by shaking moments about the engine's vertical and lateral axes, in addition to significant torsional vibration about the engine's longitudinal axis (Taylor, 1985). In this case, it may be of interest to examine the transmissibility between rotational DOFs on the supported mass, and the resultant forces and moments applied to the foundation.

REPRESENTATIVE EXAMPLES

The following examples illustrate the application of the equations presented in this paper for a single stage, two-stage and two-stage rafted system. The characteristics of each type of system are briefly discussed.

Mass and inertial parameters used for the three examples are given in Table 1; stiffness parameters are given in Table 2; and number and location of isolators are given in Tables 3 – 5. In each model, the even-numbered isolators were inclined from vertical by –30° about the x axis, and odd-numbered isolators were inclined from vertical by 30° about the x axis.

Results

Natural frequencies and eigenvectors (proportional to mode shapes) are shown for the single-stage model in Figure 7, and for the two-stage rafted system in Figure 8. A subset of the 54 natural frequencies for the two-stage system (i.e. with intermediate masses) is listed in Table 6 with a description of the motion for each listed mode.

Table 1. Mass and inertial parameters

	Single Stage System	Two-Stage System	Two-Stage Rafted System
Supported Mass (kg)	10000	10000	10000
I_{xx} (kg m^2)	2708	2708	2708
I_{yy} (kg m^2)	5208	5208	5208
I_{zz} (kg m^2)	4167	4167	4167
Intermediate mass or raft mass (kg)	-	500	12000
I_{xx} (kg m^2)	-	5.42	4160
I_{yy} (kg m^2)	-	8.33	9160
I_{zz} (kg m^2)	-	10.42	13000

Table 2. Stiffness parameters about principal elastic axes

	K_p	K_q (N/m)	K_r	K_λ	K_ξ (Nm/rad)	K_v
Upper and lower isolators	10^7	2×10^6	2.5×10^6	3×10^6	4×10^6	5×10^6

Table 3. Location of upper isolators for single-stage and two stage models (relative to CoG of supported mass; distances in metres)

	Isolator 1	2	3	4	5	6
r_{0x}	1	1	0	0	-1	-1
r_{0y}	0.5	-0.5	0.5	-0.5	0.5	-0.5
r_{0z}	-0.95	-0.95	-0.95	-0.95	-0.95	-0.95

Table 4. Location of upper isolators for two-stage rafted model (distances in metres)

	Relative to CoG of supported mass			Relative to CoG of raft		
Isolator	r_{0x}	r_{0y}	r_{0z}	r_{1x}	r_{1y}	r_{1z}
1	1	0.5	-0.95	1	0.85	0.2
2	1	-0.5	-0.95	1	-0.85	0.2
3	0	0.5	-0.95	0	0.85	0.2
4	0	-0.5	-0.95	0	-0.85	0.2
5	-1	0.5	-0.95	-1	0.85	0.2
6	-1	-0.5	-0.95	-1	-0.85	0.2

Table 5. Location of lower isolators for two-stage rafted model (relative to CoG of raft; distances in metres)

	Isolator 1	2	3	4	5	6
r_{1x}	1	1	0	0	-1	-1
r_{1y}	0.85	-0.85	0.85	-0.85	0.85	-0.85
r_{1z}	-0.2	-0.2	-0.2	-0.2	-0.2	-0.2

Two-stage isolator assemblies in the two-stage model included the same mass and stiffness properties and consequently the 36 modes dominated by motion of the intermediate masses (six intermediate masses with six DOFs each) occurred as six groups of six modes with similar natural frequencies. The average natural frequency of each group of modes is listed in Table 6 with a description of the dominant motion.

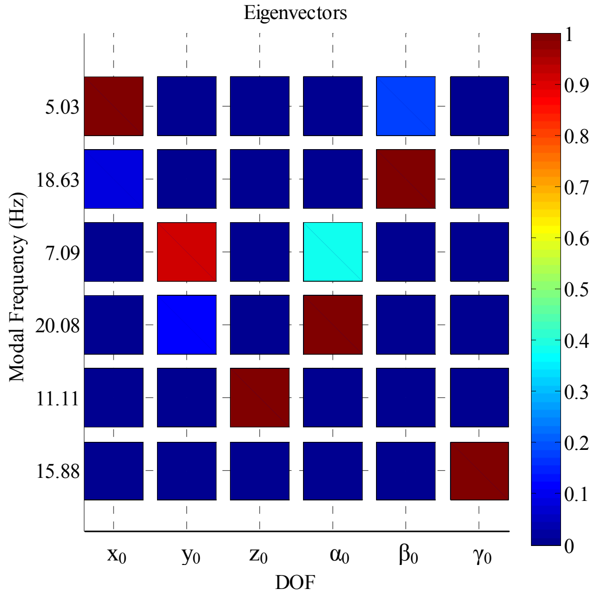


Figure 7. Natural frequencies and eigenvectors for the single stage vibration isolation model

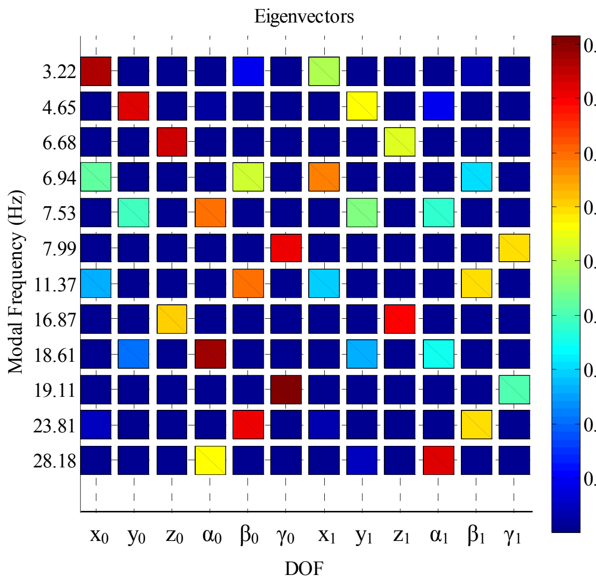


Figure 8. Natural frequencies and eigenvectors for the two-stage rafted vibration isolation model.

The eigenvectors or mode shapes provide an indication of the coupling between different degrees of freedom at each natural frequency. For example, the modes at 6.68 Hz and 16.87 Hz for the two-stage rafted system in Figure 8 include motion of the supported mass and the raft in the z direction; however, little or no motion of other degrees of freedom. The mode at 7.53 Hz has coupled motion in the y and α directions for both the raft and the supported mass.

The sum of the magnitude of vertical forces applied to the foundation due to a 1 N vertical (i.e. z direction) force applied to the supported mass is plotted versus frequency in Figure 9. Figure 10 shows the sum of the magnitude of the vertical forces applied to the foundation due to a 1 Nm moment applied to the supported mass about the x axis (i.e. a moment in the α direction).

Table 6. Natural frequencies and descriptions of modes for two-stage vibration isolation model.

Natural Frequency (Hz)	Description of Vibration Mode
3.46	x translation of m_0
4.93	y translation of m_0
7.55	z translation of m_0
9.96	γ rotation of m_0 coupled to y displacement of intermediate masses 1, 2, 5, 6
10.90	β rotation of m_0 coupled to x displacement of all intermediate masses
10.97	α rotation of m_0 coupled to y displacement of all intermediate masses
14.53	x translation of intermediate masses
17.17	Coupled y and z translation of intermediate masses
32.37	Coupled y and z translation of intermediate masses
120.83	Coupled β and γ rotation of intermediate masses
174.38	Coupled β and γ rotation of intermediate masses
193.47	α rotation of intermediate masses

Both two-stage systems have a steeper roll off of transmitted force at higher frequencies when compared to the single stage system. However, the extra degrees of freedom in the two-stage systems lead to additional modes, and the performance of the vibration isolation system is degraded for forcing frequencies in the vicinity of the natural frequencies. Therefore, care must be taken to ensure the natural frequencies associated with all the modes of vibration of the isolation system are not aligned with forcing frequencies.

A limitation of the two-stage system is that modes with significant motion of the intermediate masses occur at high frequencies relative to the modes with motion of the supported mass. This is due to the difference in mass. For example, in the two-stage isolation model, modes with significant motion of the supported mass have natural frequencies below 12 Hz (see Table 5). Modes with significant translational motion of the intermediate masses occur at approximately 15 Hz, 17 Hz, and 32 Hz, and modes with rotational motion of the intermediate masses have natural frequencies at approximately 121 Hz, 174 Hz, and 193 Hz. The effects of the modes at 17 Hz, 32 Hz, and 193 Hz can be seen in Figure 10.

Note that the models used in these examples did not include damping. The effect of damping would be to limit the magnitude of the response at resonant frequencies and broaden the resonant peaks.

CONCLUSION

Equations of motion for two-stage and two-stage rafted vibration isolator systems have been derived using a matrix methodology, based on the work of Smollen (1966). The equations of motion are valid for systems with any number of isolators with arbitrary location and position. Utilising matrix equations to describe the dynamics of the vibration isolator system overcomes some of the difficulty associated with manipulating equations in terms of scalar variables and can be easily implemented in numerical analysis software. Use of the equations has been demonstrated by analysing representative single-stage, two-stage, and two-stage rafted vibration isolation systems. Characteristics of the systems, including the level of coupling between degrees-of-freedom, and the transmissibility of the systems have been discussed.

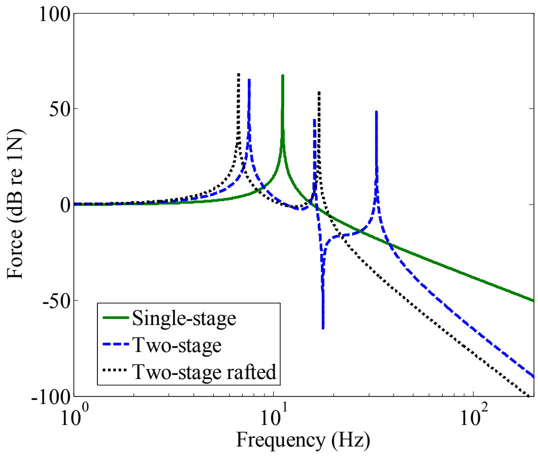


Figure 9. Summed magnitude of forces applied to the foundation in the z (vertical) direction due to a 1 N force on the supported mass in the z direction.

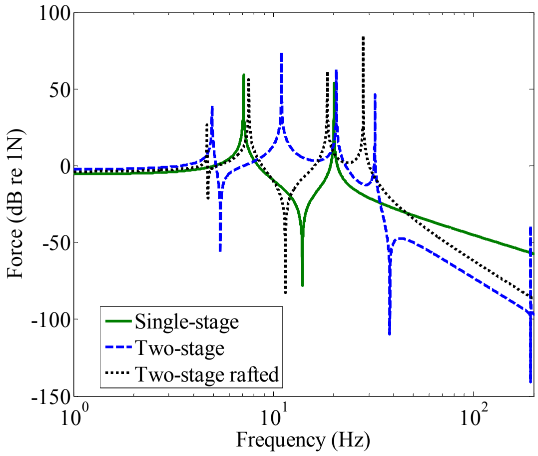


Figure 10. Summed magnitude of forces applied to the foundation in the z (vertical) direction due to a 1 Nm moment applied to the supported mass in the α direction.

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