A method of predicting variable speed rail corrugation growth using standard statistical moments

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ABSTRACT

Wear-type rail corrugation is a significant problem in the railway transport industry. Some recent work has suggested that speed control can be used as an effective tool to minimize the rate of corrugation growth. This has brought about the need to model corrugation growth under variable passing speed. Variable speed rail corrugation growth modelling normally consists of either numerical simulation of a sequence of varied speed wheel passes or direct integration of a probabilistic passing speed distribution function; both of which are computationally expensive. This paper investigates the use of the statistical moments of the speed probability density function to greatly improve the computational speed of variable speed corrugation growth models and compares results of changing standard deviation and skewness to numerical integration models. It also identifies the effects of individual statistical moments on corrugation growth to provide better insight into control methods. The new modelling method correlated well with the numerical integration models for small standard deviations in speed (less than 10%) and highlighted a need to consider kurtosis in predicting the performance of speed control based corrugation mitigation schemes. For larger standard deviations in speed, higher than 4th order effects need to be considered.

INTRODUCTION

Wear-type rail corrugation is a periodic wear pattern that develops on running rail heads with extended use. It has a characteristic wavelength of 25-400 mm and occurs in rail networks worldwide (Sato et. al., 2002). This surface irregularity grows as a function of train passages causing serious unwanted noise and vibration (see Hempelmann and Knothe, 1996). Figure 1 shows a corrugation profile after approximately 9 months of passenger rail traffic.

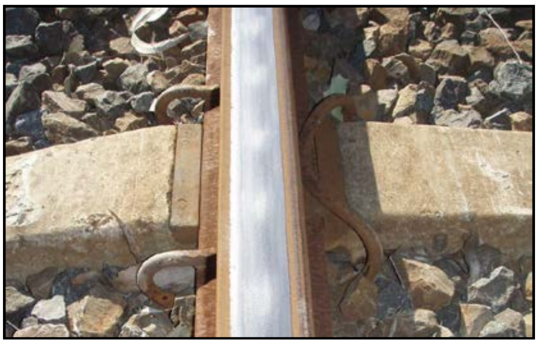

Figure 1 - 9-month-old corrugated rail profile

Fully developed corrugation on rails causes excessive noise and vibrations of the vehicle and track, inevitably leading to expensive reprofiling of the rail by grinding. This process is currently costing the Australian rail industry in excess of AU$ 10 million every year and it will cost more with predicted increases in rail traffic density unless alternative mitigation methods are put in place. Recent studies on rail corrugation have suggested that uniformity in train passing speed accelerates the growth of corrugations and that speed control offers a possible practical solution to controlling growth rate (see Bellette et. al., 2008 and Meehan et.al.,2009). Preliminary field testing has provided ranges of expected statistical spreads in speed which require accurate and efficient modelling to make predictions of changes in corrugation growth rate.

Analytical frequency domain corrugation prediction models have shown the capacity to predict dominant frequencies and amplitude growth rates of rail corrugation (see Hempelmann and Knothe, 1996, Müller, 2001, Wu and Thompson, 2005, Grassie and Johnson, 1985, Tassilly and Vincent, 1991 and Batten et. al., 2011). More complex three dimensional numerical models are computationally expensive, making effective analysis of speed distribution effects over many vehicle passes difficult.

Currently, variable speed corrugation growth modelling methods consist of either, A: direct integration of an assumed speed distribution function (Bellette et. al., 2008, Batten et. al., 2010) or B: simulation of multiple single wheel or bogie passes to find an equivalent distributed speed growth rate (Meehan et. al., 2009). Method A has also been used to model growth of corrugation on road surfaces under stochastically varying speed in Hoffmann and Misol (2008). Machine tool chatter has also been modelled with speed variance effects, showing that speed variation may increase the area of regions in parameter space where chatter does not occur. However, such chatter models only model a deterministic speed sequence, not probabilistic (see Namachchivaya and Beddini, 2003). That is, the speed is a chosen function of time, making analysis simpler but not appropriate for corrugation growth modelling as tracking a predefined speed sequence with a high enough accuracy is unrealistic in practice.

Of the two previously developed methods, integration of a speed distribution function (A) provides much more accurate predictions of growth rate but requires significant computational complexity to accurately integrate the distribution curve numerically. Simulation of successive wheelset passages (B) removes the need for numerical integration but accurate representation of a known speed distribution requires many simulated passes, also increasing the computational expense.

This paper aims to present a new method of modelling variable speed corrugation growth, based on the truncated series expansion of the expectation of the frequency domain corrugation growth spectrum in terms of the moments of the speed probability density function. Results are benchmarked against the model presented in Meehan et. al. (2009) using Gaussian distributions of speed to analyse standard deviation and triangular distributions to analyse standard skewness like those in Batten et. al., (2011). An investigation was also performed using this new modelling technique to predict and provide insight into the effects of higher order statistical speed moments (such as kurtosis) on corrugation growth.

MODELLING

The following subsections demonstrate, firstly, how an existing variable speed corrugation growth model is used to investigate standard deviation and standard skewness effects on rail corrugation growth on straight track and, secondly, how the same results can be produced using a simpler linearised model based on standard statistical moments instead of integrating a variable passing speed probability density function.

Probabilistic speed corrugation growth on straight track

The single pass corrugation growth model used in this paper is based on that presented in Meehan et. al., (2009). A single mode approximation is made of a field-measured corrugation growth spectral density plot and the parameters used to calculate the effects of a distributed passing speed. The full equation for single-mode dominant corrugation growth rate spectrum, G_r, under a distributed passing speed function, $p(x)$, is given below, where K_b is the sensitivity of wear depth to dynamic normal force, ω_n is the natural frequency, ζ is the damping ratio and K_c represents the modal sensitivity of the wheel/rail displacement to a change in longitudinal rail profile (all given in Appendix A).

$$G_r(\omega) = \left|\frac{Z_{n+1}}{Z_n}\right| - 1 = \exp\left(\int_{-\infty}^{\infty} \ln f(x,\omega) p(x) dx\right) - 1 \tag{1}$$

where,

$$f(x,\omega) = \ln\left(\frac{\left(\omega_n^2\left(1 - K_b K_c/(1+K_b)\right) - x^2\omega^2\right)^2 + (2\zeta\omega_n x\omega)^2}{(\omega_n^2 - x^2\omega^2)^2 + (2\zeta\omega_n x\omega)^2}\right) \tag{2}$$

Two different speed distribution shapes were used to observe the effect of altering standard deviation and standard skewness in speed distribution on corrugation growth rate under fixed kurtosis. For the purpose of analysing standard deviation effects, both Gaussian and triangular distributions with 4 standard deviations were integrated using equation (1). The triangular distributions were also used with a series of 5 standard skewnesses to show the effects of speed distribution asymmetry. Appendix B provides the equations and input parameters used for the Gaussian and triangular speed distributions.

Statistical moment variable speed corrugation growth

The objective here was to use a Taylor approximation of the logarithm growth expression to obtain a simpler and more tangible expression in terms of the statistical moments of the speed distribution; average, μ, standard deviation, σ, standard skewness, S, and kurtosis, k, in place of the integrated speed distribution function. By taking a third order Taylor expansion of equation (2) about the average speed, μ, the following equation is produced.

$$f(x,\omega) \approx f(\mu,\omega) + \frac{df(\mu,\omega)}{dx}(x-\mu) + \frac{d^2 f(\mu,\omega)}{dx^2}\frac{(x-\mu)^2}{2} + \frac{d^3 f(\mu,\omega)}{dx^3}\frac{(x-\mu)^3}{6} + \frac{d^4 f(\mu,\omega)}{dx^4}\frac{(x-\mu)^4}{24} \tag{3}$$

When multiplied with the distribution function, p(x), and integrated for the full domain of speeds as per equation (1), the first four statistical moments appear in the result.

$$\int_{-\infty}^{\infty} f(x,\omega) p(x) dx \approx f(\mu,\omega) + \frac{d^2 f(\mu,\omega)}{dx^2}\frac{\sigma^2}{2} + \frac{d^3 f(\mu,\omega)}{dx^3}\frac{S\sigma^3}{6} + \frac{d^4 f(\mu,\omega)}{dx^4}\frac{k\sigma^4}{24} \tag{4}$$

Where, the integral expressions for the first four statistical moments are as follows.

$$\mu = \int_{-\infty}^{\infty} x \cdot p(x) dx \tag{5}$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x) dx \tag{6}$$

$$S = \int_{-\infty}^{\infty} (x-\mu)^3 p(x) dx \Big/ \sigma^3 \tag{7}$$

$$k = \int_{-\infty}^{\infty} (x-\mu)^4 p(x) dx \Big/ \sigma^4 \tag{8}$$

The mean, μ, defines the centroid of the area enclosed by the distribution function, $p(x)$. The variance, or standard deviation squared, σ^2 measures the spread of the distribution about the mean. The skewness, S, defines the asymmetry about the mean and the kurtosis, k, compares the difference in density between the centralized coordinates (i.e. close to the mean) and the tail sections of the distribution shape. For clarity, the kurtosis and skewness effects on distribution shape are shown in Figure 2.

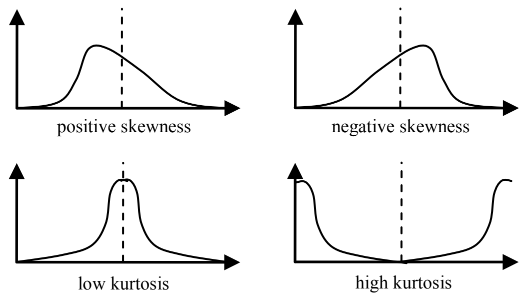

Figure 2 – Effects of kurtosis and skewness on distribution shape

Equation (4) can be substituted back into equation (1) to obtain the variable speed growth rate in terms of the first three statistical moments instead of an integral of the distribution function.

$$G_r(\omega) \approx \exp\left(f(\mu,\omega) + \frac{d^2 f(\mu,\omega)}{dx^2}\frac{\sigma^2}{2} + \frac{d^3 f(\mu,\omega)}{dx^3}\frac{S\sigma^3}{6} + \frac{d^4 f(\mu,\omega)}{dx^4}\frac{k\sigma^4}{24}\right) - 1 \tag{9}$$

So the corrugation growth rate at a given frequency for an arbitrary speed distribution shape can be approximated based on standard statistical measures, the single pass growth rate at the average speed and the double, triple and quadruple derivatives of growth rate at the average speed. For a more complex single pass corrugation growth function of speed, finite difference approximations of the double and triple derivatives may be used but for this paper the analytical derivatives of the single mode approximation were used (see Appendix B for input parameters).

The analytical equations for the first to forth derivatives were derived with the use of Matlab's symbolic functions. These were adequate for representing a highly damped frequency spectrum of corrugation growth but lower damping ratios caused the Taylor approximation to break down at a much narrower band of speeds since higher order effects were much more significant. Finite difference approximations with large step sizes would be more effective at representing these functions at larger standard deviations of speed from the average but this technique wasn't used in this paper.

The following subsections present the simulation results for both the numerically integrated distribution shapes and the equivalent statistical moment expansions.

RESULTS

Equations (1) and (2) were solved numerically using the midpoint method for 500 equally spaced speeds and 100 equally spaced frequencies in Matlab using an Intel (R) Core 2 Duo 1.73 GHz CPU with 2 GB of RAM and a Windows XP Operating System. Both triangular and Gaussian distributions with an average speed of 55 km/h were solved for with a speed range of 16 times the standard deviation used in each case. Statistical moment results were obtained by solving Equation (9) for 100 frequencies with insertion of analytical solutions of derivatives provided by Matlab's symbolic functions.

Firstly, the case of symmetric distributions of speed is investigated with both triangular and Gaussian distributions of zero skewness and varying standard deviation. 4 standard deviations were simulated and the results for the two modelling techniques compared. To test asymmetric cases, a series of 10 triangular distributions of two different standard deviations and 5 different skewnesses with equally spaced modes from $\sigma\sqrt{2}$ to $\sigma\sqrt{2}$ are then simulated.

Standard deviation and kurtosis effects on growth rate

To validate this modelling technique against predictions of increasing standard deviation on corrugation growth rate as per Meehan et. al. (2009), a series of Gaussian normal distributions and symmetric triangular distributions were integrated with the one mode single pass growth function as per equation (1) and the results of standard deviation against maximum corrugation growth rate (in frequency) were compared for the two modelling techniques. Second and third order Taylor expansions were used to observe the effect of the speed kurtosis on these predictions. The difference in the 2^{nd} and 4^{th} order approximations is directly related to the speed distribution kurtosis which provides insight into how kurtosis will affect corrugation growth rate. The results of these simulations are shown in Figure 3.

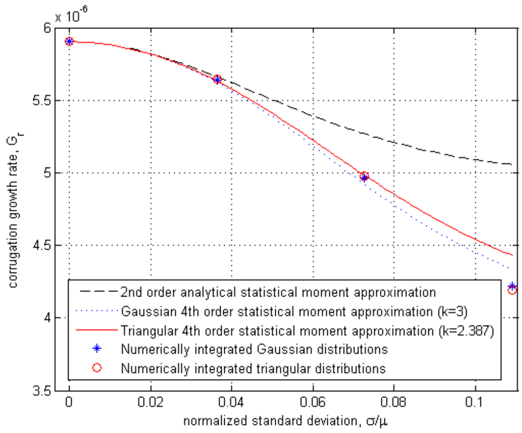

Figure 3 – Gaussian and triangular distributions with varying standard deviation and their 2^{nd} and 4^{th} order statistical moment approximations

By inspection, as would be expected, the 4^{th} order approximation matches closely to the numerically integrated solutions for narrower speed distributions but begins to diverge at higher standard deviations. By equation (9), the 4^{th} term in the statistical moment expansion is proportional to the kurtosis and the standard deviation to the power of 4. This means that the difference between the 2^{nd} and 4^{th} order approximations of the Gaussian distribution is equal to the change in corrugation growth rate for an increase in kurtosis of 3. So by inspection of Figure 3, an increase in kurtosis of 3 equates to a decrease in corrugation growth rate of 17% from that achieved with a standard deviation of 11%.

Increasing the standard deviation from zero will always cause a reduction in corrugation growth rate at the dominant wavelength since the double derivative of growth rate with speed will always be negative here due to the nature of it being a local maximum. This is because, for any given wavelength of corrugation, there will be a critical speed at which corrugation growth will be promoted most. If the probability density at the critical speed for the dominant wavelength is reduced, without significantly altering that dominant wavelength, then there are less wheel passages close to resonant/critical conditions and hence corrugation growth will be reduced.

Kurtosis may cause an increase or decrease in corrugation growth rate, depending on the sign of the fourth derivative at the dominant wavelength as per Equation (9). At small standard deviations in speed the critical speed for corrugation growth at the dominant wavelength will be close to the mean so increasing kurtosis will reduce this probability density (see Figure 2) and subsequently reduce the corrugation growth at that wavelength. For larger standard deviations, however, the most critical speed for the dominant wavelength may be far enough away from the mean that increasing kurtosis will actually increase the probability density at that speed.

Standard skewness effects on growth rate

To validate the third order statistical moment approximations of skewness effects, five triangular distributions of varying skewness were integrated using equation (1) with a constant standard deviations of 7.3% and 3.6% and the same skewness values were used in a third order statistical moment approximation of each using equation 9. A constant was added to the statistical moment results representing higher than 4^{th} order effects to account for the discrepancies seen in Figure 3 b) for the symmetric case. The results are shown below.

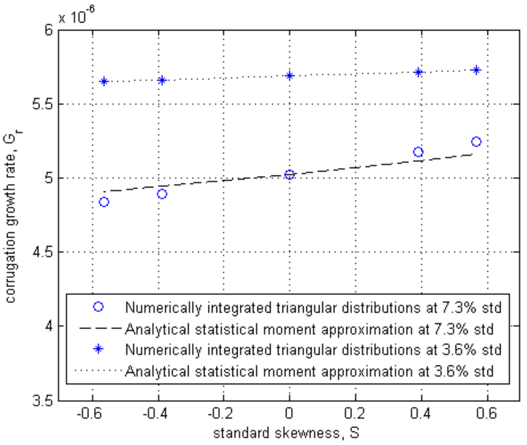

Figure 4 – Skewness effects on corrugation growth rate from integrated triangular distributions and statistical moment expansion for standard deviations of 3.6% and 7.3%

Figure 4 shows the analytical statistical moment model matches the numerical results for small standard deviations (<10%). . This indicates that higher than 3rd order statistical moment expansion is unnecessary for accurately predicting skewness effects on corrugation growth if the standard deviation is <10%. For larger standard deviations the approximation rapidly diverges from the solution obtained via integration. This indicates that the higher order effects which relate to speed distribution asymmetry, much like the third and fourth order terms in equation (9), are scaled with a factor of standard deviation.

Figure 3 also shows a slight increase in corrugation growth rate as skewness is increased. The reason that the trend in growth rate with skewness is opposite of that presented in Batten et. al. (2011) is that cornering effects are not considered and the single mode growth spectrum is dissimilar to 5 mode system presented previously.

Increasing or decreasing skewness will not always have the same effect on corrugation growth rate because it's dependent on the sign of the third derivative of growth rate with speed at the dominant wavelength. The results of these simulations and the results of the simulations performed in Batten et. al. (2011) show that the third derivative may be positive or negative, depending on site conditions. Under an increase in speed, the spectral density peaks will always widen in the spatial frequency domain as discussed in Batten et. al. (2011). If the frequency spectrum for a single pass is symmetric about its peak this will cause the third derivative of growth rate with speed at the dominant frequency to be negative. If there is a large enough reduction in slope magnitude in the wavelength domain comparing either side of the peak in growth rate then this effect may be countered, causing the third derivative at the dominant wavelength to be positive as was the case in this paper. Under cornering on under-canted track though, increasing skewness (i.e. biasing lower speeds) will most likely decrease corrugation growth rate since under-canted cornering conditions cause large increases in corrugation growth rate at higher speeds, much larger than the changes in growth rate presented here (see Batten et. al., 2011).

DISCUSSION OF RESULTS

The new model was able to produce results much quicker than the integration and multiple pass models. Using an Intel (R) Core 2 Duo 1.73 GHz CPU with 2 GB of RAM running a Windows XP Operating System it took 77.9 s to solve one corrugation growth rate via integration of a Gaussian distribution as compared to 0.012s for the analytical approximation of equation (9). This is a significant saving in computational expense of almost 4 orders of magnitude (10^4). For standard deviations of less than 10% the 4th order statistical moment approximations closely matched the numerically integrated results for symmetric Gaussian and triangular distributions. For asymmetric triangular distributions, at standard deviations below 4% the 3rd order statistical moment approximation was accurate in determining changes in corrugation growth rate with standard skewness.

As can be seen when comparing the two sets of integrated distributions, the kurtosis effect is minimal between the Gaussian and triangular distributions. The question must then be raised: can kurtosis change enough to significantly alter predicted results of increasing speed standard deviation? As part of a project undertaking field experiments to validate modelling predictions of corrugation growth under varying train speed (see Meehan et. al., 2009 and Batten et. al., 2010), a speed control system was installed on a site with a tight radius corner in suburban Brisbane that is known to rapidly corrugate on the low rail. Standard statistical measurements were taken before and after implementation of the speed control system which was designed to make the trains follow a new distribution having increased variance. The kurtosis decreased by 4.8 which is more than the difference between the 2nd and 4th order moment approximations. These approximations showed a factor of 1.8 difference in the effectiveness of widening passing speed when kurtosis was included. This means that for any speed control based corrugation mitigation system, the kurtosis change can be significant in predicting the effectiveness in reducing corrugation growth rate.

For every extra term considered in the truncated Taylor series expansion of equation (2) one extra statistical moment appears in the final growth rate equation. Every successive term included will have a diminishing effect on the corrugation growth rate function and will be scaled by a function of the first statistical moment, standard deviation. The trend in corrugation growth rate change for altering each statistical moment will be dependent upon the sign of its equivalent derivative from the single pass growth function, equation (2). The only universal effect will be that increasing standard deviation will decrease the corrugation growth rate at the dominant wavelength since the double derivative of corrugation growth at the dominant frequency will always be negative. Odd numbered terms represent effects relating to speed distribution asymmetry so these terms will equate to zero for a symmetric distributions of speed.

CONCLUSIONS

This report presents a method by which variable speed corrugation growth can be rapidly approximated without the computational burden of numerical integration of a speed distribution function or simulation of many passes. Analytical equations were used to find the first four derivatives of growth rate in the frequency domain with train speed. These were used with standard statistical moments of standard deviation, skewness and kurtosis and the results compared with numerically integrated triangular and Gaussian speed distribution functions. Results showed a good correlation between the 4th order statistical moments for the increased standard deviation simulations up to a standard deviation of 10% and between the 3rd order statistical moments and the varying skewness simulations when the standard deviation was small (less that 4% of the mean). Higher than 4th order statistical moments are expected to have increasing effect with larger standard deviations since the 4th order approximations matched closely with numerical solutions of standard deviation changes. Also, 3rd order approximations matched closely with results of skewness changes if the standard deviation was small but diverged rapidly as standard deviation was increased.

It was also shown how kurtosis can have significant effects on the performance of a speed control based corrugation mitigation scheme. Comparing the triangular and Gaussian distribution, results showed little difference but the 2nd and 4th order statistical moment results show a difference in predicted growth rate of 12% from that of the single pass growth rate. These correlated to a change in kurtosis of 3. The change in growth rate (17%) is significant when compared to the 27% caused by increasing the standard deviation to 11%. These effects will always depend on the 4th derivative of the growth rate function with speed so it is possible that under some conditions this trend will be reversed, as with those observed for skewness changes. Under cornering conditions though, the third derivative will normally be negative because of significant increases in wear sensitivity to normal force with higher speeds so reduction in skewness will cause a decrease in corrugation growth rate (see Batten et. al., 2011). The only universal effect will be that increasing standard deviation will always decrease corrugation growth at the dominant wavelength.

This paper shows that the method of modelling variable speed rail corrugation growth via statistical moment expansion will work for narrow speed distributions. As standard deviation increases however, higher order effects become more apparent and must be considered. Future work will consider the possibility of using a widely spread finite difference approximation to find the derivatives of growth rate with speed so as to improve lower order approximations at large standard deviations at the cost of diminished accuracy for small standard deviations.

ACKNOWLEDGEMENTS

The authors are grateful to the CRC for Rail Innovation (established and supported under the Australian Government's Cooperative Research Centres program) for the funding of this research as part of Project No. R3.158 Corrugation Control. The authors are also very grateful for the support of Queensland Rail, Railcorp and the Australian Rail Track Corporation.

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APPENDICES

Appendix A – Simulation parameters

Parameter	Symbol	Value
Natural frequency [rad/s]	ω_n	850
Wear sensitivity [m/N]	K_b	2.4×10^{-6}
Modal sensitivity	K_c	0.785
Damping ratio	Z	0.14
Average speed [m/s]	μ	15.3
Kurtosis of distribution, Gaussian, triangular	K	3, 2.387

Appendix B – Speed distribution functions

The Gaussian distribution function is given by the following equation.

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{B.1}$$

The triangular distribution must have a predefined mode, x_m which ranges from the root of two times the standard deviation either side of the defined mean.

$$x_{\mathrm{m\,min,max}} = \mu \pm \sigma\sqrt{2} \tag{B.2}$$

Based on this mode, the upper and lower limits of speed can be calculated.

$$x_{\mathrm{min,max}} = \frac{3\mu - x_m \pm \sqrt{-3x_m{}^2 + 6\mu x_m - 3\mu^2 + 24\sigma^2}}{2} \tag{B.3}$$

With the domain limits and speed mode set, the distribution function, $p(x)$, is then bound by the following equations.

$$p(x) = \frac{2\left(x - x_{\mathrm{min}}\right)}{\left(x_{\mathrm{max}} - x_{\mathrm{min}}\right)\left(x_m - x_{\mathrm{min}}\right)} \quad \text{If } x < x_m \tag{B.4}$$

$$p(x) = \frac{-2\left(x - x_{\mathrm{max}}\right)}{\left(x_{\mathrm{max}} - x_{\mathrm{min}}\right)\left(x_{\mathrm{max}} - x_m\right)} \quad \text{If } x > x_m \tag{B.5}$$