On the equivalence of the Ffowcs Williams - Hawkings and the non-uniform Kirchhoff equations of aeroacoustics

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ABSTRACT

The purpose of this study is to compare the well-known Ffowcs Williams and Hawkings (FW-H) equation of aeroacoustics with the non-uniform Kirchhoff equation derived previously by the author. The purpose of this comparison is the clarification of the issue of possible equivalence of the two equations, as they produce equivalent predictions for radiated sound in many cases despite having different appearance. The comparison is done by reconsidering the original derivation by Ffowcs Williams and Hawkings. It is shown that one set of conditions within the medium inside the rigid body leads to the FW-H equation, whereas another set of conditions leads to the non-uniform Kirchhoff equation. As the conditions within the rigid body can be chosen arbitrarily, it is concluded that the two equations are equivalent and the non-uniform Kirchhoff equation represents a new form of the FW-H equation.

INTRODUCTION

The development of jet aircraft in the 1950s caused significant interest from researchers and engineers in the ability to predict the noise radiated by fluid flow near surfaces of fast-moving objects. Sir James Lighthill, in his well-known paper (Lighthill 1952), achieved a breakthrough in understanding the mechanism of sound radiation by turbulent flow. He showed that this mechanism can be described by a non-uniform wave equation where the source term is determined by Lighthill's stress tensor which includes all non-acoustic stresses in the fluid. Lighthill also showed that the radiated sound had quadrupole characteristics.

Curle (1955) extended Lighthill's theory to fluid flows in the presence of rigid boundaries. He showed that, in addition to Lighthill's quadrupole sound, fluid flow near a rigid boundary radiates dipole sound which was determined by the force acting upon the flow from the boundary.

Ffowcs Williams and Hawkings (1969) extended Curle's theory to the case of moving boundaries. They showed that the sound radiated by a moving boundary is described by monopole sources on the boundary. The strength of these sources is determined by the velocity of the boundary with respect to a stationary observer. For a stationary object, Ffowcs Williams and Hawkings (FW-H) equation is reduced to Curle's equation.

Since its derivation, the Ffowcs Williams and Hawkings equation has become the foundation for one of the most frequently used methods of prediction of sound radiated by fluid flow near rigid surfaces. A brief list of applications where the FW-H equation is utilised includes rotating helicopter blades (Brentner, Farassat 2003), rotating fans (Moon et al. 2003), and flow near an airfoil (Howe 1999). This equation is also used in the prediction of noise radiated by moving ships, submarines, torpedoes, and ship propellers.

Despite being widely used, the FW-H theory has lacked accurate experimental verification, as the obtained experimental results showed a discrepancy of at least a few decibels with theoretical predictions (see Zinoviev (2010) for references). In view of this fact, Zinoviev and Bies (2004) conducted a critical analysis of the derivation of Curle's equation, where they showed that the correct application of theorems of vector analysis lead to a different equation. Later the obtained equation has been given the name "the non-uniform Kirchhoff equation" by the present author, as this equation contains Kirchhoff integrals (Stratton 1941) together with the source term which determines Lighthill's quadrupole sound.

The claim by Zinoviev and Bies (2004) that the derivation of Curle's equation contains errors caused objections from some members of the aeroacoustical community (Farassat 2005, Farassat & Myers 2006), to which the authors provided their responses (Zinoviev and Bies 2005, Zinoviev & Bies 2006). Although the present author now accepts that some criticism regarding examples of application of FW-H and Curle's equations is justified, he remains convinced that his original analysis of Curle's equation is correct.

As both FW-H and the non-uniform Kirchhoff equations are derived for the same problem with the same assumptions, there exists a possibility that they are equivalent, i.e. they are, in fact, different forms of the same equation. The first attempt to answer the question whether these equations are equivalent was made by the present author (Zinoviev 2010). He showed that the two equations were indeed equivalent if the sum of two integrals containing Lighthill's stress tensor over the rigid surface was zero. These two integrals are equivalent to the acoustic field radiated by sources determined by Lighthill's stress tensor and its spatial derivatives on the boundary.

The equivalence of the two equations was considered for a few situations by Zinoviev (2010). First of all, it was shown that the sum of the integrals was zero where Lighthill's stress tensor vanished altogether, for instance, for linear acoustical waves in an ideal fluid. Also, it was shown that, for a weakly non-linear flow in an ideal fluid near an infinite plane the equations are equivalent if the plane is stationary. If the plane was vibrating or for flow in a viscous fluid, the sum of the two integrals could be, in general, non-zero, and a more detailed investigation was shown to be required for a definite conclusion.

As Curle's and the FW-H equations are derived by different methods, it has become necessary to consider the equivalence of the FW-H and the non-uniform Kirchhoff's equations separately from the analysis conducted in Zinoviev (2010) for Curle's equation. In this paper, the derivation of the FW-H equation is reconsidered with the purpose to verify the possible equivalence of this equation and the non-uniform Kirchhoff equation.

This paper has the following structure. First, the original derivation of the Ffowcs Williams and Hawkings equation is considered. Second, it is shown that the boundary conditions on the inner side of the rigid surface can be chosen arbitrarily. Third, it is demonstrated that a different set of boundary conditions leads not to the FW-H equation, but to the non-uniform Kirchhoff equation. Last, it is concluded that these two equations are equivalent.

THE DERIVATION OF THE FFOWCS WILLIAMS AND HAWKINGS EQUATION

To derive their equation, Ffowcs Williams and Hawkings (1969) considered a volume of fluid, V, bounded by a surface, Σ (Figure 1).

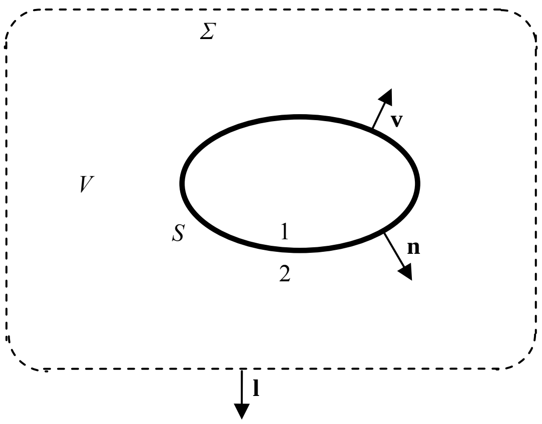

Figure 1. Layout of the fluid volume, V, with the surface of discontinuity, S

The volume V is divided into regions 1 and 2 by a closed surface, S, moving towards the region 2 with the velocity, $\mathbf{v}$. Note that the surface S is a surface of *discontinuity*, i.e. sources of mass and momentum can be located on S, so that the vectors of mass and momentum flows can become discontinuous. Assume that $\mathbf{l}$ is the outward normal to Σ, and $\mathbf{n}$ is the normal to S directed from the region 1 towards the region 2. As in the original derivation by Ffowcs Williams and Hawkings, in the analysis that follows the superscripts 1 and 2 refer to values in the corresponding regions, and an overbar implies that the variable is defined in the total volume V.

If ρ is the fluid density and t is time, then the rate of change of mass in the volume V equals the sum of rates of change of mass in the two regions

$$\frac{\partial}{\partial t}\int_V \overline{\rho} dV = \frac{\partial}{\partial t}\int_V \rho^{(1)} dV + \frac{\partial}{\partial t}\int_V \rho^{(2)} dV. \tag{1}$$

Due to the moving boundary S, for each region the rate of change of mass equals the sum of mass flows through the surfaces Σ and S

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{V^{(1,2)}} \rho^{(1,2)} dV = \\ &-\int_\Sigma \left(\rho u_i\right)^{(1,2)} l_i d\Sigma - \int_S \left[\rho\left(u_i - v_i\right)\right]^{(1,2)} n_i dS,\end{aligned} \tag{2}$$

In Equation (2), u_i are components of the vector of fluid velocity, where the index i=1,2,3 refers to one of the axes in the three-dimensional Cartesian space. Summation over repeating indices is assumed. The rate of change of the total fluid mass within the volume V can now be written as

$$\begin{aligned}&\frac{\partial}{\partial t}\int_V \overline{\rho} dV = -\int_\Sigma \left(\overline{\rho u_i}\right) l_i d\Sigma + \\ &\int_S \left[\rho\left(u_i - v_i\right)\right]^{(2)} n_i dS - \int_S \left[\rho\left(u_i - v_i\right)\right]^{(1)} n_i dS.\end{aligned} \tag{3}$$

According to the divergence theorem, the flux of a vector over a closed surface equals the integral of the divergence of the vector over the volume bounded by the surface. By applying the divergence theorem to the first term on the right in Equation (3) the following can be obtained

$$\begin{aligned}&\int_V \left(\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho u_i}\right)\right) dV = \\ &\int_S \left[\rho\left(u_i - v_i\right)\right]^{(2)} n_i dS - \int_S \left[\rho\left(u_i - v_i\right)\right]^{(1)} n_i dS.\end{aligned} \tag{4}$$

Rewriting Equation (4) for an elementary volume of the fluid leads to the mass conservation law for this volume

$$\begin{aligned}&\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho u_i}\right) = \left[\rho\left(u_i - v_i\right)\right]^{(2)} n_i \delta(S) - \\ &\left[\rho\left(u_i - v_i\right)\right]^{(1)} n_i \delta(S),\end{aligned} \tag{5}$$

where $\delta(S)$ is Dirac's delta-function determined for the surface S

$$\delta(S) = \delta\left(\left|\mathbf{x} - \mathbf{x_0}\right|\right), \quad \mathbf{x_0} \in S. \tag{6}$$

The momentum conservation law for the elementary volume can be derived analogically. It has the following form

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\overline{\rho u_i}\right) + \frac{\partial}{\partial x_j}\left(\overline{\rho u_i u_j} + \overline{p}_{ij}\right) = \\ &\left[p_{ij} + \rho u_i\left(u_j - v_j\right)\right]^{(2)} n_j \delta(S) - \\ &\left[p_{ij} + \rho u_i\left(u_j - v_j\right)\right]^{(1)} n_j \delta(S).\end{aligned} \tag{7}$$

Here p_{ij} is the compressive stress tensor (Lighthill 1952)

$$p_{ij} = p\delta_{ij} + \mu\left[-\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} + \frac{2}{3}\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}\right], \tag{8}$$

where μ is the coefficient of viscosity, p is the pressure, and δ_{ij} is Kronecker's delta.

Note that Equations (5) and (7) are derived for a general fluid volume V. The only assumptions used during the derivation are the validity of mass and momentum conservation laws everywhere in the fluid except on the surface S. To derive an equation for a rigid surface in the fluid, it is necessary to make additional assumptions regarding the boundary conditions on the rigid surface.

Following Ffowcs Williams and Hawkings (1969), assume that the region 1 is the region inside the rigid surface S. Due to the rigidity of the surface S it can be considered impenetrable for the surrounding fluid. Therefore, it is logical to consider the first boundary condition to be the equality of the normal component of the fluid in the outside region and that of the surface

$$u_n^{(2)}\Big|_S = v_n. \tag{9}$$

In addition, Ffowcs Williams and Hawkings (1969) assume that the fluid inside the rigid object (i.e. in the region 1) is at rest and, as a result, all thermodynamic variables in that region have their equilibrium values. This leads to the following boundary conditions for the region 1

$$\rho^{(1)} = \rho_0 = const, \tag{10}$$

$$p_{ij}^{(1)} = 0, \tag{11}$$

where p_{ij} is interpreted as the difference of the stress tensor from its mean value.

After the substitution of Equations (10) and (11) into Equations (5) and (7), the latter two equations take the following form

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho u_i}\right) = \rho_0 v_i \delta(S) n_i, \tag{12}$$

$$\frac{\partial\left(\overline{\rho u_i}\right)}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\rho u_i u_j} + \overline{p}_{ij}\right) = p_{ij}\delta(S) n_j. \tag{13}$$

By eliminating $\overline{\rho u_i}$ from Equations (12) and (13) one can obtain Ffowcs Williams and Hawkings equation in the differential form

$$\begin{aligned} &\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x_i^2}\right)(\rho - \rho_0) = \\ &\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} - \frac{\partial}{\partial x_i}\left(p_{ij}\delta(S) n_j\right) + \frac{\partial}{\partial t}\left(\rho_0 v_i \delta(S) n_i\right), \end{aligned} \tag{14}$$

where T_{ij} is Lighthill's stress tensor

$$T_{ij} = \rho u_i u_j + p_{ij} - c^2(\rho - \rho_0)\delta_{ij}. \tag{15}$$

Equation (14) is a non-uniform wave equation for fluid density fluctuations where the right-hand part determines the acoustic sources in the fluid. The solution of such an equation can be written via the integral of the source term over the fluid volume. With the assumption of low Mach numbers (v/c<<1), such a solution for Equation (14) takes the following form (Ffowcs Williams and Hawkings 1969)

$$\begin{aligned} &4\pi c^2 \rho'(\mathbf{x},t) = \\ &\frac{\partial^2}{\partial x_i \partial x_j}\int_V \frac{T_{ij}(t - r/c)}{r} dV(\mathbf{y}) - \\ &\frac{\partial}{\partial x_i}\int_S \frac{p_{ij}(t - r/c) n_j}{r} dS(\mathbf{y}) + \\ &\rho_0 \frac{\partial}{\partial t}\int_S \frac{v_n(t - r/c)}{r} dS(\mathbf{y}), \end{aligned} \tag{16}$$

where $\rho'(\mathbf{x},t) = \rho(\mathbf{x},t) - \rho_0$ are density fluctuations, $r = |\mathbf{x} - \mathbf{y}|$ is the distance between the source point $\mathbf{y}$ and the observation point $\mathbf{x}$.

Equation (16) is Ffowcs Williams and Hawkings equation in the integral form. The first term in the right-hand part determines Lighthill's quadrupole sources in the fluid volume, the second term is responsible for dipole sources on the surface due to forces acting between the fluid and the surface, and the third term describes the monopole sources due to the surface motion.

ARBITRARINESS OF BOUNDARY CONDITIONS ON THE INNER SIDE OF THE RIGID SURFACE

From the start of the derivation, the boundary S is assumed to be the surface of discontinuity, and indeed, Equations (9), (10) and (11) considered together imply discontinuous vectors of mass and momentum flows on S, as the normal components of these vectors are, in general, different on both sides of S.

At the same time, the condition of the impenetrable boundary determined by Equation (9) is, in fact, the definition of the rigid boundary and considering this condition only is sufficient to solve any problem of sound radiation and scattering by a rigid boundary. The conditions determined by Equations (10) and (11) on the *inner* side of the rigid boundary are different in this respect. The sources of mass and momentum, which are allowed on the surface S, can account for *any* pressure and velocity fields on the *inner* side of S and this will not affect the boundary condition on the *outer* side of S determined by Equation (9). It is possible to say that the fluid inside S is effectively shielded from the fluid outside S by the sources of mass and momentum on S.

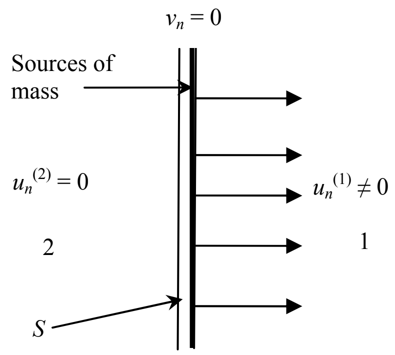

Figure 2. De-coupling of the velocity fields on both sides of the surface S due to sources of mass on S.

The independence of the velocity fields in the regions 1 and 2 is demonstrated in Figure 1. The Figure shows that the normal velocity of the surface and that of the fluid in the region 2 (the outer side of S) are equal according to Equation (9). Note that these velocities may not be zero, as shown in Figure 2, but they must be equal. At the same time, the normal velocity in the region 1 (the inner side of S) is not zero. This discontinuity in the normal velocity can be attributed to the sources of mass in the region 1. These sources of mass affect only the field in this region, as any flow of mass from this sources towards the region 2 is prohibited by the condition of the rigid boundary (Equation (9)).

A similar argument can be put forward about the momentum flows, or, which is the same, the stress fields in the fluid. Any difference in stress fields on both sides of S can be compensated by force acting from the surface on the fluid.

This argument leads to the inevitable conclusion that the two regions should be considered as decoupled and non-interacting, which means that the boundary condition inside should have no bearing on the field outside. As a result, the choice of boundary conditions on the inner side of the boundary S should not affect the result of this derivation, i.e. the equation for the sound wave radiated by the fluid flow near this boundary.

ALTERNATIVE BOUNDARY CONDITIONS AND THE NON-UNIFORM KIRCHHOFF EQUATION

Consider, for instance, the boundary conditions on the inner side of the rigid surface, which imply continuous normal component of velocity and the compressive stress tensor across the surface S. Such conditions are commonly used at boundaries between two media, as, in many cases, there are no sources of mass and momentum on such boundaries. The continuous boundary conditions on the surface S can be written as

$$u_n^{(1)}\Big|_S = u_n^{(2)}\Big|_S = v_n, \tag{17}$$

$$p_{ij}^{(1)}\Big|_S = p_{ij}^{(2)}\Big|_S. \tag{18}$$

Equations (17) and (18) will now take place of the Equations (10) and (11). After the substitution of Equations (17) and (18) into Equations (5) and (7), the right-hand parts of these equations vanish, and the following equations will take place of Equations (12) and (13)

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho u_i}\right) = 0, \tag{19}$$

$$\frac{\partial\left(\overline{\rho u_i}\right)}{\partial t} + \frac{\partial}{\partial x_j}\left(\overline{\rho u_i u_j} + \overline{p}_{ij}\right) = 0. \tag{20}$$

By eliminating, as above, $\overline{\rho u_i}$ from Equations (19) and (20) it is possible to obtain the following non-uniform wave equation for density fluctuations

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x_i^2}\right)\left(\rho - \rho_0\right) = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}. \tag{21}$$

According to well-known theorems of the potential theory (Korn 1971), the solution of such an equation in the presence of rigid boundaries can be written as

$$\begin{aligned} &4\pi c^2 \rho'(\mathbf{x},t) = \\ &\frac{\partial^2}{\partial x_i \partial x_j}\int_V \frac{T_{ij}\left(t - r/c\right)}{r} dV(\mathbf{y}) - \\ &c^2 \int_S \frac{\partial}{\partial n}\frac{\rho'\left(t - r/c\right)}{r} dS(\mathbf{y}) + \\ &c^2 \int_S \frac{1}{r}\frac{\partial}{\partial n}\rho'\left(t - r/c\right) dS(\mathbf{y}). \end{aligned} \tag{22}$$

Note that, during the derivation of Equation (22), derivatives in the first term in the right-hand part have been interchanged due to the property of such integrals mentioned by Ffowcs Williams and Hawkings (1969).

Equation (22) is the non-uniform Kirchhoff equation. It is clear that it differs in its appearance from the FW-H Equation (16). Whereas the first terms in the right-hand parts of these equations, which are responsible for Lighthill's quadrupole sound, are identical, the second and the third terms are different. In the FW-H equation, the second and third terms are determined by the total force and the total normal velocity on the boundary respectively. In the non-uniform Kirchhoff equation, the second and third terms are determined by the density fluctuations and their normal derivative on the boundary or, in other words, by *acoustic* components of the force and the normal velocity.

Despite their different appearance, the FW-H and the non-uniform Kirchhoff equations are derived with equivalent boundary conditions. Therefore, they do not contradict each other and, in fact, are different forms of the same equation. This statement is the main result of this paper.

Being equivalent, both equations can be used for predicting the noise radiated by gas or fluid flows near solid boundaries. As stated above, examples of such flows may be found in various civilian and military applications in air and water. Whereas in different applications any of these equations can be more convenient for use than the other, an investigation of this question for any specific examples is considered to be outside of the scope of this work.

CONCLUSIONS

In this paper, the original derivation of the Ffowcs Williams and Hawkings equation is considered. It is shown that the boundary conditions used by these authors on the inner side of the rigid boundary can be chosen arbitrarily, as the regions outside and inside the rigid boundary are shielded from each other by the boundary. It is also demonstrated that, if the boundary conditions implying continuous thermodynamic variables across the boundary are used, the derivation leads to the non-uniform Kirchhoff equation. As a result, the conclusion about the equivalence of the two equations is made.

Due to their equivalence, these two equations can be used for predicting flow noise parameters in similar circumstances. The question as to which of the equations is more applicable to any particular fluid flow requires further investigation.

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