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Sound Decisions: Moving forward with Acoustics

Interchangeability of two existing formulations of sound propagation in isothermal surface ducts

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ABSTRACT

Sound propagation in isothermal ducts is traditionally described by a formulation based on the n^2-linear sound speed profile (SSP) where the acoustic field is expressed via Airy functions. The present author recently suggested an alternative formulation based on the n^2-exponential SSP with acoustic field described by Bessel functions. In this paper, it is shown that the n^2-exponential SSP is a better approximation of the linear SSP, but both approximations are valid, as their difference with the linear SSP is small. Also, vertical pressure profiles are calculated by several methods combining elements of the two approximations. It is demonstrated that the methods involving exact calculations of the horizontal wavenumber produce most accurate results. Overall, it is shown that the two formulations are mutually interchangeable and their various elements can be combined together and utilised in acoustic propagation modelling depending on convenience of their use in a particular environment.

1 INTRODUCTION

Acoustic surface ducts are formed in the top layer of the ocean if the vertical sound speed gradient within the layer is positive, so that sound waves are refracted towards the ocean surface. In an isothermal duct, water temperature within the layer is uniform and the sound speed increases linearly with depth due to the increase in hydrostatic pressure. In such a duct, successive reflections from the surface lead to a significant increase in propagation range.

Since an analytic solution to the wave equation in a medium with linearly changing sound speed is not known, the linear SSP is commonly approximated by the n^2-linear SSP, where the inverse square of the sound speed depends linearly on depth and the acoustic pressure is determined via Airy functions (Jensen et al, 2011).

The present author previously proposed an alternative formulation that can be utilised to describe sound propagation in a duct with a linear SSP (Zinoviev, 2018). As the inverse square of the sound speed depends exponentially on depth in this formulation, it was referred to as the "n^2-exponential SSP". The acoustic pressure in this formulation is expressed via Bessel functions.

This paper is devoted to a comparison of the two existing approximations of the linear SSP. Section 2 provides a general description of a pressure field in a vertically stratified and range-independent medium. Section 3 describes the linear SSP and its two approximations. Section 4 deals with accuracy of the SSP approximation, whereas accuracy of pressure field calculations is investigated in Section 5.

2 ACOUSTIC FIELD IN A VERTICALLY STRATIFIED MEDIUM

If an acoustic medium is a fluid half space where the sound speed, $c(z)$, depends only on depth, z, and the harmonic time dependence is assumed, then the acoustic pressure in the far field, $P(r,z,t)$, is usually considered to be of the form

$$P(r,z,t) = \frac{A}{\sqrt{r}} e^{i(k_r r - \omega t)} p(z). \tag{1}$$

In this formulation, r is the horizontal range, t is time, A is the complex pressure amplitude, k_r is the horizontal wavenumber, ω is the angular acoustic frequency, and $p(z)$ satisfies the depth-separated Helmholtz equation for the medium (Jensen et al., 2011):

$$\left[\frac{d^2}{dz^2} + \left(\frac{\omega^2}{c^2(z)} - k_r^2\right)\right] p(z) = 0. \quad (2)$$

Eq. (2) is supplemented by the pressure-release boundary condition at $z = 0$.

3 THE LINEAR SOUND SPEED PROFILE AND ITS APPROXIMATIONS

3.1 THE LINEAR SOUND SPEED PROFILE

Within the linear SSP, the dependence of sound speed, c_{lin}, on depth, z, is determined by a simple formula,

$$c_{lin}(z) = c_s + gz, \quad (3)$$

where c_s is the sound speed at the surface and g is the constant sound speed gradient. The surface duct is present if $g > 0$. As mentioned above, a closed-form solution of Eq. (2) within the SSP described by Eq. (3) is not known.

3.2 THE *N*²-LINEAR SOUND SPEED PROFILE

Traditionally, the linear dependence of sound speed on depth is approximated by the *n*²-linear SSP, where the square of sound speed, $c_{nsl}(z)$, is inversely proportional to depth:

$$\frac{1}{c_{nsl}^2(z)} = az + b, \quad a = \text{const}, \quad b = \text{const}. \quad (4)$$

If the constants a and b are expressed via the parameters c_s and g, the sound speed $c_{nsl}(z)$ takes the form of

$$c_{nsl}(z) = \frac{c_s}{\sqrt{1 - \frac{2g}{c_s} z}}. \quad (5)$$

It is clear that, in all realistic ducts, $2gz/c_s \ll 1$. Then the right-hand part of Eq. (5) can be expanded into Taylor series:

$$c_{nsl}(z) = c_s + \text{gz} + \frac{3}{2}\frac{g^2 z^2}{c_s} + \cdots. \quad (6)$$

It is clear that this approximation of the linear SSP is valid as the third and subsequent terms in Eq. (6) are much smaller than the sum of the first two.

In a fluid half-space with the *n*²-linear SSP, the closed-form solution for the pressure field is expressed via Airy functions $\text{Ai}(\zeta)$ as follows (Jensen et al., 2011, Zinoviev, 2018):

$$p_{nsl}(z) = \text{Ai}\left(\frac{k_r^2 - k_s^2\left(1 - \frac{2g}{c_s} z\right)}{\left(-k_s^2 \frac{2g}{c_s}\right)^{2/3}}\right), \quad k_s = \frac{\omega}{c_s}. \quad (7)$$

The pressure-release boundary condition on the surface leads to a discrete spectrum for horizontal wavenumbers, $k_{r,n}$:

$$k_{r,n} = k_s \sqrt{1 + \left(\frac{2g}{\omega}\right)^{2/3} \zeta_n}, \; n = 1, 2, 3 \ldots, \quad (8)$$

where n is the number of a normal mode of the medium, and ζ_n are zeros of the Airy function satisfying the following equation:

$$\text{Ai}(\zeta_n) = 0. \quad (9)$$

It is well-known that ζ_n can be approximated as

$$\zeta_n \approx -\left(\frac{3\pi(n-1/4)}{2}\right)^{2/3}, \tag{10}$$

with the accuracy of this approximation improving with increasing mode number n.

3.3 THE *N*[2]-EXPONENTIAL SOUND SPEED PROFILE

The present author previously showed that, if the inverse square of the sound speed exponentially tends to a constant value with increasing depth, the solution for the acoustic pressure can be expressed via Bessel functions (Zinoviev, 2016). The author called such SSP the "n^2-exponential SSP" by analogy with the "n^2-linear SSP". He also showed that, if the constant asymptotic value for the sound speed is infinite, the vertical dependence of the sound speed $c_{nse}(z)$ and the corresponding solution for the acoustic pressure $p_{nse}(z)$ can be expressed via Bessel functions as follows (Zinoviev, 2018):

$$c_{nse}(z) = c_s e^{\frac{gz}{c_s}}, \tag{11}$$

$$p_{nse}(z) = J_{k_r c_s/g}\left(\frac{\omega}{g} e^{-\frac{gz}{c_s}}\right), \tag{12}$$

where $J_{k_r c_s/g}$ is a Bessel function of the first kind of the order $k_r c_s/g$. The condition for determining the horizontal wavenumber spectrum $k_{r,n}$ takes the form of

$$p_{nse}(0) = J_{k_{r,n} c_s/g}\left(\frac{\omega}{g}\right) = 0. \tag{13}$$

As $gz/c_s \ll 1$ in all practical situations, the right-hand side of Eq.(11) can be expanded into Taylor series:

$$c_{nse}(z) = c_s + \mathrm{gz} + \frac{1}{2}\frac{g^2 z^2}{c_s} + \cdots. \tag{14}$$

It can be seen from Eq. (14) that it approximates well the linear SSP determined by Eq. (3).

4 ACCURACY OF THE TWO APPROXIMATIONS OF SOUND SPEED PROFILE

The present author previously estimated the accuracy of the two approximations by visually comparing their dependencies of the sound speed on depth (Zinoviev, 2018). In this paper, their accuracy is demonstrated by comparing the Taylor expansions determined by Eqs. (6) and (14). In these equations, the first two terms represent the linear SSP, whereas the third term determines the accuracy of the approximations. The fourth and following terms are neglected here as they are much smaller than any of the first three terms.

As the third term in Eq. (6) is three times larger than the third term in Eq. (14), the difference between the sound speeds determined by the n^2-exponential SSP and by the linear SSP is by the same factor smaller than the corresponding difference for the n^2-linear SSP. Zinoviev (2018) arrived at a similar conclusion by visually comparing the two SSPs. Therefore, the n^2-exponential SSP can be considered a better approximation of the linear SSP than the n^2-linear SSP. At the same time, as the third term in both Eqs. (6) and (14) are much smaller in comparison with the other two terms, both approximations of the linear SSP are valid.

5 ACCURACY OF PRESSURE FIELD CALCULATIONS

Due to the validity of both approximations of the linear SSP, Eqs. (1) to (14) allow one to define multiple ways of calculating the pressure field within an environment with the linear SSP. For example, values for $k_{r,n}$, which need to be substituted into Eq. (12) for the n^2-exponential SSP, can be calculated either via the roots of Eq. (13) or by means of Eq. (8) for the n^2-linear SSP, where the zeros ζ_n of Airy functions can be calculated either exactly using Eq. (9) or approximately by Eq. (10). Table 1 shows definitions of five solutions for the pressure field, $p_m(z)$, where z is depth and $m = 0,1,\ldots,4$. The solutions are defined by methods of their evaluation and normalised by their maximum value within the depth interval under consideration.

Table 1. Definitions of solutions describing the pressure field in a duct with the linear SSP

Notation	Function used	Calculation of $k_{r,n}$	Calculation of ζ_n
$p_0(z)$	Airy, Eq. (7)	Eq. (8)	Exact, Eq. (9)
$p_1(z)$	Airy, Eq. (7)	Eq. (8)	Approximate, Eq. (10)
$p_2(z)$	Bessel, Eq. (12)	Exact, Eq. (13)	N/A
$p_3(z)$	Bessel, Eq. (12)	Eq. (8)	Exact, Eq. (9)
$p_4(z)$	Bessel, Eq. (12)	Eq. (8)	Approximate, Eq. (10)

For the purpose of evaluating accuracy of calculations of the pressure field, a semi-infinite medium with a linear SSP is considered. The sound speed gradient in the medium is 0.016 s^{-1}, which is a typical value for isothermal surface ducts. The acoustic frequency is set to 500 Hz. Calculations are carried out for the first two modes, for which vertical profiles are calculated by means of the function $p_0(z)$. The profiles are shown in Figure 1.

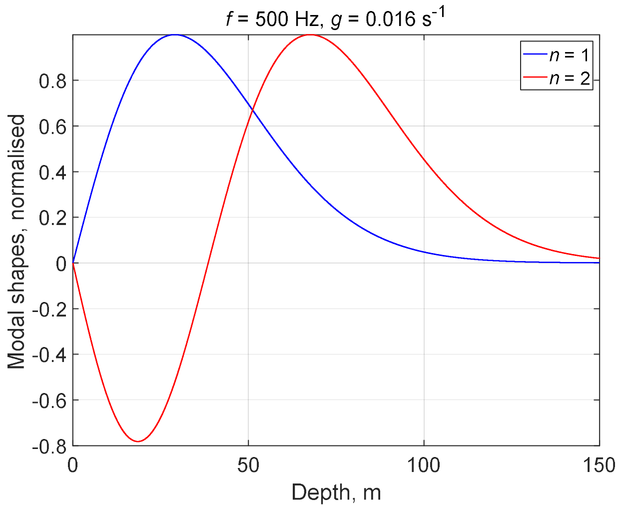

Figure 1: Vertical profiles of the first two modes in the medium under consideration; f, g and n are the acoustic frequency, the sound speed gradient, and the mode number respectively

As calculations of the pressure field by functions $p_m(z)$ produce quite similar results, the following differences are introduced to demonstrate the accuracy of calculations:

$$\Delta p_m(z) = p_m(z) - p_0(z), \quad m = 1, 2, 3, 4. \qquad (15)$$

Eq. (15) represents differences between the pressure field calculated by one of the functions $p_m(z)$ and that calculated by the function $p_0(z)$. Figure 1 shows that the function $p_0(z)$ is an accurate solution of the depth-separated wave equation, as the pressure-release boundary condition at $z = 0$ is satisfied with good accuracy.

The results of calculations of $\Delta p_m(z)$ are shown in Figure 2. In this figure, the solution represented by the green line is entirely based on the n^2-exponential SSP formulation ($p_2(z)$ in Table 1 above), whereas the red line solution is based on the n^2-linear SSP formulation with approximate calculation of the Airy function zeros ζ_n ($p_2(z)$ in Table 1). The blue and cyan line solutions ($p_3(z)$ and $p_4(z)$ in Table 1) can be called "mixed", as they utilise elements of both formulations: they use Bessel functions with the horizontal wavenumbers $k_{r,n}$ calculated with the use of ζ_n using a formula derived on the basis of n^2-linear SSP formulation (Eq. (8)).

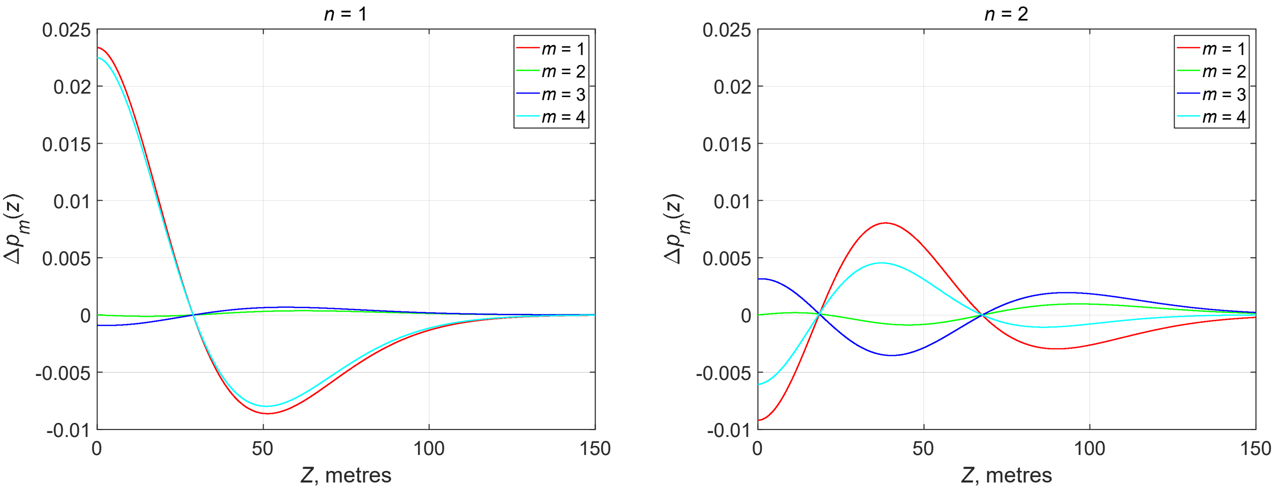

Figure 2: Differences between normalised vertical pressure profiles determined by Eq. (15) for the first and second normal modes

Figure 2 shows that the solution for the pressure based on the *n*²-exponential SSP with exact values for $k_{r,n}$ (green line) is the closest one to the traditional solution based on *n*²-linear SSP that utilises Airy functions with exact calculations of ζ_n. It can be seen that the difference between these two solutions is not more than 0.001 for the second mode and even smaller for the first mode. This proves that the formulations utilising the *n*²-linear and *n*²-exponential profiles are, essentially, equivalent at least for the parameter values used in the calculations.

Although the other three solutions considered here are less accurate, their accuracy can still be considered reasonable, as their difference with the base solution $p_0(z)$ is no more than 0.025. Out of these solutions, the one based on Airy functions with approximate values of ζ_n (red line) produces results with worse accuracy than the other two solutions based on Bessel functions. It can also be noted that the two “mixed” solutions, which use elements of both formulations (blue and cyan lines), also produce results with good accuracy. These solutions can utilise the best characteristics of both formulations, such as the mathematical compactness of the *n*²-exponential formulation as well as calculation of horizontal wavenumbers via tabulated zeros of Airy functions. As expected, the solution making use of the exact values for ζ_n is more accurate than the one based on approximate calculations.

6 FINAL REMARKS

In this paper, approximations of the linear Sound Speed Profile (SSP) based on the *n*²-linear and *n*²-exponential SSPs are considered. It is shown that, for realistic ducts, the equations for the SSP in both approximations can be expanded into Taylor series leading to the linear SSP with some discrepancy. As this discrepancy is three times smaller for the *n*²-exponential SSP, it can be considered a better approximation of the linear SSP.

Solutions determining pressure field, which are based on both approximations, are evaluated for a medium with the linear SSP. It is shown that, out of all solutions considered, the ones with exact evaluation of the horizontal wavenumbers produce closest results. Also, all functions including those that use elements of both approximations are shown to produce similar results for the pressure field, thus making the two approximations interchangeable at least for the parameters used in the calculations.

REFERENCES

Jensen, Finn B., Henrik Schmidt, Michael B. Porter, and William A. Kuperman. 2011. *Computational Ocean Acoustics*. 2nd ed. New York.

Zinoviev, Alex 2016. ‘Easy-to-use closed-form equations for modal cut-on frequencies of a surface duct with an exponential sound speed profile’. In *Proceedings of ACOUSTICS 2016*, Brisbane, Australia.

Zinoviev, Alex 2018. ‘The *n*²-exponential sound speed profile and its use for modelling sound propagation in an isothermal surface duct’. In *Proceedings of ACOUSTICS 2018*, Adelaide, Australia.