Longitudinal magnetoelastic Riemann wave in a rod

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ABSTRACT

The propagation of longitudinal magnetoelastic waves in a rod is under our consideration. Magnetoelasticity is a scientific branch which arose on the junction of mechanics of deformable bodies, electrodynamics and acoustics. It studies dynamic processes arising during interaction between electromagnetic and deformational fields. The nonlinear Bernoulli's model of a rod has been used for describing longitudinal oscillations. The rod assumed an ideal conductor. For the research we've got the evolutionary equation from the system of equations of magneto-elasticity. For that we entered a small parameter into the system. The obtained evolutionary equation represents Riemann equation with regard to axial deformation. Profile of the Riemann wave is corrupting along with propagation because different wave's pieces have different velocity. That is why at a certain moment of time the wave turns over. Under this model the time when the wave turns over depends on the value of the external magnetic field. The profile of the wave has been taken as a sine at initial moment of time. The moment of the wave's inversion grows with increasing of the value of the external magnetic field.

The effect of magnetoelasticity was discovered by the Italian physicist E. Villari in 1865. However magnetoelasticity as a scientific branch began to develop at the end of the 50-ies of the XX century. It arose on the junction of mechanics of deformable bodies, electrodynamics and acoustics. The first works were initiated by problems in geophysics. It was necessary to describe wave dynamics of deep layers of Earth taking into account its conductivity and interaction with the geomagnetic field. Since that time dynamic processes arising during interaction between electromagnetic and deformational fields have been intensively studied. It is connected with different physical, technical and technological applications. Among them is the problem of durability of constructions operating in strong electromagnetic fields when Ampere forces have significant influence on strength properties. Noncontact actuation of oscillations and waves helps to solve different problems in crack detection and vibratory processing of stiffing fusions.

Including fields of different physical nature into mechanical systems opens new opportunities for technical and technological development. Effects of magnetoelasticity appear in strong magnetic fields when produced strains have major influence on wave and dissipative characteristics of medium or thin bodies: rods, planes, membranes. Anisotropy of properties is typical of boundless magnetoelastic medium in a magnetic field. In medium with finite conductivity a magnetic field leads to additional dissipation mechanism. Such features of magnetoelastic systems open new possibilities of practical application.

In magnetoelasticity, the influence of a magnetic field on a deformational field is described employing the Lorentz forces

$$\vec{F}_m = \rho_e \vec{E} + \vec{j} \times \vec{B}, \tag{1}$$

which enter equations of motion of an elastic body

$$\rho \frac{\partial^2 \vec{u}}{\partial \tau^2} = (\lambda + \mu)\, grad\, div\, \vec{u} + \mu \Delta \vec{u} + \vec{F}_{nonlinear} + \vec{F}_m . \tag{2}$$

Here $\vec{E}$ is the intensity of the magnetic field ; $\vec{j}$ is the vector of the electric current density, $\vec{B}$ is the magnetic induction vector; ρ_e is the volumetric density of the electric charges; $\vec{u}$ is displacements vector ; λ, μ are the Lame constants ; ρ is the density of material; τ is the time.

The force $\vec{F}_{nonlinear}$ includes elements which result from the consideration of elastic nonlinearity. If only the quadratic nonlinearity is taken into account, then components of the vector can be represented through the gradients of displacements as follows [7]:

$$F_i = \left(\mu + \frac{A}{4}\right)\left(u_{l,kk}u_{l,i} + u_{i,l}u_{l,kk} + 2u_{l,k}u_{i,lk}\right) +$$
$$+\left(\lambda + \mu + \frac{A}{4} + B\right)\left(u_{l,k}u_{l,ik} + u_{i,l}u_{k,lk}\right) + (\lambda + B)\, u_{l,l}u_{i,kk} + \tag{3}$$
$$+(B + 2C)\, u_{l,l}u_{k,ki} + \left(\lambda + \mu + \frac{A}{4} + B\right)\left(u_{l,k}u_{l,ki} + u_{i,l}u_{k,lk}\right)$$

Here A, B, C are the Landau constants; (index after comma means differentiation with respect to the corresponding coordinate; repeating indices mean summation.

From Maxwell equations, one can obtain equations for electric and magnetic inductions$\left(\vec{D}\right)$ and$\left(\vec{B}\right)$, respectively:

$$\frac{\partial \vec{D}}{\partial \tau} = rot\, \vec{H} - \vec{j}\,, \tag{4}$$

$$\frac{\partial \vec{B}}{\partial \tau} = rot\left[\frac{\partial \vec{u}}{\partial \tau} \times \vec{B}\right] + \frac{c^2}{4\pi\sigma} \Delta \vec{B}, \tag{5}$$

which along with electromagnetic equations of state

$$\vec{j} = \sigma \vec{E}\,, \vec{D} = \varepsilon_e \vec{E}\,, \vec{B} = \mu_e \vec{H}\,, \tag{6}$$

need to be added to equations (1), (2). Here $\vec{H}$ is intensity of the magnetic field, σ is the conductivity, ε_e is the permittivity and μ_e is the magnetic conductivity, c is the electromagnetic constant.

In magnetoelasticity, are neglected both biasing current and electric field. Due to this, equations of magnetoelasticity can be written as follows:

$$\rho \frac{\partial^2 \vec{u}}{\partial \tau^2} = (\lambda + \mu)\, grad\; div\, \vec{u} + \mu \Delta \vec{u} +$$
$$+ \vec{F}_{nonlinear} + \frac{1}{4\pi}\left(rot\, \vec{H} \times \vec{H}\right), \tag{7}$$
$$\frac{\partial \vec{H}}{\partial \tau} = rot\left[\frac{\partial \vec{u}}{\partial \tau} \times \vec{H}\right] + \frac{c^2}{4\pi\sigma} \Delta \vec{H}\,.$$

Along with classic models there are the so-called "précised" or non-classic models in rod dynamics. Those models consider additional factors that affect the dynamic processes or are free from certain hypotheses used in engineering theories and limiting their application areas. Bishop's model generalizes the classic theory of Bernoulli that's used to describe longitudinal oscillations of the rod. It additionally considers the kinetic energy of transverse deformations.

We consider propagation of the longitudinal waves in a homogeneous nonlinear elastic rod placed in an external magnetic field. Let us suppose that external constant magnetic field with intensity H_0 is transverse to the direction of the waves' propagation (see Figure. 1).

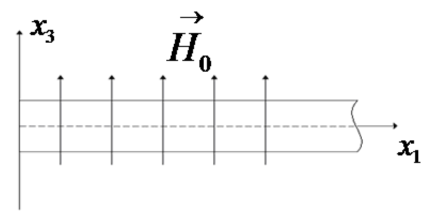

Figure 1. Rod in an external magnetic field

Generally, magnetic field which results from the interaction between external constant magnetic field and the deformation field can be represented as follows:

$$\vec{H} = \vec{H}_0\, \vec{n} + \vec{h}, \tag{8}$$

where $\vec{h}$ is a small disturbance of the magnetic field, $\vec{n}$ is the normal vector.

For longitudinal elastic waves in the rod and for the magnetic field, we obtain the following expressions:

$$\vec{u} = (u_1, 0, 0), \vec{h} = (h_1, h_2, h_3), \vec{H} = (h_1, h_2, H_0 + h_3). \tag{9}$$

The system of equations of magnetoelasticity, according to the Bernoulli's model of the ideal conductive rod, can be written as follows:

$$\frac{\partial^2 \vec{u}}{\partial \tau^2} - c_0^2\left(1+\frac{6\alpha_1}{E}\frac{\partial \vec{u}}{\partial x_1}\right)\frac{\partial^2 \vec{u}}{\partial x_1^2} + \frac{1}{4\pi\rho}\vec{H}\frac{\partial \vec{H}}{\partial x_1} = 0,$$
$$\frac{\partial \vec{H}}{\partial \tau} + \vec{H}\frac{\partial^2 \vec{u}}{\partial x_1 \partial \tau} + \frac{\partial \vec{H}}{\partial x_1}\frac{\partial \vec{u}}{\partial \tau} = 0. \qquad (10)$$

Here $c_0 = \sqrt{E/\rho}$ is the velocity of longitudinal waves. According to the model under the consideration, only a transverse component of the magnetic field (h_3) is taken into the account. Other items in the system (10), which include components $(h_1),(h_2)$ have smaller order than others and therefore can be neglected. Thus expressions (9) can be presented as follows:

$$\vec{u} = (u_1, 0, 0) \equiv u(x_1, \tau), \vec{h} = (0, 0, h_3) \equiv h(x_1, \tau),$$
$$\vec{H} = (0, 0, H_0 + h_3) \equiv H_0 + h(x_1, \tau). \qquad (11)$$

For the further research, we need to obtain the evolutionary equation. To achieve that, we make the change of the variables and introduce a small parameter. Let us rewrite system (10) in the form

$$\frac{\partial G}{\partial \tau} - c_0^2\left(1+\frac{6\alpha_1}{E}Q\right)\frac{\partial Q}{\partial x_1} + \frac{1}{4\pi\rho}(H_0+h)\frac{\partial h}{\partial x_1} = 0,$$
$$\frac{\partial Q}{\partial \tau} - \frac{\partial G}{\partial x_1} = 0, \qquad (12)$$
$$\frac{\partial h}{\partial \tau} + (H_0+h)\frac{\partial G}{\partial x_1} + G\frac{\partial h}{\partial x_1} = 0.$$

Here $Q = \frac{\partial u}{\partial x}$ is the deformation, $G = \frac{\partial u}{\partial \tau}$, ν is the Poisson coefficient , $R = \sqrt{J_0/F}$ is the polar radius of gyration, $J_0 = \iint_F (x_2^2 + x_3^2) dF$ is the polar moment of inertia, F is the area of cross-section of the rod, $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is the modulus of elasticity, $\alpha_1 = \frac{E}{2} + \frac{3\lambda}{2} + A + B(1-2\nu) + \frac{C}{3}(1-6\nu)$ is the elastic nonlinearity coefficient.

We introduce dimensionless variables

$$U = Q, V = \frac{G}{c_0}, W = \frac{h_3}{H_0},$$
$$\tilde{x}_1 = \frac{1}{\nu R}x_1, \tilde{\tau} = \frac{c_0}{\nu R}\tau \qquad (13)$$

as well as a moving reference system:

$$x = \tilde{x}_1 - V_p\tilde{\tau}, t = \varepsilon\tilde{\tau} \qquad (14)$$

where V_p is characteristic wave velocity, not known in advance, ε is a small parameter.

Substituting (13) and (14) into (12) and omitting items one of the second order and higher, we obtain the following systems of equations:

$$-V_p\frac{\partial V}{\partial x} - \frac{\partial U}{\partial x} + \frac{c_A^2}{c_0^2}\frac{\partial W}{\partial x} = 0,$$
$$-V_p\frac{\partial U}{\partial x} - \frac{\partial V}{\partial x} = 0, \qquad (15)$$
$$-V_p\frac{\partial W}{\partial x} + \frac{\partial V}{\partial x} = 0.$$

$$\frac{\partial V}{\partial t} - \frac{6\alpha_1}{E}U\frac{\partial U}{\partial x} + \frac{c_A^2}{c_0^2}W\frac{\partial W}{\partial x} = 0,$$
$$\frac{\partial W}{\partial t} + W\frac{\partial V}{\partial x} + V\frac{\partial W}{\partial x} = 0. \qquad (16)$$

which represent zero and first dimensionless approximations of the system (12), respectively. Using the 2nd and 3rd equations in (15), we obtain the connection between the functions:

$$U = -W, V = -V_pU, \qquad (17)$$

and from the 1st equation we determine the velocity:

$$V_p = \sqrt{1+\frac{c_A^2}{c_0^2}}. \tag{18}$$

Here $c_A = \sqrt{H_0^2 / 4\pi\rho}$ is the Alfven wave velocity.

Substituting (17) and (18) into (16) and summing the obtained equations, we transform it to the equation of the form

$$\frac{\partial U}{\partial t} - \alpha U \frac{\partial U}{\partial x} = 0. \tag{19}$$

Here $\alpha = \left(\frac{c_A^2}{c_0^2} - \frac{6\,\alpha_1}{E} + 2V_p\right) \Big/ \left(1+V_p\right).$

Equation (19) represents Riemann equation and it is equivalently to the following characteristic system:

$$\frac{dx}{dt} = -\alpha U, \frac{dU}{dt} = 0, \tag{20}$$

which has the following complete integral:

$$x + \alpha U t = \psi(U), \tag{21}$$

where $\psi(U)$ – arbitrary function.

The general solution of (19) can be written as follows:

$$U(x,t) = F(x+\alpha U t), \tag{22}$$

where *F* – is the inverse function with respect to ψ and can be found from initial and boundary conditions.

Expression (22) is known as a simple wave or the Riemann wave. Profile of the Riemann wave is corrupting along with propagation because different wave's pieces have different velocity. The set of the profile's points where $U(x,t)=0$ will be stationary.

Let's consider in more detail a nonlinear evolution of a wave which initially was assigned as a sinusoidal wave $U(x,0)=U_0 sin(kx).$. In this case (22) will look as follows:

$$U(x,t) = U_0 sin(kx + \frac{U}{U_0} t_1), \tag{23}$$

here $t_1 = k\alpha U_0 t$ – dimensionless time.

The graphical analysis of nonlinear corruptions of the simple wave (23) is shown in Figure. 2:

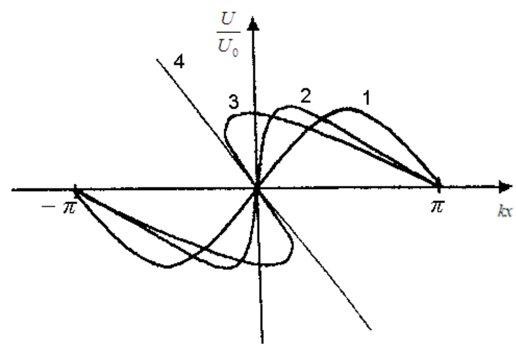

Figure 2. Simple wave's profile corruption: $1 - t_1 = 0$, $2 - t_1 = 1$, $3 - t_1 = 2$, $4 - kx = -t_1 \frac{U}{U_0}$.

From the Figure.2 one can see that at a certain moment of time $t_1 = t_{1*}$ the wave has infinite tangent and then (when $t_1 > t_{1*}$) it becomes multivalued. Such phenomenon sometimes calls wave capsizing. It begins when on the wave's profile a point with vertical tangent appears. Parameters of upset point x_*, t_{1*}, U_* can be found from the following system:

$$\begin{aligned} \left(\frac{\partial x}{\partial U}\right)_* &= \left[\frac{1}{k\sqrt{U_0^2 - U_*^2}} - \frac{t_{1*}}{kU_0}\right] = 0, \\ \left(\frac{\partial^2 x}{\partial U^2}\right)_* &= \frac{U_*}{k(U_0^2 - U_*^2)^{3/2}} = 0. \end{aligned} \tag{24}$$

Solving the system (24) we get the capsizing happens when $U_* = 0, t_{1*} = 1, x_* = 0$. Returning to "real" dimensional time we obtain that capsizing happens at the following moment:

$$t_* = \frac{1}{k\alpha U_0} \qquad (25)$$

which depends on external magnetic field intensity Figure.3:

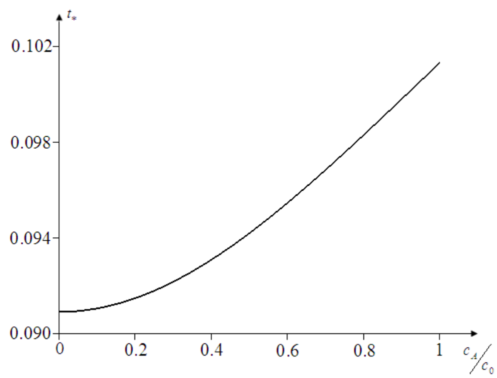

Figure 3. Capsizing time dependency on magnetic field

For condensed media in magnetic fields under 10 tesla, Alfven wave velocity is smaller than a longitudinal wave velocity. That is why, changing of all parameters is shown on $0 \le c_A / c_0 < 1$ interval.

From Figure.3 one can see that the increase of magnetic field leads to the increasing of capsizing time.

In this paper, we have obtained the evolutionary equation for the system of magnetoelasticity equations according to the Bernoulli's model of the ideal conductive rod; studied the influence of an external magnetic field on longitudinal Riemann wave propagation. The analysis performed proves that the magnetic field affects the time of wave capsizing.

REFERENCES

1 Allen, S.J., Silva, C.N.de and Kline, K.A. Theory of Simple Deformable Directed Fluids, *Phys. Fluids*, 10, pp. 2551 – 2555 (1967)

2 Ambartsumyan, S.A. *Theory of Bimodular Elasticity* (Nauka, Moscow (in Russian), 1982)

3 Baser, J. and Ericson, W. (1974). Nonlinear Motion in Magnetoelasticity, *Arch. Rat. Mech. Anal*. 55, pp. 124 – 192

4 Beliy, I.V., Fertik, S.M. and Klimenko, L.T. *Reference Book on Magneto-Pulse Treatment of Metals*(Vitcha Skola, Kharkov (in Russian), 1997)

5 Donato, A. The Burger's Equation in Magnetotermoelasticity with OneDimensional Deformation, *ZAMP* 27, 2, pp. 281– 284 (1976)

6 Erofeyev, V.I. Magnetoelastic Nonlinear Waves in Solids with Microstructure, *Appl. Mech. Rev*. 46, 12. Part 1, p. 557 (1993)

7 Erofeyev, V., Kazhaev, V. and Kovalev, S. Longitudinal MagnetoElastic Waves in Solids with Microstructure, *IUTAM Symposium on Mechanical and Electromagnetic Waves in Structured Media*, ed. R.C. McPhedran, L.C. Botten and N.A. Nicorovici. Kluwer Academic Publishers. Printed in the Netherlands, pp. 149-158 (2001)

8 Karpman, V.I. (1978). Non-Linear Waves in Dispersive Media. Pergamon Press. New York

9 Landau, L.D. and Lifshits E.M. *Theory of Elasticity*, (Pergamon Press, 1986)

10 Pospelov, L.A. Nonlinear Magneto-Elastic Waves, *PMM*, 27, 5, pp. 153-156 (1963) (in Russian)