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Thermoacoustic wave propagation in a narrow channel subject to temperature gradient

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ABSTRACT

This paper examines a linear propagation of thermoacoustic waves in a gas enclosed in a narrow channel subject to temperature gradient axially and extending infinitely. An analysis is made to derive a wave equation by assuming that a typical axial length is much longer than the channel width but a thickness of thermoviscous diffusion layer is arbitrary relative to the width. It is shown that the system of equations is reduced to a spatially one-dimensional wave equation for a pressure in the form of an integro-differential equation due to memory by the thermoviscous effects. Approximations of the wave equation are discussed based on a Deborah number. For a short time after a disturbance is given, i.e. a large Deborah number, the equation is shown to be simply the one derived by the boundary-layer theory, while for a long time, i.e. small Deborah number, it is reduced to a diffusion-wave equation. It is unveiled from this that the thermoviscous effects combined with the temperature gradient give rise to wave propagation in the positive direction of the gradient. If the gradient is steep, they give rise to negative diffusion so that the convective instability will occur.

INTRODUCTION

Acoustic waves propagating in a gas enclosed in a channel (or in a tube) are subject to damping due to thermoviscous diffusive effects. But when an appropriate temperature gradient is imposed along the channel wall, it is known that the acoustic waves become unstable by those diffusive effects. Then the waves grow by gaining energy from heat flux flowing in the background through the wall and the gas. To exploit such an instability, it is desired to specify marginal conditions for it.

It was Rott [1,2] that derived first marginal conditions of a column of helium gas in a tube, i.e. the onset of Taconis oscillations in cryogenics. But analyses are usually very complicated and difficult. The author has been concerned with simple derivation of the conditions in a framework of a boundary-layer theory [3]. It turns out that the theory is applicable not only to derive the conditions but also to describe ensuing self-excited, Taconis oscillations, when it is extended to a weakly nonlinear case [4,5]. The theory will also be applicable to another classically known example of thermoacoustic oscillations in resonators like a Sondhauss tube [6].

But modern thermoacoustic devices use a so-called stack comprising of narrow channels, in which a typical thickness of diffusion layer is comparable with the span length [7]. To this case, the boundary-layer theory will be inapplicable and a new theory is required. The purpose of this paper is to examine a linear propagation of thermoacoustic waves in a gas enclosed in a channel subject to temperature gradient axially and without any restriction on the thickness of the diffusion layer in comparison with the channel width.

But use is made of a ‘narrow-tube approximation’, which stipulates that the ratio of a typical axial length of a temperature gradient and a wavelength is much longer than the channel width. It is demonstrated that the system of equations is reduced to a spatially one-dimensional wave equation for a pressure. Approximations of this equation for a short and long time after a disturbance is given are discussed based on a Deborah number.

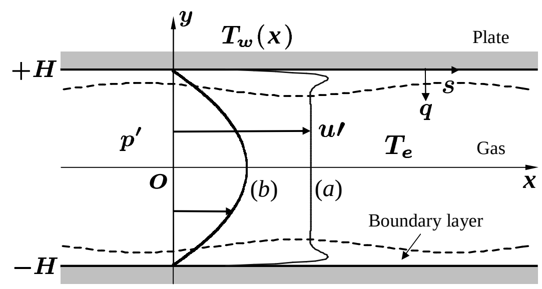

Figure 1: Illustration of a two-dimensional, gas-filled channel subject to temperature gradient axially. The curves labeled (a) and (b) show profiles of the axial velocity u' in cases of a very thin thermoviscous diffusion layer, i.e. boundary layer, and of a very thick one, respectively.

BASIC EQUATIONS

Let us consider a two-dimensional wave propagation in a gas enclosed in a channel of two parallel plates extending infinitely, as shown in Figure 1. Take the x-axis along the direction of propagation and the y-axis normal to it with the origin of the coordinates at a midpoint of the two plates, separated by $2H$. Suppose that the temperature of the plates varies along the x-axis and that the plates have large heat capacity. The temperature of the upper and lower plates at a position x is equal with each other, and is denoted by $T_w(x)$. No gravity is assumed.

In a quiescent equilibrium state, the pressure in the gas takes a uniform value p_0, while the temperature of the gas T_e may be set equal to that of the plates, if the channel width is narrow enough in comparison with a typical axial length of the temperature variation of T_w. The subscript e implies a value in the quiescent state, which is a function of T_e. For an ideal gas, the density of the gas ρ_e is proportional to $1/T_e$.

Letting a typical axial length be L, the ratio H/L is assumed to be much smaller than unity. A reduction of equations based on this assumption is called a narrow-tube approximation. But no restrictions on a typical thickness of a viscous diffusion layer $\sqrt{\nu_e/\omega}$ are made, ν_e being a local kinematic viscosity and ω a typical angular frequency. Although a thermal diffusion layer $\sqrt{\kappa_e/\omega}$ may be taken, κ_e being a thermal diffusivity, its thickness is comparable with that of the viscous one as long as the Prandtl number Pr $(=\nu_e/\kappa_e)$ is of order unity. Therefore the thermal diffusion layer is represented by the viscous one.

Linearizing equation of continuity, Navier-Stokes equation, equations of energy and of state for the ideal gas around the quiescent state, and exploiting the narrow-tube approximation ($H/L \ll 1$), it follows that

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x}(\rho_e u') + \frac{\partial}{\partial y}(\rho_e v') = 0, \tag{1}$$

$$\rho_e \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} + \mu_e \frac{\partial^2 u'}{\partial y^2}, \tag{2}$$

$$0 = -\frac{\partial p'}{\partial y}, \tag{3}$$

$$\rho_e c_p \left(\frac{\partial T'}{\partial t} + u' \frac{\mathrm{d}T_e}{\mathrm{d}x} \right) = \frac{\partial p'}{\partial t} + k_e \frac{\partial^2 T'}{\partial y^2}, \tag{4}$$

$$\frac{p'}{p_0} = \frac{\rho'}{\rho_0} + \frac{T'}{T_e}, \tag{5}$$

in $-\infty < x < \infty$, $-H < y < H$ and $-\infty < t$, where ρ', p' and T' denote, respectively, disturbance in density, pressure and temperature, while u' and v' denote, respectively, x- and y-components of the velocity, the prime implying disturbance; μ_e and k_e denote, respectively, the shear viscosity and the thermal conductivity; c_p denotes a heat capacity at constant pressure. For (1)-(5), the boundary conditions at the plate surfaces require that

$$u' = v' = T' = 0 \ \text{ at } \ y = \pm H. \tag{6}$$

Here it is pointed out that temperature dependence of μ_e and k_e is taken into account through the power laws: $\mu_e/\mu_0 = (T_e/T_0)^\beta$ and $k_e/k_0 = (T_e/T_0)^\beta$, β being a constant between 0.5 and 0.6, and the subscript 0 implying a reference state.

DERIVATION OF WAVE EQUATION

It is found immediately from (3) that the excess pressure p' is uniform over a cross-section of the channel so that p' is a function of x and t, i.e. $p'(x,t)$. This is the key point of the narrow-tube approximation. Exploiting this, we consider to express the other quantities in terms of p'. Use of the Fourier transform with respect to t enables us to solve (1), (2), (4) and (5) to obtain u', T', ρ' and v'. For the details, see the paper [7].

Averaging (1), (2) and (4) over the width, i.e. integrating each equation over y from $-H$ to H with (6), and dividing it by $2H$, it follows that

$$\frac{\partial \bar{\rho}'}{\partial t} + \frac{\partial}{\partial x}\left(\rho_e \bar{u}'\right) = 0, \tag{7}$$

$$\rho_e \frac{\partial \bar{u}'}{\partial t} = -\frac{\partial p'}{\partial x} + \frac{s}{2H}, \tag{8}$$

$$\rho_e c_p \left(\frac{\partial \bar{T}'}{\partial t} + \bar{u}' \frac{\mathrm{d}T_e}{\mathrm{d}x} \right) = \frac{\partial p'}{\partial t} + \frac{q}{2H}, \tag{9}$$

where quantities with over-bar designate the mean values over the width, and s and q denote, respectively, shear stress acting on the gas at the plate surfaces and heat flux flowing into the gas through them, which are given respectively by

$$s = \mu_e \left.\frac{\partial u'}{\partial y}\right|_{y=+H} - \mu_e \left.\frac{\partial u'}{\partial y}\right|_{y=-H}, \tag{10}$$

and

$$q = k_e \left.\frac{\partial T'}{\partial y}\right|_{y=+H} - k_e \left.\frac{\partial T'}{\partial y}\right|_{y=-H}. \tag{11}$$

Equations (7) to (9) are combined to eliminate $\bar{\rho}'$, $\bar{u}'$, and $\bar{T}'$ into a single equation for p':

$$\frac{\partial^2 p'}{\partial t^2} - \frac{\partial}{\partial x}\left(a_e^2 \frac{\partial p'}{\partial x}\right) = \frac{a_e^2}{c_p T_e} \frac{\partial}{\partial t}\left(\frac{q}{2H}\right) - \frac{\partial}{\partial x}\left(a_e^2 \frac{s}{2H}\right). \tag{12}$$

While the left-hand side represents a wave equation for a lossless propagation in the gas non-uniform in temperature, the first and second terms on the right-hand side represent, respectively, the effects due to the heat flux in the form of a monopole and the shear stress in the form of a dipole. They are expressed by the solutions in terms of p'.

Using the expressions of u' and T' obtained to calculate s and q, it follows from (12) that

$$\begin{aligned} &\frac{\partial^2 p'}{\partial t^2} - \frac{\partial}{\partial x}\left(a_e^2 \frac{\partial p'}{\partial x}\right) + \frac{\partial}{\partial x}\left[\frac{a_e^2 \sqrt{\nu_e}}{H} \mathscr{M}\left(\frac{\partial p'}{\partial x}\right)\right] \\ &+ \frac{\gamma - 1}{\sqrt{\mathrm{Pr}}} \frac{\sqrt{\nu_e}}{H} \mathscr{M}_{\mathrm{P}}\left(\frac{\partial^2 p'}{\partial t^2}\right) - \frac{a_e^2}{T_e} \frac{\mathrm{d}T_e}{\mathrm{d}x} \frac{\sqrt{\nu_e}}{H} \\ &\times \left[\frac{1}{1-\mathrm{Pr}} \mathscr{M}\left(\frac{\partial p'}{\partial x}\right) - \frac{1}{(1-\mathrm{Pr})\sqrt{\mathrm{Pr}}} \mathscr{M}_{\mathrm{P}}\left(\frac{\partial p'}{\partial x}\right)\right] = 0, \end{aligned} \tag{13}$$

where $a_e(x)$ denotes the adiabatic sound speed given by $\sqrt{\gamma p_0/\rho_e}$, γ being the ratio of the specific heats, and $\mathscr{M}_{\mathrm{P}}(\phi)$ denotes a functional of a function $\phi(x,t)$ defined by

$$\mathscr{M}_{\mathrm{P}}\left[\phi(x,t)\right] \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{t} \frac{G[\nu_e(t-\tau)/\mathrm{Pr}H^2]}{\sqrt{t-\tau}} \phi(x,\tau) \mathrm{d}\tau, \tag{14}$$

with

$$G(t) = 1 + 2\sum_{n=1}^{\infty} (-1)^n \exp\left(-\frac{n^2}{t}\right). \tag{15}$$

The functional $\mathscr{M}(\phi)$ denotes $\mathscr{M}_{\mathrm{P}}(\phi)$ with Pr set equal to unity formally.

PROPERTIES OF RELAXATION FUNCTIONS

It is found in (13) that p' is no longer governed by a differential equation but an integro-differential equation. The time-integral in (14) represents a memory by the diffusion effects. The function $G/\sqrt{\pi t}$ may be called a relaxation function, which measures a weight of dependence of $\mathscr{M}_{\mathrm{P}}(\phi)$ at t on a past value of $\phi(\tau)$ tracked back by τ from the present time t.

For $\mathscr{M}(\phi)$, $\nu_e t/H^2$ appears in the argument of G. This dimensionless quantity is the inverse of a Deborah number De defined by

$$\mathrm{De} \equiv \frac{H^2}{\nu_e t}. \tag{16}$$

This quantity measures a ratio of a viscous relaxation time H^2/ν_e to a time t concerned. Of course, a thermal relaxation time H^2/κ_e $(=\mathrm{Pr}H^2/\nu_e)$ may also be concerned but it is represented by the viscous one from the reason mentioned before.

To examine a behavior of the relaxation function, the following formula is used:

$$1+2\sum_{n=1}^{\infty}(-1)^n z^{n^2} = \prod_{n=1}^{\infty}(1-z^{2n})\prod_{n=1}^{\infty}(1-z^{2n-1})^2 = \sqrt{\frac{2}{\pi}k'K(k)}, \quad (17)$$

where $z=\exp(-1/t)$, $\log(1/z)=\pi K(k')/K(k)$, k and k' being the modulus of the complete elliptic integral $K(k)$ and its complementary modulus $k'^2=1-k^2$. Using this, G is expressed as

$$G(t) = \prod_{n=1}^{\infty}\left[1-\exp\left(-\frac{2n}{t}\right)\right]\prod_{n=1}^{\infty}\left\{1-\exp\left[-\frac{(2n-1)}{t}\right]\right\}^2. \quad (18)$$

If $t \ll 1$, then G may be approximated as

$$\frac{1}{\sqrt{\pi t}}G(t) \approx \frac{1}{\sqrt{\pi t}}, \quad (19)$$

while for $t \gg 1$, G may be approximated as

$$\frac{1}{\sqrt{\pi t}}G(t) \approx 2\exp\left(-\frac{\pi^2}{4}t\right), \quad (20)$$

where the last expression of (17) has been used [7].

Figure 2(a) plots $G(t)/\sqrt{\pi t}$ numerically calculated as a function of t but it lie on either one of the asymptotic expressions (19) for $t \ll 1$ or (20) for $t \gg 1$ too closely to be visible. The function tends to diverge as $t \to 0$, whereas it tends to vanish exponentially as $t \to \infty$. Figure 2(b) blows up an interval $0.1 \le t \le 0.4$, where the dots represent $G(t)/\sqrt{\pi t}$ numerically calculated. For $t < 0.15$, they lie on the curve (19), while for $t > 0.30$, they lie on the curve (20). The transition between the two asymptotic expressions occurs over a narrow interval $0.15 < t < 0.30$ centered around $t \approx 0.2$. This suggests that $G(t)/\sqrt{\pi t}$ may be approximated substantially by either one of the two asymptotic expressions.

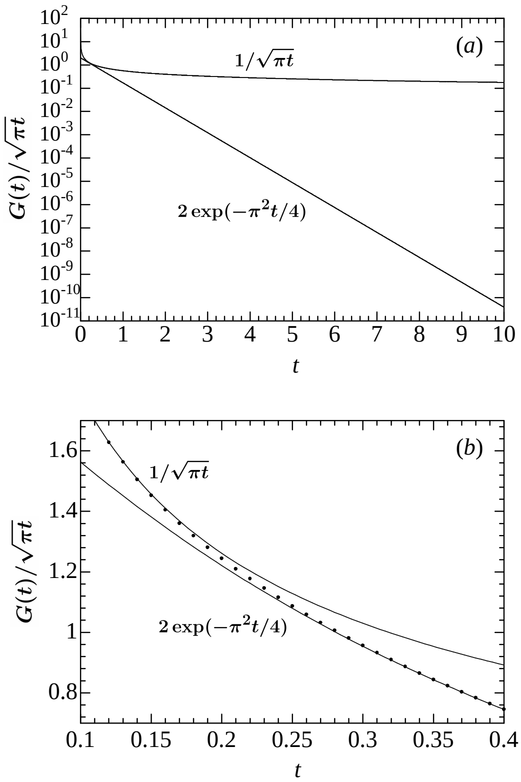

Figure 2: Graph of $G(t)/\sqrt{\pi t}$ by dots together with the two asymptotic expressions (19) and (20) by solid curves.

APPROXIMATIONS OF THE WAVE EQUATION

Thanks to such an amazing behavior of the relaxation function, (13), called a thermoacoustic-wave equation, may be simplified. For De > 0.2, $G(\nu_e t/H^2)/\sqrt{\pi t}$ may be set to be $1/\sqrt{\pi t}$. It then follows that

$$\frac{\partial^2 p'}{\partial t^2} - \frac{\partial}{\partial x}\left(a_e^2\frac{\partial p'}{\partial x}\right) + \frac{a_e^2\sqrt{\nu_e}}{H}\left[C\frac{\partial^{-\frac{1}{2}}}{\partial t^{-\frac{1}{2}}}\left(\frac{\partial^2 p'}{\partial x^2}\right) + \frac{(C+C_T)}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial^{-\frac{1}{2}}}{\partial t^{-\frac{1}{2}}}\left(\frac{\partial p'}{\partial x}\right)\right] = 0, \quad (21)$$

where C and C_T are defined as

$$C = 1+\frac{\gamma-1}{\sqrt{\mathrm{Pr}}} \quad \text{and} \quad C_T = \frac{1}{2}+\frac{\beta}{2}+\frac{1}{\sqrt{\mathrm{Pr}}+\mathrm{Pr}}, \quad (22)$$

and the derivative of minus half-order is defined by

$$\frac{\partial^{-\frac{1}{2}}\phi}{\partial t^{-\frac{1}{2}}} \equiv \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t}\frac{\phi(x,\tau)}{\sqrt{t-\tau}}\mathrm{d}\tau. \quad (23)$$

For a circular tube of radius R, H should be replaced by a hydraulic radius $R/2$, R being a tube radius. This equation is simply the one derived by the boundary-layer theory [3]. It should be remarked, however, that this equation is applicable to any situation for a short time after a disturbance is given at $t=0$, but it tends to be invalid as De becomes less than 0.2.

For a long time, on the other hand, since $G(t)/\sqrt{\pi t}$ diverges as $t \to 0$ and decays exponentially as $t \to \infty$, it may be regarded roughly as a delta function as $t \to \infty$. It is verified indeed that $G(t)/\sqrt{\pi t}$ satisfies

$$\int_0^{\infty}\frac{1}{\sqrt{\pi t}}G(t)\mathrm{d}t = 1, \quad (24)$$

where G vanishes for $t < 0$. The long-time behavior corresponds formally to a case of an extremely narrow tube as $H \to 0$. For a long but finite value of time, a higher-order correction than the delta function is necessary. Such a correction is systematically obtained by expanding the thermoacoustic-wave equation transformed in terms of H [7].

Doing so, the lowest asymptotic equation is obtained as follows:

$$\frac{\partial p'}{\partial t} - \frac{\partial}{\partial x}\left(\alpha\frac{\partial p'}{\partial x}\right) + \frac{\alpha}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial p'}{\partial x} = 0, \quad (25)$$

where α denotes a new diffusivity defined by

$$\alpha = \frac{a_e^2 H^2}{3\gamma\nu_e}. \quad (26)$$

For the circular tube, α is given by $a_e^2R^2/8\gamma\nu_e$. It is found that (25) is a differential equation and the memory disappears. If the temperature gradient is absent, the third term drops out so that p' is subject to diffusion simply. But when the gradient is present, the third term can give rise to a wave propagation and the coefficient

$$\frac{\alpha}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x} \quad (27)$$

provides a propagation velocity V. If the temperature dependence of μ, i.e. β vanishes, α is the constant independent of the temperature. Supposing that a channel width $2H = 0.1$ mm filled with atmospheric air, V takes a value of about 50 m/s.

Thus it is newly revealed that the diffusive effects under the temperature gradient can give rise to a wave propagation in the positive direction of the gradient.

A solution to the diffusion-wave equation (25) is easily expected. If a moving frame with V is introduced, (25) is reduced to a simple diffusion equation. Thus p' is simply diffused in this frame. So we are now interested in higher-order equation than (25). It is obtained as follows:

$$\frac{\partial p'}{\partial t} - \frac{\partial}{\partial x}\left(\alpha \frac{\partial p'}{\partial x}\right) + \frac{\alpha}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial p'}{\partial x} + \left[\frac{6}{5}\gamma - (\gamma-1)\mathrm{Pr}\right] \times \frac{\alpha}{a_e^2}\frac{\partial^2 p'}{\partial t^2} - \frac{2}{5}(1+\mathrm{Pr})\frac{\alpha H^2}{\nu_e T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}\frac{\partial^2 p'}{\partial t \partial x} = 0. \tag{28}$$

While (25) takes account of H up to its quadratic terms, the quartic terms are taken in (28). Equation (28) may also be regarded as an equation for a case of a thick diffusion layer but of finite thickness. At the same time, it corresponds to the equation for a long but finite value of t.

Effects of the higher-order terms are not readily seen. Suppose a case where T_e varies exponentially and β vanishes for sake of simplicity. Then V takes a constant value. If the temperature gradient is very large, the lowest balance in (28) takes place between the first and third terms as $\partial p'/\partial t + V \partial p'/\partial x \approx 0$. Using this relation, (28) may be expressed as

$$\frac{\partial p'}{\partial t} + V\frac{\partial p'}{\partial x} = D\frac{\partial^2 p'}{\partial x^2}, \tag{29}$$

where D is given by

$$D = \alpha\left\{1 - \left[\frac{6}{5}\gamma(2+\mathrm{Pr}) - (\gamma-1)\mathrm{Pr}\right]\frac{V^2}{a_e^2}\right\}. \tag{30}$$

For air with $\gamma = 1.4$ and $\mathrm{Pr} = 0.72$, there may appear a region in which D takes a negative value if V/a_e is large. In this case, it is anticipated that a negative diffusion will occur so that a convective instability will appear.

Finally the diagram indicating spatial and temporal domains for each approximate equation to be valid is shown in Figure 3. Equation (13) is valid everywhere in this diagram. A state prior to $t = 0$ is assumed to be quiescent so that the lower bound of the integrals may be set equal to zero. The horizontal axis represents a dimensional length measured by a typical diffusion length ν/U, ν being a typical kinematic viscosity and U a typical axial speed. This axis may be interpreted as an inverse of the Reynolds number Re ($\equiv UH/\nu$). The vertical axis represents a dimensional time measured by a typical diffusion time H^2/ν and also corresponds to an inverse of the Deborah number De.

Equation (21) is valid as long as the Deborah number remains less than about 0.2. Although (21) is the same as the one derived by the boundary-layer theory for a thin diffusion layer $\mathrm{Re}^{-1} \ll 1$, it now turns out that it holds for any value of the horizontal axis. On the other hand, (25) is valid for a domain $\mathrm{Re}^{-1} \to \infty$ and $\mathrm{De}^{-1} \to \infty$. In other words, it holds when the diffusion layer is extremely thick. If finiteness of Re^{-1} and De^{-1} is taken into account, then (28) is used. In an intermediate domain with $\mathrm{Re}^{-1} \approx \mathrm{De}^{-1} \approx 1$, (13) should be used. But it may be simplified by approximating $G/\sqrt{\pi t}$ as

$$\frac{1}{\sqrt{\pi t}}G(t) = \begin{cases} \dfrac{1}{\sqrt{\pi t}} & \text{for } t < M, \\ 2\exp\left(-\dfrac{\pi^2 t}{4}\right) & \text{for } t > M, \end{cases} \tag{31}$$

where M is chosen to satisfy (25), i.e. $M \approx 0.213$. But note that the small jump of 0.04 occurs at $t = M$.

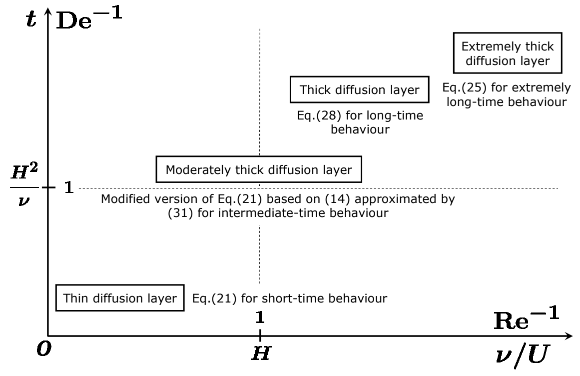

Figure 3: Diagram of spatial and temporal domains in which each approximate equation of (13) is to be valid.

CONCLUSION

Thermoacoustic wave propagation in a two-dimensional, gas-filled channel has been examined in the framework of the linear theory and the narrow-tube approximation. The thermoacoustic-wave equation for the excess pressure has been derived, and it is applicable to any situations. Approximations of this equation have been discussed for a short and long time.

It has been revealed that for the short time, the equation is reduced to the equation derived by the boundary-layer theory, while for the long time, a new diffusion-wave equation is derived. The former is a reason why the boundary-layer theory can yield the marginal condition for initial instability. It has also been unveiled that the diffusive effects can give rise to wave propagation when the temperature gradient is present. If the gradient is steep enough, a negative diffusion will occur and a convective instability will occur.

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