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Nonlinear elastic properties of solids with defects

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ABSTRACT

The meaning of the experimentally measured nonlinear parameters of a medium is discussed. The difference in meaning between the local nonlinearity, which is measured in the vicinity of a single defect and depends on the size of the region of averaging, and the effective volume nonlinearity of the medium containing numerous defects is emphasized. The local nonlinearity arising at the tip of a crack is calculated; this nonlinearity decreases with an increase in the region of second harmonic generation. The volume nonlinearity is calculated for a solid containing spherical cavities. The volume nonlinearity is also calculated for a medium containing infinitely thin cracks in the form of circular disks, which assume the shape of ellipsoids in the course of the crack opening.

One of the interesting and perspective trends if modern acoustics is research of nonlinear processes caused with presence of mesoscale inhomogenuities and material defect structure. Presence of mesoscale inhomogenuities in solids leads to appearance of some new physical properties not presented in homogeneous solids. The example for that are such quantum phenomena as negative magnetoresistance, quantum galvanomagnetic effect, etc. The experiments conducted by number of authors have shown the defects of supramolecular structure of solids give rise to the so-called structure nonlinearity, which has local behavior and may exceed the physical nonlinearities due two lattice anharmonicity by two or three orders of magnitude [1-5]. However, there still is no universally accepted definition of the quantitative characteristics of structure nonlinearity, such as, e.g., the nonlinear acoustic parameter is for traveling waves [6, 7].

Experimental data allow estimating nonlinearity of only definite mediums or objects. Measurements of high local nonlinearity in the vicinity of a single crack are described in [8, 9]. In [9] estimation of spatial distribution of nonlinear parameters in thin cylindrical plate with defects has been made basing on measured Lamb wave magnitudes on one of the resonant frequencies and its harmonics (Figure 1).

Anomalously large nonlinear acoustic parameters were found in several local regions, including the artificial defect region. These results of acoustic measurements and the results of the X-ray diffraction analysis allow for the conclusion that the spatial distribution of the nonlinear parameter is identical to the distribution of defects in the tested sample. It is important to note that the spatial distribution of the nonlinear acoustic parameter makes it possible to obtain more significant information on defects in a tested material proceeding, than visualization of the shape of acoustic harmonic amplitudes. Nevertheless, the quantitative values of obtained nonlinear parameters are based on experimental data and only allow comparing the parameters value in the investigated sample and give no information if it is the same in case an object with the same defects would have another shape.

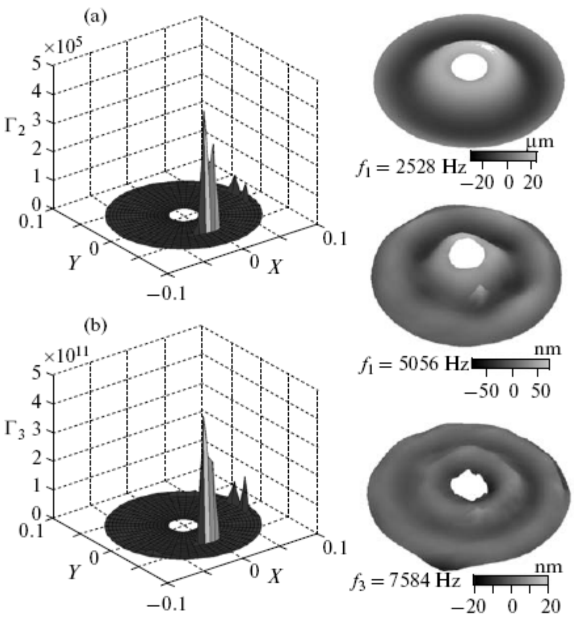

Figure 1. Spatial Distribution of the (a) quadratic and (b) cubic nonlinearity parameters calculated from the vibration's shape of the sample with defects for the fundamental frequency of 2.5 kHz [9].

Thus, numerous experiments only reveal the tendency and allow no quantitative comparison of the results. At the same time, understanding the meaning of the results of measurements may provide the basis for interpreting quantitative experimental data, which are primarily necessary for nonlinear acoustic diagnostics. Below, we state our views on how to solve this problem.

Descriptions of supramolecular objects characterized by nonlinear behavior can be found in many papers and reviews (see, e.g., [1-5, 10]). For definiteness, we consider a solid medium containing defects in the form of cracks. It is well known that the sharp edge of a crack is a region of high stress concentration [11]. If we assume that, away from the crack, the stress is τ , near the edge of the crack it is estimated as [12]

$$\tau_* = \tau(1+G), \qquad G = 2\sqrt{l/r_0} \,. \quad (1)$$

where $2l$ is the length of the crack and r0 is the curvature radius of its edge. For a crack with a length of 1 mm and a curvature radius of 1 nm (several interatomic distances), the amplification coefficient is $G \sim 10^3$. This means that relatively weak acoustic vibrations $\sigma(t)$ can be amplified in the vicinity of the crack to such an extent that nonlinear effects described by the function $\tau_*(t)$ (and primarily represented by higher harmonic generation) become noticeable. Since, with distance from the crack tip, the stress decreases from $\tau_*(t)$ to $\tau(t)$ corresponding to a homogeneous medium, i.e., the coefficient *G* decreases, the manifestations of nonlinear properties should also decrease. Hence, in the presence of a single crack, the quantitative characteristic of nonlinearity should depend on the coordinates. However, local measurements on the nanometric scale can only be performed on the surface by using tunnel or atomic-force microscopes. In the volume of the medium where the crack is located, such measurements present considerable difficulties. Therefore, it is necessary to understand what kind of macroscopic manifestations observable in ultrasonic experiments can be caused by the nonlinearity that is generated by nano- and micro-inhomogeneities of the medium.

Let the dependence of stress τ on strain *e* be described by the function

$$\frac{\tau}{\tau_0} = \Phi\left(\frac{e}{e_0}\right), \qquad (2)$$

which, in the region of small values of e/e_0 , has a linear segment corresponding to the Hooke law. For actual media, one usually has to take into account hereditary properties, relaxation, and hysteresis [1, 2], but, for simplicity, we assume that dependence (2) is algebraic.

The constants τ_0 and e_0 correspond to the characteristic values of stress and strain in dependence (2) and serve for normalization. From Eq. (2), we obtain

$$\frac{\tau_*}{\tau_0} = \frac{\tau}{\tau_0}(1+G) = \Phi\left(\frac{e_*}{e_0}\right),$$

$$\frac{e_*}{e_0} = \Phi^{-1}\left[\frac{\tau}{\tau_0}(1+G)\right].$$

Assuming that, away from the crack, the stress is small and lies on the linear part of curve (2), we arrive at the formula

$$\frac{e_*}{e_0} = \Phi^{-1}\left[\frac{E}{\tau_0}(1+G)\cdot e\right], \qquad (3)$$

Where *E* is Young's modulus. Formula (3) gives a nonlinear dependence of the strain e_* near the crack on e the strain away from the crack.

To perform calculations and to obtain some estimates, we specify dependence (2) by the function

$$\Phi(x) = \ln(1+x), \ \Phi^{-1}(x) = \exp(x) - 1,$$

which approximately describes the tendencies in the behavior of nonlinear properties. In this case, Eq. (3) takes the form

$$\frac{e_*}{e_0} = \exp\left[\frac{E}{\tau_0}(1+G)\cdot e\right] - 1, \qquad (4)$$

Let the strain caused by an acoustic wave away from the crack be a harmonic function of time:

$$e = A_1 \cos(\omega t).$$

Then, Eq. (4) can be expanded in harmonics:

$$\frac{e_*}{e_0} = I_0\left[\frac{E}{\tau_0}(1+G)\cdot A_1\right] - 1 + \\ + 2\sum_{n=1}^{\infty} I_n\left[\frac{E}{\tau_0}(1+G)\cdot A_1\right]\cos(n\omega t).$$

Here, I_n are modified Bessel functions. The second harmonic amplitude, which is most often measured in the experiments, is

$$\frac{A_2}{e_0} = 2I_2\left[\frac{E}{\tau_0}(1+G)\cdot A_1\right]. \quad (5)$$

For a weak nonlinearity, when the argument of the Bessel function in Eq. (5) is small, the following approximate formula is valid:

$$\frac{A_2}{e_0} = \left[\frac{E}{2\tau_0}(1+G)\cdot A_1\right]^2. \quad (6)$$

It is possible to determine the nonlinear parameter ε_0 of local nonlinearity by considering dependence (6) away from the crack, i.e., in the region where G=0. In this case,

$$\varepsilon_0 = \frac{4A_2}{A_1^2} = \frac{e_0}{\tau_0^2}E^2. \qquad (7)$$

The stress concentration coefficient depends on the coordinates. We assume that the axis of a cylindrical coordinate system coincides with the crack front line and ignore the dependence on the polar angle in our qualitative analysis. Then [12], we have

$$G = 2\sqrt{l/r}, \qquad (8)$$

where $r > r_0$ is the radial coordinate. Since the harmonic is measured by a sensor that has a finite size, the measurement data represent averaged quantities:

$$\frac{\overline{A_2}}{e_0} = \left(\frac{EA_1}{2\tau_0}\right)^2 \overline{(1+G)^2} . \qquad (9)$$

Let us calculate the average denoted by the overbar in Eq. (9) and assume that the averaging is performed over a certain volume with the characteristic size R:

$$\overline{(1+G)^2} = \frac{2}{R^2 - r_0^2} \int_{r_0}^{R} \left(1 + 2\sqrt{\frac{l}{r}}\right)^2 r\, dr . \quad (10)$$

Substituting Eq. (10) in Eq. (9), we obtain the expression for the nonlinear parameter defined by Eq. (7):

$$\varepsilon = \varepsilon_0 \left[1 + \frac{16}{3}\sqrt{l}\, \frac{R + \sqrt{Rr_0} + r_0}{(R + r_0)\left(\sqrt{R} + \sqrt{r_0}\right)} + \frac{8l}{R + r_0}\right] . (11)$$

One can see that the magnitude of nonlinearity depends on the size of the region of averaging that is performed in measuring the second harmonic amplitude. If this size is minimal, i.e., $R = r_0$, from Eq. (11) we obtain the maximal value

Away from the crack tip, at $R >> r_0$, the magnitude of the nonlinear parameter given by Eq. (11) decreases with increasing distance R and tends to ε_0 given by Eq. (7).

Another situation takes place when the sample contains many cracks and it is necessary to estimate their total contribution to the increase in the nonlinearity of the medium. Evidently, in this case, it is necessary to determine the second harmonic amplitude generated by all of the cracks in a unit volume of the medium. In other words, it is necessary to integrate Eq. (9) over a volume that is small compared to the wavelength and divide the integral by the value of this volume:

$$\frac{A_2}{e_0} = \left(\frac{EA_1}{2\tau_0}\right)^2 \frac{1}{V} \int (1+G)^2\, dV . \qquad (12)$$

When no cracks are present and *G=0*, from Eq. (12) we obtain previous expression (7) for the nonlinear parameter of the homogeneous medium. In the presence of an ensemble of cracks, we have

$$\varepsilon = \frac{4A_2}{A_1^2} = \varepsilon_0 \left(1 + n \int \left(2G + G^2\right) dv\right) . (13)$$

Here, *n* is the number of cracks per unit volume and the integration is performed over the spatial region where the stress is caused by a single crack. We assume that the number of cracks is not too large and that the cracks little affect each other.

The proposed approach encounters a difficulty related to the fact that dependence (8) of the stress concentration coefficient on the coordinates is valid only in the vicinity of the crack. Substitution of such dependences in Eq. (13) yields diverging integrals. Still, there is one case for which the aforementioned computational scheme can be successfully implemented. This case corresponds to spherical cavities. Evidently, the stress concentration near a spherical cavity is much smaller than the stress concentration near a crack with a sharp edge. However, the result obtained for the spheres illustrates the idea and may serve as the basis for qualitative generalizations.

Thus, our subsequent calculations are based on the solution to the problem of stresses arising in a solid that contains a spherical cavity (Figure 2). We assume that, away from the cavity, the stress is τ_{ik}^0, while the total stress in the presence of the cavity is τ_{ik}. The vector $\vec{n}$ is directed from the center of the sphere to the point of observation, r_0 is the radius of the sphere, and σ is the Poisson's ratio of the surrounding medium. The solution obtained to this problem by R. Southwell and J. Goodier is rather cumbersome (it is partially represented in [13, 14]):

$$\begin{aligned}
\tau_{ik} = &\left(\tau_{ik}^0 - \frac{1}{3}\tau_{ll}^0\delta_{ik}\right)\left[1 + \frac{5(1-2\sigma)}{7-5\sigma}\left(\frac{r_0}{r}\right)^3 + \frac{3}{7-5\sigma}\left(\frac{r_0}{r}\right)^5\right] + \\
&+\left(\tau_{il}^0 n_k n_l + \tau_{kl}^0 n_i n_l - \frac{2}{3}\tau_{ll}^0 n_i n_k\right)\frac{15}{7-5\sigma}\left(\frac{r_0}{r}\right)^3\left[\sigma - \left(\frac{r_0}{r}\right)^2\right] + \\
&+\left(\tau_{lm}^0 n_l n_m n_i n_k - \frac{1}{3}\tau_{ll}^0 n_i n_k\right)\frac{15}{2(7-5\sigma)}\left(\frac{r_0}{r}\right)^3\left[-5 + 7\left(\frac{r_0}{r}\right)^2\right] + \\
&+\left(\tau_{lm}^0 n_l n_m \delta_{ik} - \frac{1}{3}\tau_{ll}^0\delta_{ik}\right)\frac{15}{2(7-5\sigma)}\left(\frac{r_0}{r}\right)^3\left[1 - 2\sigma - \left(\frac{r_0}{r}\right)^2\right] + \\
&+\frac{1}{3}\tau_{ll}^0\left[\delta_{ik} + \frac{1}{2}\left(\frac{r_0}{r}\right)^3\left(\delta_{ik} - 3n_i n_k\right)\right].
\end{aligned}$$

(14).

Let the medium be subjected to uniform deformation, as is shown in Figure 2, and let only one strain tensor component τ_{zz}^0 be nonzero away from the cavity. Then, setting $i = k = z$, and $n_z = \cos\theta$, at the surface of the sphere $r = r_0$ we obtain

$$\tau_{zz} = \frac{15}{7-5\sigma}\tau_{zz}^0 \sin^2\theta\left[\sin^2\theta - \frac{1+5\sigma}{10}\right] . (15)$$

One can see that the stress at the sphere surface is zero along the direction of force application, i.e., along the z-axis, which corresponds to $\theta = 0$. At the same time, the stress is maximal in the transverse direction: at $\theta = \pi/2$, it is

$$\tau_{zz} = \frac{3(9-5\sigma)}{2(7-5\sigma)}\tau_{zz}^0 \qquad (16)$$

Formula (16) coincides with the result given in [14] (see the last formula of problem 12, item 7 of [14]). Stress distribution (15) in the azimuth angle is shown in Figure 2, where the higher stress region corresponds to the lighter area of the contour pattern. According to Eq. (16), for the spherical cavity, the stress amplification is relatively small; for example, for copper ($\sigma = 0.35$) $\tau_{zz} / \tau_{zz}^0 \approx 2.1$. This value is much smaller than that for a crack with a sharp edge (Eq. (1)).

For arbitrary distances from the center of the sphere, from solution (14), we obtain

$$\frac{\tau_{zz}}{\tau_{zz}^0} = 1 + G, \tag{17}$$

$$G = \frac{(4-5\sigma) + 3(11+5\sigma)\cos^2\theta - 75\cos^4\theta}{2(7-5\sigma)}\left(\frac{r_0}{r}\right)^3 + \frac{3\left(3 - 30\cos^2\theta + 35\cos^4\theta\right)}{2(7-5\sigma)}\left(\frac{r_0}{r}\right)^5. \tag{18}$$

Calculation of the volume integral

$$\int G dv = 2\pi \int_0^{\pi} \sin\theta \, d\theta \int_{r_0}^{\infty} G(r,\theta) r^2 \, dr \tag{19}$$

of the function *G* given by Eq. (18) yields zero. Thus, only the integral of type (19) of the function G^2 contributes to the increase in nonlinearity (13). Calculations yield the following result:

$$\frac{\varepsilon}{\varepsilon_0} = 1 + n\int G^2 d\nu = 1 + n\nu_0 \frac{807 - 730\sigma + 175\sigma^2}{35(7-5\sigma)^2} \tag{20}$$

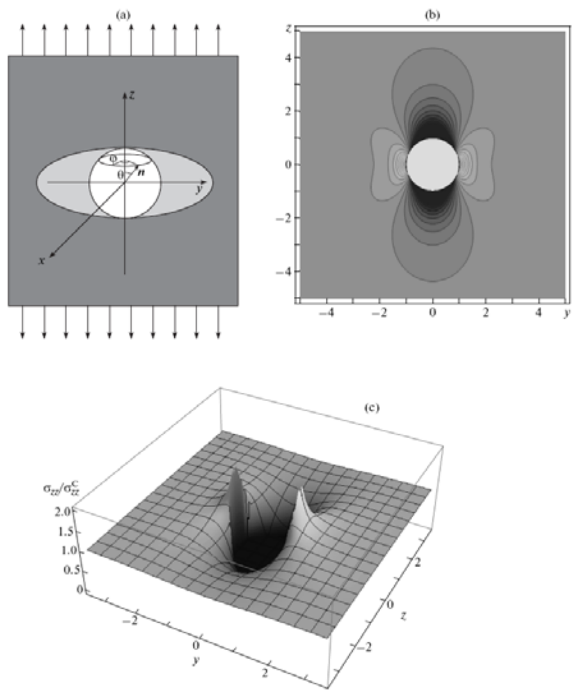

Figure 2. (a) Cavity and the related spherical coordinates. (b) The contour pattern of stress distribution in the YZ plane. The sampling interval is 0.074; lighter areas represent the distribution of increased stress at the sphere surface. (c) The 3D pattern of stress distribution in the YZ plane.

Here, $\nu_0 = 4\pi r_0^3 / 3$ is the volume of one spherical cavity and $n\nu_0$ is the portion occupied by spherical cavities in the volume of the medium. For example, for copper, the coefficient in the term proportional to $n\nu_0$ is about 0.6, which testifies to a small increase in the nonlinearity of the medium with spherical cavities, as compared to the homogeneous medium.

We also know another, less general, result of the theory of elasticity, which can be used to calculate the acoustic nonlinearity in the case of cracks with sharp edges. In [15], the problem of stress distribution around an infinitely thin disk-shaped crack of radius a was solved. In this problem, the origin of cylindrical coordinates is at the center of the disk. The z-axis is normal to the disk plane. The solution is obtained for the tensor component τ_{zz} and for the case when the stress away from the crack has the component τ_{zz}^0; i.e., the tensile force acts along the z-axis. Under tension, the crack opens and its axial section takes the form of an ellipse with its major axis in the direction of the r-coordinate. According to [15], for $z \geq 0$, the stress amplification coefficient G (see Eq. (17)) in the vicinity of this crack is expressed as

$$G = \frac{2}{\pi}\int_0^{\infty}\left(\cos x - \frac{\sin x}{x}\right)\left(1 + \frac{x}{a}z\right)\exp\left(-\frac{x}{a}z\right)J_0\left(\frac{x}{a}r\right)dx \tag{21}$$

Here, J_0 is the zero-order Bessel function.

Integral (21) can be calculated:

$$\frac{\pi}{2}G = \frac{\left(A^2-1\right)^{1/2} - \alpha\left(A^2-\alpha^2-\beta^2\right)^{1/2}}{\left[\alpha^2+(\beta+1)^2\right]^{1/2}\left[\alpha^2+(\beta-1)^2\right]^{1/2}} - \arcsin\frac{1}{A} + \frac{\alpha A^6}{\left(A^4-\beta^2\right)^{3/2}}\cos\left(3\arctan\frac{\sqrt{A^2-\beta^2}}{A\sqrt{A^2-1}}\right). \tag{22}$$

where

$$\alpha = \frac{z}{a}, \ \beta = \frac{r}{a}, \quad 2A = \left[\alpha^2 + (\beta+1)^2\right]^{1/2} + \left[\alpha^2 + (\beta-1)^2\right]^{1/2} \tag{23}$$

In the plane z=0 (for $|\beta| > 1$), in which the crack lies, from solution (22) we obtain the simple expression given in [14]:

$$\frac{\pi}{2}G = \left(\frac{r^2}{a^2} - 1\right)^{-1/2} - \arcsin\frac{a}{r}. \tag{24}$$

According to this formula, at the point $r = a, \ z = 0$, the stress goes to infinity. Setting $r = a + r_0$ (r_0 is the minimal curvature radius of the crack edge) in Eq. (24), we obtain the formula for the maximal stress amplification near the edge:

$$G = \frac{1}{\pi}\sqrt{\frac{2a}{r_0}}\,. \qquad (25)$$

This formula coincides with general formula (1) correct to notations and numerical factor.

The constant-level lines corresponding to constant values of stress amplification $\dfrac{\tau_{zz}}{\tau_{zz}^{0}} = 1 + G$ were calculated using solution (22)–(24); these lines are shown in Figure 3.

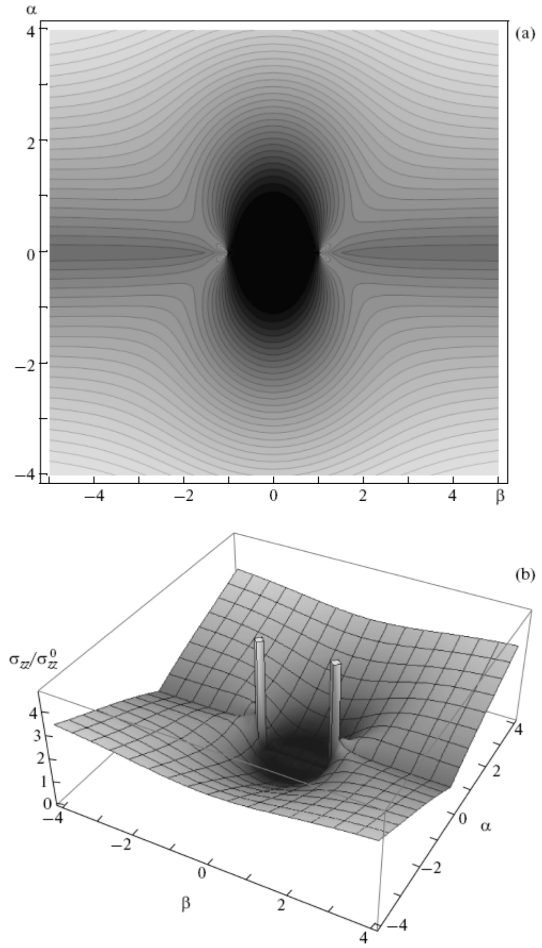

Figure 3. Stress amplification around an infinitely thin disk_shaped crack of radius . (a) The contour pattern of stress distribution in the YZ plane. The sampling interval is 0.086. (b) The 3D pattern of stress distribution in the YZ plane.

Now, let us calculate the acoustic nonlinearity by using Eq. (13). First, we calculate the volume integral of G using solution (21). We note that the integral over the radial cylindrical coordinate diverges at the upper limit (at infinity). Therefore, to calculate this integral, it is necessary to change to the spherical coordinate system, where the exponential factor in Eq. (21) eliminates the aforementioned divergence. Calculating the integral over the polar angle, we obtain that the volume integral of *G* is zero, as in the case of a spherical cavity (Eqs. (18), (19)). Hence, to calculate the acoustic nonlinearity, it is necessary to perform integration over the volume for the function G^2:

$$G^2 = \frac{4}{\pi^2}\int_0^\infty \frac{d}{dx_1}\left(\frac{\sin x_1}{x_1}\right)\frac{d}{dx_2}\left(\frac{\sin x_2}{x_2}\right)\left(1+\frac{x_1}{a}z\right)\left(1+\frac{x_2}{a}z\right)\times$$
$$\times\exp\left(-\frac{x_1+x_2}{a}z\right) J_0\left(\frac{r}{a}x_1\right) J_0\left(\frac{r}{a}x_2\right) x_1 x_2\, dx_1 dx_2. \qquad (26)$$

In Eq. (26), the integrand involves the product of two Bessel functions and, therefore, the change to the spherical coordinate system is not necessary. Using the formulas

$$\int_0^\infty J_0\left(\frac{r}{a}x_1\right) J_0\left(\frac{r}{a}x_2\right) r\,dr = a^2\,\frac{\delta(x_1 - x_2)}{x_1},$$
$$\int_0^\infty (x\cos x - \sin x)^2\,\frac{dx}{x^4} = \frac{\pi}{6},$$

we calculate the volume integral of Eq. (26) and obtain a simple expression for the effective nonlinear parameter:

$$\frac{\varepsilon}{\varepsilon_0} = 1 + n\int G^2\,dV = 1 + n\frac{10}{3}a^3. \qquad (27)$$

Comparing Eq. (27) with Eq. (20) derived earlier, we see that, in the presence of an ensemble of disk shaped cracks (the disks are parallel to each other), the amplification of nonlinearity does not depend on Poisson’s ratio and linear elastic moduli of the medium. In addition, expression (27) does not depend on the relative volume occupied by the cracks in the solid. Hence, in the case described by Eq. (27), the increase in nonlinearity in the presence of cracks can be greater than the nonlinearity increase in the presence of spherical cavities (i.e., in the case described byEq. (20)).

In closing, we note that solutions (14) and (21) to the problems of the theory of elasticity, which were used by us for calculating the nonlinear acoustic parameter of the medium, are in fact unique results obtained by the classics of mechanics. One can hardly expect that, in the future, it will be possible to obtain exact results for cracks of some other types. Therefore, the approach developed in this paper can be mainly used for approximate calculations to obtain qualitative estimates of the acoustic nonlinearity of solid media with defects.

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