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On the selection and implementation of sources for finite-difference methods

Jonathan Botts (1), Alex Bockman (1), Ning Xiang (1)

(1) Graduate Program in Architectural Acoustics, Rensselaer Polytechnic Institute, Troy, USA

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ABSTRACT

Finite-difference methods are becoming increasingly popular in the acoustics community, and the importance of higher-order methods has been acknowledged. However, the choice of an appropriate source for these methods has been largely overlooked in the acoustics literature. Published literature acknowledges the importance of selecting a continuous function, but it does not consider whether the function's derivatives are also continuous. The error resulting from discontinuous higher-order derivatives can contaminate a finite difference simulation with unnecessary, low-order, dispersive error, diminishing the order of accuracy of the overall scheme. This problem is discussed in the context of simple, wave equation solvers of various order with a variety of sources.

INTRODUCTION

For many acoustics problems, finite difference methods are often dismissed for being too computationally intensive; however, with increasing computing power, they are starting to gain tractability. These methods are advantageous for room acoustics in that they directly solve the wave equation or an equivalent linear system. Thus they are not limited by the high-frequency assumptions of statistical or geometrical acoustics. Non-linear formulations are also common, but this paper will focus on simple, wave equations solvers.

In order to extend the utility of these methods, many have explored techniques to increase stability and accuracy, using implicit (Tam and Webb 1992, Zheng et al. 1999) or higher-order schemes (Georgakopoulos et al. 2002, Sakamoto 2007). In order to maintain numerical stability, limits are put on the discretization of the problem, diminishing efficiency. Many implicit methods are unconditionally stable, removing this restriction, but they must still overcome the issue of accuracy. Higher-order methods afford coarser discretization while maintaining accuracy, which generally improves efficiency.

While a variety of approaches have the potential to enhance these methods, careful implementation is required. Many small details have the capacity to contaminate the performance of an entire scheme, and one often-overlooked detail is the implementation of acoustic sources. Especially in the applied finite-difference literature, improper source implementation reduces the potential accuracy of the scheme and could easily be remedied. Attention has been given to the directivity of sources (Escolano et al. 2009) or the scattering properties of sources (Schneider et al. 1997), but not to their smoothness and influence on accuracy. This paper will demonstrate that acoustic source implementations introduce error as a result of insufficient smoothness.

SIGNAL PROBLEM

To discuss the issue of source implementation, consider the one-dimensional, linear wave equation

$$u_{tt} = c^2 u_{xx}, \quad 0 \le x \le l, \quad t \ge 0, \tag{1}$$

where the subscripts represent partial derivatives, and c is the speed of sound. The solution $u(x,t)$ could represent pressure, particle velocity, or some other quantity that is described by the wave equation. The problem is a simplified test problem to simulate source signal propagation into a domain, perhaps a waveguide. The solution, or sound field, is initially at rest, $u(x,0) = u_t(x,0) = 0$, and we prescribe Dirichlet boundary conditions,

$$u(l,t) = 0, \quad u(0,t) = s(t), \tag{2}$$

where $s(t)$ is the source function used to excite the domain. It could be located at any grid point in the mesh, but again, for simplicity, it is located on the left boundary. It should be noted that several authors use an initial spatial distribution to excite the mesh. For example, Tam (1995), Fung et al. (1996), and Sakamoto (2007) utilize an initial spatial Gaussian distribution to excite the mesh. This is equivalent to driving it with some time function at a given point or set of points. Similar results will apply since identical issues arise with both spatial and temporal differencing.

SCHEMES

Since the focus of the paper is source implementation, simple, second and fourth-order, explicit difference schemes are used. Let the grid function be $v_j^n = v(x_j, t_n) = v(j\Delta x, n\Delta t)$ where $j = 0, 1, \ldots, N$, $n = 0, 1, 2, \ldots$, and Δx and Δt are the spatial and temporal discretization steps respectively. The discretization will be uniform for the purposes of this paper. The second-order difference scheme in time-marching form is

$$v_j^{n+1} = 2v_j^n - v_j^{n-1} + \sigma^2 \left(v_{j+1}^n - 2v_j^n + v_{j-1}^n \right), \tag{3}$$

where $\sigma = c\Delta t/\Delta x$. This scheme is well-known and can be shown to be neutrally stable for $\sigma \le 1$. The truncation error can also be shown to be $\mathcal{O}\left(\Delta x^2, \Delta t^2\right)$. Since it is of even order, the scheme is dispersive, and the error will evidence itself as ripples behind and in front of the wavefronts.

For the fourth-order scheme, a correction term is added that eliminates the second-order terms in the truncation error. In time marching form, the scheme is

$$v_j^{n+1} = 2v_j^n - v_j^{n-1} + \sigma^2 w_n^n + \frac{\sigma^2}{12}\left(\sigma^2 - 1\right)\left(w_{j+1}^n - 2w_j^n + w_{j-1}^n\right),$$

where $w_j^n = v_{j+1}^n - 2v_j^n + v_{j-1}^n$. Proving stability and consistency is straightforward; by using the wave equation to exchange temporal derivatives for spatial derivatives, truncation error can be shown to be $\mathcal{O}\left(\Delta x^4\right)$. Since these are one-dimensional schemes, when $\sigma = 1$, the numerical solution is exact to machine precision. As with all finite difference schemes, the initial and boundary conditions must be implemented to the desired accuracy, and in this case, it suffices to let all initial data be zero. The only other issue to worry about is the accuracy of the source implementation. If the source is sufficiently smooth, the global error should be either second or fourth order, depending on the scheme.

SOURCES

The types of sources discussed in this paper are those that approximate an impulse. These are used because they are temporally compact and have low-pass frequency characteristics, which is a requirement of the differencing. High frequency content in a signal is generally transformed into dispersive error, which may contaminate the calculations. A pulse is the most efficient way to introduce the entire usable frequency content to the problem domain. Extended signals could be used, but they only exacerbate the issue of computational efficiency.

The two common source types that are typically reported in the literature are based on sinusoidal and exponential functions. Since they are to be impulse-like, only one half of the sinusoid is used. The sinusoidal source is of the form

$$s_s(t) = \sin^p(2\pi kct), \tag{4}$$

where $s(t) = 0$ after one half-period, and p and q determine the frequency range and smoothness of the source. The tempered gaussian sources are of the form

$$s_e(t) = t^p \exp\left(\frac{-t^q}{2\alpha^2}\right), \tag{5}$$

where α is related to the width of the pulse and thus the frequency content of the signal. Each source is assumed to be turned on at $t = 0$ for simplicity. This exponential source is the most common type of source reported in the literature. In Botteldooren (1994; 1995), and Bayliss et al. (1986), the source is roughly in the form of this exponential source with $p = 1$ and $q = 2$. This is proportional to the first derivative of a Gaussian distribution. In order to model explosion pulse propagation, Liu and Albert (2006) uses a similar source with $p = 2$ and $q = 1$. Sakamoto (2007) shows that the initial Gaussian distribution results in an equivalent exponential source with $p = 1$ and $q = 2$.

Clearly the sources start at zero to ensure continuity with the onset and offset conditions, but the derivatives of $s(t)$ are non-zero and thus discontinuous especially for low values of p and q. Usually, the highest value of p or q reported for these types of sources is 2.

The important observation to make is that the higher-order derivatives of these functions are non-zero when the source is switched on, disrupting smoothness. Consider the sinusoidal source with $p = 2$ as an example. Similar results can be developed for the exponential sources. The first few derivatives are

$$s'(t)|_{t=0} = 4\pi kc\cos(2\pi ct)\sin(2\pi kct)|_{t=0} = 0 \tag{6}$$

$$s''(t)|_{t=0} = 2(2\pi kc)^2\cos(4\pi kct)|_{t=0} = 2(2\pi kc)^2 \tag{7}$$

The odd derivatives will be zero, but the even derivatives will contribute discontinuities.

$$s^{(4)}(t)|_{t=0} = -8(2\pi kc)^4\cos(4\pi kct)|_{t=0} = -8(2k\pi c)^4 \tag{8}$$

In order for these errors to be neglected, they must enter below the order of the truncation error introduced by the differencing. This implies that higher-order difference methods require smoother source functions. Upon closer examination, many papers that report higher-order methods use source functions that likely introduce low-order error. The higher-order scheme will then propagate the low-order error with greater accuracy. As will be shown below, for an exponential source, $p = 1$, $q = 2$ is insufficient to match the error of a fourth order scheme, yet this type of source is reported in several references.(Bayliss et al. 1986, Fung et al. 1996, Sakamoto 2007)

Observing that high-frequency signal content is a large contributor to contaminating dispersion, one might consider a source whose frequency characteristics approach a unit step function, the ideal low-pass filter. The response of a high-order, low-pass filter, convolved with a unit impulse is another source candidate. Unlike the other sources, these are not contained to single pulses, but the frequency content drops off very quickly above a certain target frequency. These sources do provide very specific frequency content, but they do not serve to substantially reduce the overall error of the scheme. Changing the order of the filter does not necessarily change the smoothness of the time-signal, so the error is not considerably improved as in the case of the other sources.

This set of source types is certainly not complete, but it gives insight into the issues involved with source implementation for finite-difference schemes. Just as with finite-difference schemes themselves, the most appropriate choice will depend on the specific problem. The performance of each source type is discussed in the context of the signal problem described above.

NUMERICAL EXPERIMENTS

Description

First, the signal problem is used to show the properties of sources within each class. We consider a simplified acoustics problem, with parameters $l = 50$ m, $c = 343$ m/s, and $\Delta x = 50$ m / 250 pts $= 0.2$ m. The calculation is allowed to run until $t = 0.13$ s. For the sinusoidal and exponential sources, the exact solution is known and compared to the numerical solution. The maximum error is calculated and compared to the size of Δx since the truncation error for both schemes is expected to behave like powers of Δx. Another measure of accuracy can be obtained by refining the mesh and repeating the calculation. The rate that the error decreases relative to the mesh refinement, gives an indication of how the error is behaving with respect to Δx. Thus, the two measures of accuracy for each scheme are the magnitude of the error in the infinity-norm and the rate of error reduction with respect to mesh refinement.

The first set of calculations utilizes the second order scheme with sinusoidal sources. The exponent, p, ranges from 1 to 8, and calculations are done for both $\sigma = 0.5$ and $\sigma = 0.99$. The size of σ directly influences the accuracy of the scheme, but it does not affect the rate of decrease of error with respect to mesh refinement. As σ approaches 1, the numerical solution approaches the exact solution, and when $\sigma = 1$, the error is on the order of machine epsilon. Letting σ approach 1 lets the error continually decrease, despite the accuracy of the differencing.

For these calculations, the parameter, k, affecting the width of the pulse is adjusted so that the source has similar frequency characteristics over all of the trials. It is chosen such that the point where the frequency magnitude spectrum drops below $10\log_{10}\left(\Delta x^2\right)$ at 200 Hz. The particular threshold and frequency range are relatively unimportant, but they serve to standardize the problem. The 200 Hz wavelength should be well-resolved on the grid introducing no unnecessary error.

Next, similar calculations are done with the exponential sources. Since these sources depend on one more parameter, many more combinations are possible. For brevity, one parameter is kept constant while the other is varied. In this case, p is 2 while q ranges from 1 to 32 logarithmically. Similar results are achieved while holding q constant and varying p.. The third parameter, α is adjusted exactly as k in the previous problem in order to standardize the problem.

Finally, calculations are done with the fourth-order scheme and the same sinusoidal sources used in the first problem. The source function is implemented appropriately to fourth-order accuracy on the boundary, assigning the necessary values to the secondary grid function w_0^n. Results from these three problem sets are reported in the following section.

Results

The results from the first set of calculations are shown in Figure 1. Calculations are performed twice using $\sigma = 0.5$ and $\sigma = 0.99$. The rate of error reduction is calculated using $N = 250$ points and $2N = 500$ points, denoted $(N : 2N)$. The order of the error relative to Δx is determined from the maximum error on the grid when the calculation is complete.

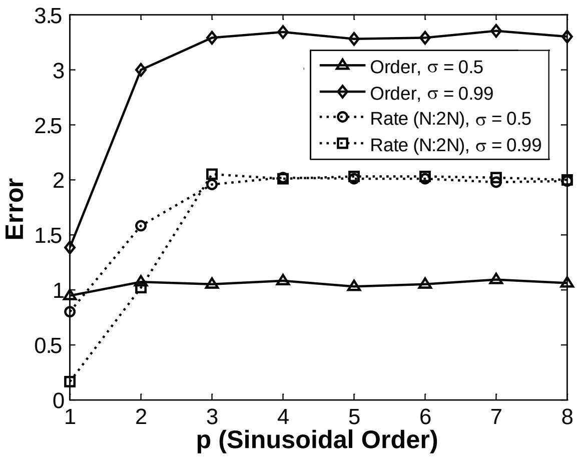

Figure 1: Maximum order of error and rate of error reduction with mesh refinement, both relative to Δx, for calculations using $\sigma = 0.5$, $\sigma = 0.99$, and sinusoidal sources.

The data in Figure 2 are produced using the second-order difference scheme, with $\sigma = 0.99$, and an exponential source. The parameter, p, is fixed at 2, and q is varied. With the initial number of points, $N = 250$, the error rates are calculated using N, $2N$, and $4N$.

Figure 3 shows the data from calculations with the fourth-order difference scheme, sinusoidal sources, and $\sigma = 0.99$. The rates are calculated as described above.

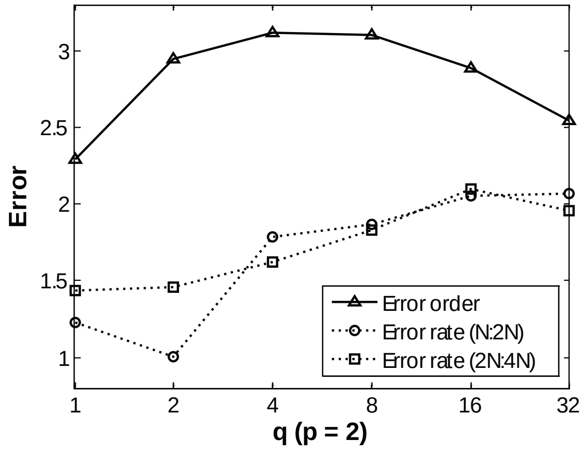

Figure 2: Maximum order of error and rate of error reduction with mesh refinement, both relative to Δx, for calculations using $\sigma = 0.99$ and exponential sources.

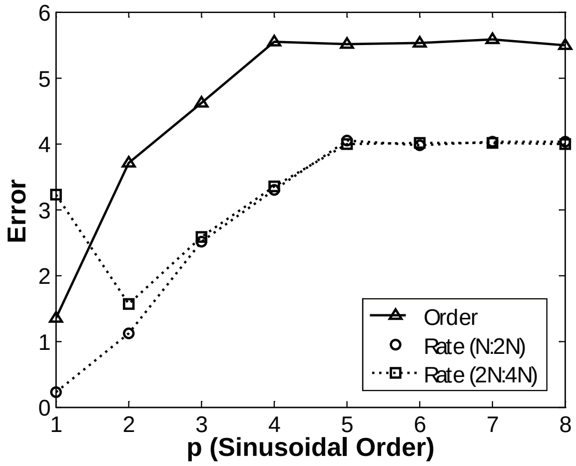

Figure 3: Maximum order of error and rate of error reduction with mesh refinement, both relative to Δx, for calculations using $\sigma = 0.99$, sinusoidal sources, and the fourth-order difference scheme.

DISCUSSION

Numerical Experiments

Figures 1 and 3 illustrate several points very clearly. It is apparent that the largest factor determining the maximum error of the calculation is the choice of σ. In both cases, when $\sigma = 0.99$, the net order of error is considerably higher than when $\sigma = 0.5$. With $\sigma = 0.99$, the error is higher than what is indicated by the truncation error, but this is only because as σ approaches 1 the error approaches machine epsilon. This overall error can be made arbitrarily small within the bounds of machine precision by adjusting σ. It should be noted that this is true only for the special one-dimensional case where the characteristics may exactly pass through the grid points.

The error rates, and not the maximum error, reflect the behavior of the truncation error when other extraneous error does not interfere. In this case the extraneous error is coming from insufficient smoothness in the source. Consider once again the example of the sinusoidal source with $p = 2$. The error from a discontinuous second derivative dominates the error produced only by the differencing, so when the mesh is refined, the error does not behave like Δx^2; it is dominated by the lower-order error from the source. This indicates that the error introduced into the scheme by the source is unnecessarily high; a proper source should not introduce further error.

Using a sinusoidal source and $p = 3$, the second derivative is now zero, and the lower-order error no longer dominates. The error rate is then governed by the error of the differencing, not the source. For the sinusoidal sources, if $p = n$, the n^{th} derivative will be non-zero at $t = 0$. This means that for a m^{th}-order scheme, in order for the error to be governed by the differencing, p must be greater than $m + 1$.

This is supported by the data in Figures 1 and 3. For the second order scheme, the error rate for $p < 3$ is less than 2, but for $p \geq 3$ it is almost exactly 2. Similarly for the fourth-order scheme, for $p \geq 5$ the error rate is effectively four, the order of the differencing.

Exponential sources have more parameters and consequently their behavior is slightly more complicated. The data Figure 2 do not exhibit the same straightforward behavior as the sinusoidal sources where after a certain order the error levels off to some maximum. It is apparent, however, that the error introduced for small values of q substantially influences the accuracy of the scheme. These data suggest that significant improvements in accuracy can be achieved by selecting appropriate q.

These sources are typical examples used to illustrate the importance of source selection. The computational resources needed to implement any of these sources in lieu of another is negligible, so careful consideration should be exercised. The following section provides description of other properties of potential sources that may also influence selection of sources.

Other source properties

The first property of these sources to see is the time response. Figure 4 shows the three types of sources mentioned above: sinusoidal, exponential, and filter impulse-response. All sources are designed to have roughly the same frequency content. The Gaussian and sinusoidal sources are pulse-like while the filter response exhibits some ringing. Though nearly all of the frequency content of the filter source is well-resolved on the grid, the smoothness of its time response is no greater than that of low-order sinusoidal or Gaussian sources. It is also slightly more difficult to implement which will often make it less useful than the others.

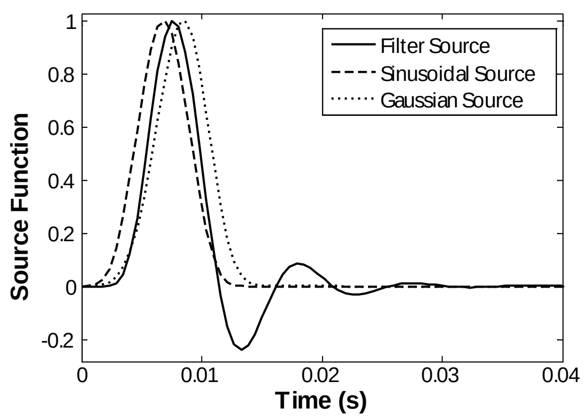

Figure 4: Three typical source functions of the types described in this paper.

The error indicators above do not necessarily give the complete picture of error introduced by sources. It is helpful to visualize the error at each grid point in space. Figure 5 shows typical error distributions for a sinusoidal source with high and low p. For high p, the maximum error is lower, as is the error everywhere in the domain; much of it is approaching zero, or machine epsilon. The acoustic analogue is increasing the signal to noise ratio of the problem by reducing the noise floor. High-order Gaussian and sinusoidal sources exhibit this behavior, but the filter sources generally do not. The filter source error behaves much like low-order sinusoidal or Gaussian sources.

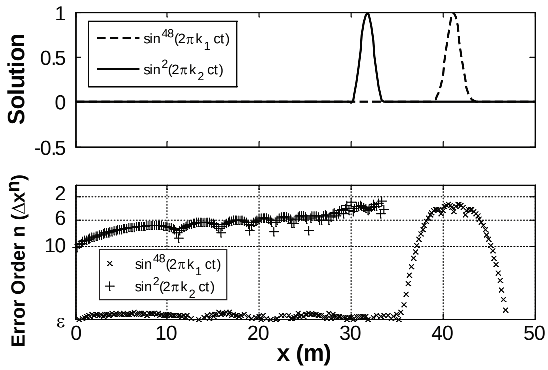

Figure 5: Spatial error distributions for a high and low-order sinusoidal source.

All of these properties can influence the performance of a source, given a specific problem. In addition to maximum error and error rate, these other more qualitative properties should also be kept in mind.

CONCLUDING REMARKS

In acoustics finite-difference problems, source choice as a source of error is often overlooked. Improper implementation has been shown to substantially reduce the accuracy of a scheme. A simple, one-dimensional problem provided illustration of common source error types. Future work should seek to identify source types that introduce minimal error.

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