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Cylindrical harmonic expansion of the sound field due to a rotating line source.

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ABSTRACT

The motion of an acoustic source relative to some fixed frame produces a Doppler shifting of the source frequency at a fixed point relative to that frame. For linear motion of the source greater than the speed of sound, the radiated sound forms a shock wave whose angle relative to the direction of motion varies with source speed. Some applications in acoustics involve a sound source rotating around a fixed point in space. For example, in surround sound systems, it may be desirable to generate the sound due to a sound source which moves around the listener. As another example, the Leslie speaker is a rotating loudspeaker system designed to produce amplitude and frequency modulation effects. In aeroacoustics, the noise produced by rotating propellers or rotors is of interest and the linear wave equation solution for a rotating source has some relevance. The description of rotating sources also has applicability in other disciplines such as electromagnetism and astronomy. This paper develops a cylindrical harmonic expansion for the sound field produced by a rotating line source. The expansion has a simple form and reverts to the standard expression for a fixed line source when the rotation speed is zero. For rotational speeds where the source is supersonic, the sound field produced by the expansion produces features similar to those demonstrated for rotating supersonic point sources, such as a Mach cone emanating from the source position, a spiral cylinder within which the field produces a spiralling pattern, and an inner cusp where the circular wavefronts converge. The expansion is implemented in matlab using a truncated form of the expansion, and examples of sound fields are given for both subsonic and supersonic cases.

INTRODUCTION

The motion of an acoustic source produces a Doppler shifting of the source frequency which is dependent on the source's motion relative to a listener [1]. The sound field produced by the source can be visualised in discrete time as a sequence of pulses radiating spherically from the source weighted by the signal amplitude at that time [2]. For source velocities greater than the speed of sound, the radiated sound forms a shock wave whose angle from the direction of motion varies with source speed.

Some applications in acoustics involve a sound source rotating around a fixed point in space. For example, in surround sound systems, it may be desirable to generate the sound due to a sound source which moves around the listener. As another example, the Leslie speaker is a rotating loudspeaker system designed to produce amplitude and frequency modulation effects [3]. In aeroacoustics, the noise produced by rotating propellers or rotors is of interest and the effects of turbulence and aerodynamics must be taken into account [4-7]. The equations describing the sound field are nonlinear. However, linear approximations have some relevance and the sound field due to a rotating point source (the spiral Green function) is therefore of interest [5].

We consider here a description of a rotating sound source without the turbulence or aerodynamic effects caused by the high speed motion of a physical source. In this case the linear wave equation is sufficient to describe the sound field. For simplicity of analysis we consider a line source emitting a single frequency. The source is assumed to be parallel to the vertical z-axis and to rotate about it at a radius r_s and at a rotation frequency f_R . The corresponding angular velocity is ω_R and the Mach number is $\mathfrak{M} = r_s\omega_R / c = k_R r_s$ where k_R is the rotational wavenumber

The sound field produced by this rotating source can be derived by first approximating the sound field using a number of stationary line sources using the principles of surround sound systems. To represent a moving source the stationary line sources must have time-varying amplitudes which produces a sound field with a multi-line spectrum. Letting the number of stationary sources tend to infinity then produces the exact representation.

THEORY

The interior sound field in a region of space can be represent in cylindrical coordinates as [8]

$$p(r,\phi,z,\omega) = \sum_{m=-\infty}^{\infty} e^{im\phi} \int_{-\infty}^{\infty} C_m(k_z,\omega) J_m(k_r r) e^{ik_z z} dk_z \tag{1}$$

Where $J_m(k_r r)$ is the cylindrical Bessel function, k_r is the radial component of the wavenumber k, $k_z = \sqrt{k^2 - k_r^2}$ and $C_m(k_z, \omega)$ the *mth* expansion coefficient.

For the case where the sound field is independent of z the expansion simplifies to

$$p(r,\phi,\omega) = \sum_{m=-\infty}^{\infty} C_m(\omega) J_m(kr) e^{im\phi} \tag{2}$$

and the sound field is represented as a sum of "phase modes" $\exp(im\phi)$ with coefficients $C_m(\omega)$ which are independent of z.

Similarly, the z-independent sound field exterior to a region containing sources has the expansion

$$p(r,\phi,\omega) = \sum_{m=-\infty}^{\infty} D_m(\omega) H_m^{(1)}(kr) e^{im\phi} \tag{3}$$

where $H_m^{(1)}(kr)$ is the cylindrical Hankel function of the first kind.

The sound field produced by a line source radiating a single complex negative frequency ω_0 is

$$\begin{aligned} p_{line}(r,\phi,\omega_0,t) &= e^{-i\omega_0 t} H_0^{(1)}\left(k\left|\vec{r}-\vec{r}_s\right|\right) \\ &= e^{-i\omega_0 t} \begin{cases} \sum_{m=-M}^{M} J_m(kr) H_m^{(1)}(kr_s) e^{im(\phi-\phi_s)}, & r < r_s \\ \sum_{m=-M}^{M} J_m(kr_s) H_m^{(1)}(kr) e^{im(\phi-\phi_s)}, & r > r_s \end{cases} \end{aligned} \tag{4}$$

The summation is limited to the finite range $[-M, M]$ for a finite value of r because the Bessel functions $J_m(kr)$ are only significant over a finite range of argument. For the interior expansion, at radius $r < r_s$ the required order of the summation is $M \approx kr$ and for the exterior expansion the required order is $M \approx kr_s$ [9].

For a positive frequency, the sound field has a form obtained from the conjugate of Eq. (4)

$$\begin{aligned} p_{line}(r,\phi,\omega_0,t) &= e^{i\omega_0 t} H_0^{(2)}\left(k\left|\vec{r}-\vec{r}_s\right|\right) \\ &= e^{i\omega_0 t} \begin{cases} \sum_{m=-N}^{N} J_m(kr) H_m^{(2)}(kr_s) e^{im(\phi-\phi_s)}, & r < r_s \\ \sum_{m=-N}^{N} J_m(kr_s) H_m^{(2)}(kr) e^{im(\phi-\phi_s)}, & r > r_s \end{cases} \end{aligned} \tag{5}$$

where $H_m^{(2)}(kr)$ is the cylindrical Hankel function of the second kind.

Consider first the approximation of the interior sound field of a stationary line source at (r_s, ϕ_s) as described by Eq. (4) using a fixed array of line sources at radius r_L and angles $\phi_l = 2\pi l / L$. Since the truncated modal expansion has $2M+1$ terms the number of sources must exceed $2M+1$ to allow accurate reproduction of the modes. We will assume $L = 2M+1$ in what follows.

The required line source amplitude weights are determined by requiring that the weighted sum of line source sound fields equals that of the desired source. For the interior field, using Eq. (4), this implies

$$\begin{aligned} &\sum_{l=0}^{L-1} w_l(\phi_s) \sum_{m=-M}^{M} J_m(kr) H_m^{(1)}(kr_L) e^{im(\phi-\phi_l)} \\ &= \sum_{m=-M}^{M} J_m(kr) H_m^{(1)}(kr_s) e^{im(\phi-\phi_s)} \end{aligned} \tag{6}$$

If the virtual source is confined to $r_s = r_L$ the Hankel function terms cancel and Eq. 6 implies, for each m, that

$$\sum_{l=0}^{L-1} w_l(\phi_s) e^{-im\phi_l} = e^{-im\phi_s} \tag{7}$$

For L odd and for $r_L = r_s$, the solutions to this set of equations are the angular sinc interpolation ("panning") functions [10]

$$w_l(\phi_s) = \frac{1}{2M+1} \sum_{n=-M}^{M} e^{in(\phi_l-\phi_s)} = \frac{\sin\left[L(\phi_l-\phi_s)/2\right]}{L\sin\left[(\phi_l-\phi_s)/2\right]} \tag{8}$$

A similar result applies for even L, which we ignore here for simplicity [10]. The same interpolation functions apply to the reproduction of the exterior field.

If we want the sound source to rotate at a constant rate around the listener with rotation frequency f_R, then the source position ϕ_s becomes a linear function of time, $\phi_s(t) = 2\pi f_R t = \omega_R t$ and therefore the panning functions in Eq. (8) become periodic functions of time. This means that each loudspeaker signal in Eq. 6 at single frequency ω_0 (Eq. 4) is modulated by $2M+1$ frequencies $n\omega_R$ for $n \in [-M, M]$ with phases $n\phi_l$, and therefore each loudspeaker radiates $2M+1$ frequencies $\omega_n = \omega_0 + n\omega_R$, each with its corresponding spatial frequency

$$k_n = k_0 + nk_R \tag{9}$$

For $n < -\lceil k_0 / k_R \rceil$ the spatial frequency k_n becomes negative, and so the sound field produced by the L loudspeakers becomes

$$\begin{aligned} &p(r,\phi,\omega_0,\omega_R,t) \\ &= e^{-i\omega_0 t} \sum_{l=0}^{L-1} \sum_{n=-M}^{M} e^{in(\phi_l-\omega_R t)} \sum_{m=-\infty}^{\infty} J_m(|k_n| r) H_m^{(\delta_n)}(|k_n| r_s) e^{im(\phi-\phi_l)} \end{aligned} \tag{10}$$

where

$$\delta_n = \begin{cases} 1, k_n > 0 \\ 2, k_n < 0 \end{cases} \tag{11}$$

Rearranging, and noting that

$$\frac{1}{L}\sum_{l=0}^{L-1} e^{i(n-m)\phi_l} = \begin{cases} 1, n-m = qL \\ 0, \text{ otherwise} \end{cases} \tag{12}$$

for integers q, yields

$$p(r,\phi,\omega_0,\omega_R,t) = \sum_{n=-M}^{M}\sum_{q=-\infty}^{\infty} e^{-i(\omega_0+n\omega_R)t} e^{i(n-qL)\phi} J_{n-qL}(|k_n|r) H_{n-qL}^{(\delta)}(|k_n|r_s) \tag{13}$$

for $r < r_s$. If we now let the number of loudspeakers L, and hence M, tend to infinity, then the terms for $q \neq 0$ tend to zero, yielding the continuous result

$$p(r,\phi,\omega_0,\omega_R,t) = \sum_{m=-\infty}^{\infty} J_m(|k_m|r) H_m^{(\delta_m)}(|k_m|r_s) e^{im\phi} e^{-i(\omega_0+m\omega_R)t} \tag{14}$$

for $r < r_s$. For $r > r_s$ a similar analysis yields

$$p(r,\phi,\omega_0,\omega_R,t) = \sum_{m=-\infty}^{\infty} J_m(|k_m|r_s) H_m^{(\delta_m)}(|k_m|r) e^{im\phi} e^{-i(\omega_0+m\omega_R)t} \tag{15}$$

These expansions are generalisations of the cylindrical harmonic addition theorem for a stationary line source and Eq.s (14) and (15) equal Eq. (4) for $\omega_R = 0$.

As for the stationary case, the summation limits can be derived from the properties of the Bessel function. However in the rotating source case, the argument of each Bessel function varies with mode number m. We now derive approximate limits for the index of $J_m(kr)$ assuming $m \approx kr_S$, for the subsonic and supersonics cases [9].

Subsonic source

Since each term in the exterior expansion of the rotating source has the term $J_m(|k_0 + mk_R|r_s)$, the negative limit for the expansion, m_1, will be different to the positive limit m_2. The negative limit m_1 is obtained from the solution to $(k_0 + m_1 k_R) r_s = -m_1$, or

$$m_1 = -k_0 r_s / (1 + k_R r_s) = -k_0 r_s / (1 + \mathfrak{M}) \tag{16}$$

Note that the spatial frequency k_m is zero for $m_0 \approx -[k_0/k_R] < m_1$ and so negative frequencies do not contribute significantly to the sound field in the subsonic case.

The upper mode limit is obtained from the solution to $(k_0 + m_2 k_R) r_s = m_2$ or

$$m_2 = k_0 r_s / (1 - k_R r_s) = k_0 r_s / (1 - \mathfrak{M}),\ \mathfrak{M} < 1 \tag{17}$$

Supersonic source

For speeds above Mach 1, $\mathfrak{M} = k_R r_s > 1$ and the slope of the line $(k_0 + mk_R) r_s$ exceeds 1. There is no upper limit to the summation and so $m_2 = \infty$. For negative mode indices the solution to $-(k_0 + m_1 k_R) r_s = -m_1$ yields an additional intersection

$$m_3 = -k_0 r_s / (k_R r_s - 1) = -k_0 r_s / (\mathfrak{M} - 1) \tag{18}$$

where $m_3 < m_1$. Between m_3 and m_1 the mode magnitudes are smaller, but they are significant outside this range and the expansion order is unbounded. In practice the mode magnitudes are a decreasing function of $|m|$ and so a finite summation over a symmetrical range [$-M$, M] can be used with reasonable results provided M is large enough.

SIMULATIONS

The expansion in Eq.s (14) and (15) were simulated in Matlab. For reference, the mode magnitudes in dB versus mode number and $k_0 r$ are shown in Fig. (1) for a stationary 400 Hz source (Mach 0) at a radius of r_s=1 metre. The mode distribution is symmetric, as expected from Eq. (4).

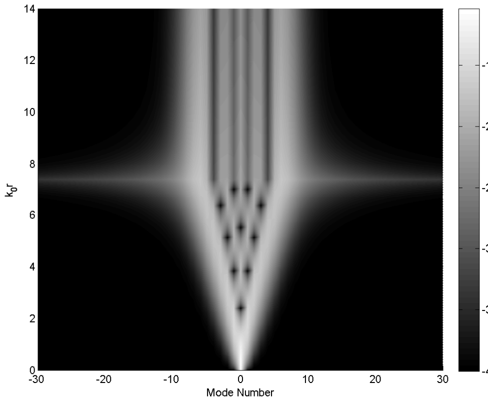

Figure 1: Mode magnitudes for a stationary sound source (Mach 0) with spatial frequency k_0=7.4.

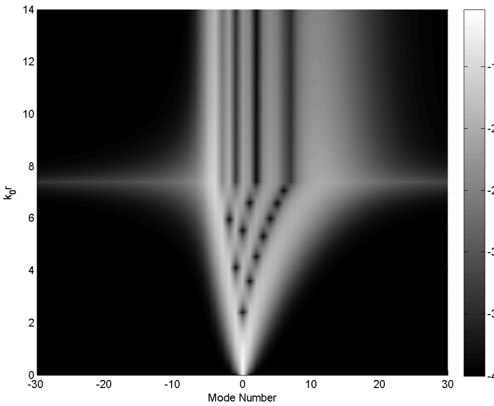

Figure 2: Mode magnitudes for a sound source rotating at Mach 0.5 with spatial frequency k_0=7.4.

Fig. (2) shows the mode magnitudes for the source rotating at Mach 0.5, (a rotational frequency of 27 Hz, with $k_0 r_s$ = 7.4). The distribution is now skewed towards positive mode indices as expected. The mode range calculated from Eq.s (16) and (17) was –5 to 15, but the modes are calculated over [–30,30] to allow the mode magnitudes over a wider range to be seen. For distances far from the source radius, the modes rapidly attenuate below the lower mode limit of –5 and are small above $m = 15$. In practise a higher limit than 15 would further reduce the truncation error. At radii close to the source radius ($k_0 r = 7.4$), the mode bandwidth tends to becomes large due to the discontinuity produced by the source and the field produced with a finite summation limit will be less accurate.

The mode magnitudes for Mach 1.5 (rotational frequency 81 Hz) are shown in Fig. (3) for a mode range of [–100,100]. At small radii the interior expansion requires a low asymmetric order, but at radii approaching r_s, the interior mode bandwidth again becomes large. However, the modal bandwidth also increases for $r < r_s$. This occurs because a cusp forms in the sound field at a radius $r_c = r_s / \mathfrak{M}$, ($k_0 r_c = 4.9$) which requires a high modal bandwidth to represent it [5]. For radii greater than r_s, the mode bandwidth is also large because the sound field displays a cylindrical shockwave that radiates outwards in a spiral, creating a strong discontinuity in the field as a function of angle.

The negative mode limits calculated from Eq. (18) and Eq. (16) were $m_3 = -15$ and $m_1 = -3$, which defines a region within which the mode magnitudes are reduced. The wave number k_m is approximately zero for $m_0 \approx -5$ and the mode magnitude is a minimum here, showing that zero frequency waves do not contribute significantly to the field. However, for negative wavenumbers m nearing m_3 the mode magnitudes increase again. Hence, a large symmetrical range of at least [–100,100] is required to accurately represent the field.

The exterior mode spectrum has a large bandwidth, and is relatively insensitive to the field radius, since the shock wave is a similar function of angle at all exterior radii.

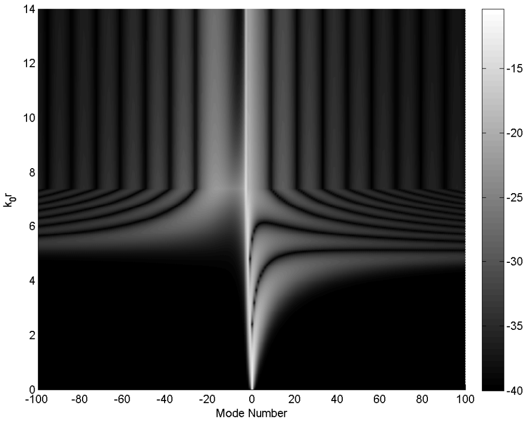

Figure 3: Mode magnitudes for a sound source rotating at Mach 1.5 with spatial frequency k_0=7.4.

The real part of the sound field produced at $t = 0$ for a 400 Hz source travelling anticlockwise at Mach 0.5 is shown in Fig. (4) for a mode range of [–5,20]. The sound field is consistent with that expected for a moving source with the wavefronts compressed in the forward direction and expanded in the reverse direction. The field is similar in appearance to that of a source rotating at Mach 0.5, emitting pulses at a rate of 2 kHz which expand cylindrically at speed c, as shown in Fig. (5) [7].

The effects of the mode truncation are most apparent at the source radius of 1 metre, where the wavefronts are not perfectly circular. At this radius $k_0 r_s = 7.4$ and the required mode order is high as shown in Fig. (2).

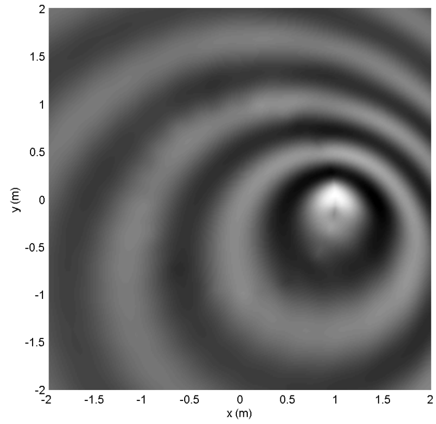

Figure 4: Sound field of a 400 Hz line source rotating at Mach 0.5, mode range –5 to 20.

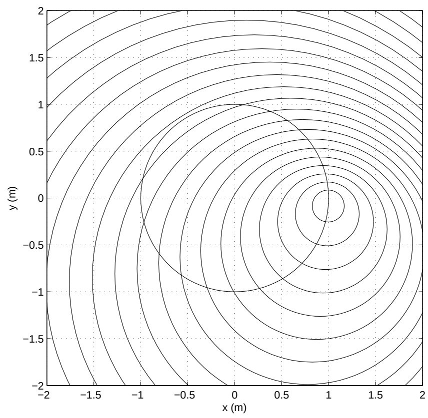

Figure 5 Expanding wavefronts from a source moving at Mach 0.5 emitting pulses at a 2 kHz rate.

The sound field of a 400 Hz source traveling at Mach 1.5 is shown in Fig. (6), for a mode range of [–100,100]. This field demonstrates the typical features of a supersonic rotating source, which are discussed in detail in [5]. The source produces a finite-width cylindrical region which extends outward in a spiral, which contains those field points which are caused by multiple retarded times. The width of the cylindrical spiral increases with Mach number and reduces to a single sheet at Mach 1.0.

At the head of the cylindrical spiral is a shock wavefront which would be a Mach cone for linear motion of a point source in the 3D case but which for a 2D rotating line source is curved and is of infinite extent in z. The interior shock wave forms a cusp at $r_c = r_s / \mathfrak{M} = 0.67m$ [5]. This cusp forms at a convergence of the circular wavefronts that emerge from the source at previous times, as shown in Fig. (7) for a line source rotating at Mach 1.5 emitting pulses at a rate of 6 kHz [7].

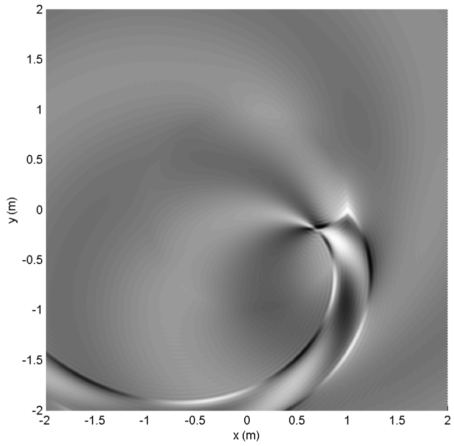

Figure 6: Sound field of a 400 Hz line source rotating at Mach 1.5, mode range –100 to 100.

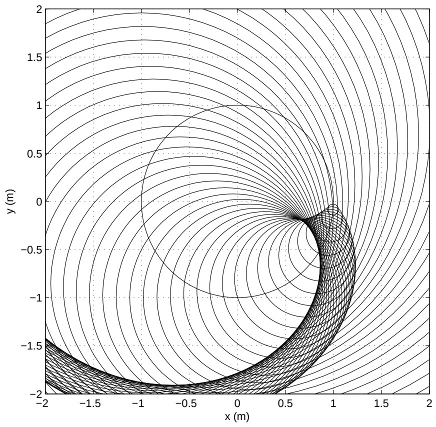

Figure 7: Expanding wavefronts from a source moving at Mach 1.5 emitting pulses at a 6 kHz rate.

CONCLUSIONS

This paper has derived a generalised form of the cylindrical harmonic addition theorem for a line source rotating at constant rate. The modal expansion represents the rotating field as a sum of fixed spatial modes each of which oscillates at a different frequency. This description offers an alternative way of describing the characteristics of the field to the parametric approach in [5]. Furthermore, the approach can be extended to a description of general rotating sources in 2D or 3D coordinates.

The modal expansion for a line source allows the properties of more general rotating sources to be studied since these can be expressed as an integral over a density of line sources.

The linear analysis ignores turbulence effects, and the nonlinear effects that occur at large sound pressures, and so will be most relevant at low Mach numbers. However, it will be more directly relevant to the simulation of a rotating source using a circular discrete array of line sources, where the rotation can be achieved electronically, and may also be relevant to the electromagnetic case.

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