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Critique of Biot-related theories of acoustic waves in porous media

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ABSTRACT

Biot's theory of porous media is discussed in detail and critically. It is pointed out that the use of two and only two displacement fields has a certain arbitrariness, and that models with additional displacement fields are possible. Biot's expression for the strain-energy per unit volume is justified in part, but it is pointed out that additional terms might be included. The theory in the low-frequency limit is discussed in detail, and the partitioning of the disturbance into three distinct types of fields is discussed. It is shown that there is sufficient latitude in the choice of coefficients in the Biot low-frequency model that the coefficients can be adjusted to fit all the major parameters associated with the three types of disturbances, but it is conjectured that the model will lead to inconsistencies for prediction of minor parameters. Unless measurements of such minor parameters are known from independent experiments, the model cannot be tested quantitatively. The use of the Biot model at higher frequencies is discussed, and it is shown that in the high-frequency limit there are always two propagating modes where the displacement fields have zero curl. It is also shown that the model predicts the attenuation at high frequencies to be independent of frequency. It is pointed out that the reported observations of the second wave were for situations where the artificial porous medium was perfectly periodic. If the medium is not periodic, it is doubtful that a second propagating wave exits.

INTRODUCTION

The papers on propagation in a fluid-saturated porous solid which Biot published in the *Journal of the Acoustical Society* in 1956 ([1], [2]) rank among the most highly cited papers in the history of acoustics. The Google Scholar web site (June 2010) shows 3089 citations for the second paper. [The manner by which Google Scholar counts citations is somewhat superficial; most authors tend to cite both papers as a unit, so the first paper is presumably cited as much or more than the first.] These papers, along with a third [3] in 1962, can be taken as what is called the Biot theory. For marine sediments, this, plus modifications due to Stoll and his colleagues, have come to be be known as the Biot-Stoll theory. [The principal account can be found in a 1970 paper by Stoll and Bryan [4]. An extensive discussion can be found in a 1989 book [5] by Stoll.] The present paper argues that the apparent current whole-hearted acceptance of the Biot theory, and especially of the later modifications associated with Stoll, has been made with insufficient critical thought. The derivations, while appealing, are heuristic, and there is little reason to expect broad applicability.

The piecing together of Biot's assumptions in the present paper is retrospective. Biot wrote a large number of papers on the subject of porous media (many of which were reprinted in a book [6] published by the Acoustical Society of America). The references in the later Biot papers were almost exclusively to earlier papers by Biot, and Biot restated his assumptions differently in subsequent papers and he also revised his notation.

The present document is a work in progress and is incomplete. It is being submitted at this time (June 10, 2010) to meet the deadline for inclusion in the ICA conference proceedings. It is intended that a more fuller account will be available by the time of the conference and that it will be submitted for journal publication.

SEPARATE FIELDS FOR SOLID AND FLUID

What appears to be the principal assumption underlying Biot's models is that the description of all phenomena of interest can be cast in terms of two (and only two) displacement fields. These were denoted by $\boldsymbol{u}$ and $\boldsymbol{U}$. In retrospect, these can be defined as the local averages of the particle displacements of the solid and fluid matter, so that

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{1}{V_s}\int' \boldsymbol{u}_{\rm mic}(\boldsymbol{x}+\boldsymbol{\xi})dV_s, \tag{1}$$

$$\boldsymbol{U}(\boldsymbol{x}) = \frac{1}{V_f}\int'' \boldsymbol{U}_{\rm mic}(\boldsymbol{x}+\boldsymbol{\xi})dV_f. \tag{2}$$

Here the integrations are taken over small volumes centered as the observation point $\boldsymbol{x}$. The subscript "mic" is an abbreviation for microscopic. In the integration for the determination of the locally-averaged solid displacement, the integration only includes the volume V_s occupied by solid material (this restriction being represented by a prime on the integral sign). In the integration for the determination of the locally-averaged fluid displacement, the integration only includes the volume V_f occupied by fluid material (this restriction being represented by a double-prime on the integral sign). The total volume being taken into consideration for this averaging process is

$$V_{\rm total} = V_{\rm s} + V_{\rm f}. \tag{3}$$

The size of the averaging volume $V_{\rm total}$ is not precisely defined, but it is implicitly assumed that one can choose such a volume sufficiently large that the averages, $\boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{U}(\boldsymbol{x})$, are insensitive to the changes of the size by factors of, say, 2. Radii of the averaging volumes should be substantially greater that what one might consider to be a representative grain size or a representative pore size. They also should be substantially less than length scales such as representative wavelengths that one might associate with a dynamic disturbance. The assertion that the local averages can be defined in a meaningful manner is related to the assertion that the porous medium, on a small scale, is statistically homogeneous. Such a procedure of defining local averages is generally well-accepted in models of continuous media, and dates back to the kinetic theory of gases. The two displacement fields may vary with position, but they should vary negligibly over distances comparable to the size of the averaging volume.

If the solid material should be composed of grains of materials with different densities, the definition as given above would be modified so that there were appropriate mass weighting, such as

$$\boldsymbol{u}(\boldsymbol{x}) = \left(\frac{1}{\int' \rho_{s,\text{mic}} dV_s} \right) \int' \rho_{s,\text{mic}} \boldsymbol{u}_{\text{mic}}(\boldsymbol{x}+\boldsymbol{\xi}) dV_s, \quad (4)$$

where the local microscopic density might vary with position $\boldsymbol{x}+\boldsymbol{\xi}$. This weighting is needed if the locally-averaged displacements are to be used in any formulation that involves anything consistent with Newton's second law, where the time rate of change of momentum is to be involved.

The basic idea of two displacement fields, rather than one, was present in an earlier theory (1944) by Frenkel [7]. Frenkel cast his theory in terms of velocities, rather than displacements, and used $\mathbf{v}_1$ to denote the the "mean macroscopic velocity of the particles of the solid phase," $\mathbf{v}_2$ to denote the "mean velocity" of the fluid matter. Biot [1] was aware of Frenkel's earlier work, but dismissed it with the statement: "the subject is summarily treated and important features are neglected." The terminology in this statement is not clarified in Biot's subsequent development.

While one could conceivably argue that a two-displacement field model is a reasonable basis for a tentative model of the relevant acoustics of porous media, one could also argue that the choice of two is to some extent arbitrary. A simpler model, but perhaps without as many implications, would be a theory based on a single displacement field, where the appropriate field variable would be

$$\boldsymbol{u}_{\text{eff}} = \frac{\rho_s V_s \boldsymbol{u} + \rho_f V_f \boldsymbol{U}}{\rho_s V_s + \rho_f V_f} \quad (5)$$

or, equivalently, with the ratio of V_f to V_{tot} replaced by the porosity χ. [In the first of his two 1956 papers, Biot uses the symbol β for porosity.]

Alternately, a model with more than two displacement fields can be envisioned. One could classify the solid grains by their sizes: those that have sizes falling in the first range of sizes would have an average displacement denoted by $\boldsymbol{u}_I$; those that have sizes falling in the second range of sizes would have an average displacement denoted by $\boldsymbol{u}_{II}$. Similarly, fluid matter could be classified by the "sizes" of the pores in which it is found: fluid matter within pores that have sizes falling in the first range of sizes would have an average displacement denoted by $\boldsymbol{U}_I$, fluid matter in pores with sizes falling in the second range of sizes would have an average displacement denoted by $\boldsymbol{U}_{II}$. One could also distinguish grains by their shapes and pores by their shapes. Other possibilities for classifying the solid matter and the fluid matter can be thought of. Having more than two displacement fields in the theory might be an unnecessary complication, but such can not be dismissed out-of-hand.

As discussed further below, the choice of two displacement fields has a direct theoretical consequence: the possibility of a second propagating wave type in some ranges of frequencies. A model with three displacement fields would lead to the theoretical possibility of two additional propagating wave types. Whether such waves are real or just an artifact of the model, arising because of the choice of the number of displacement fields, is also discussed further below.

HYPOTHESIS THAT AVERAGES IMPLY DETAILS

While the following hypothesis is not explicitly stated in Biot's papers, it has been identified after an extensive careful reflection. The hypothesis is that if one knows the local geometry completely, including the location of solid grains and of the surfaces separating solid matter from fluid matter, and if one knows the local averages and their derivatives, then one knows the microscopic displacement at every point in the local region. A limiting case is that where both average displacements are the same, and this would be so if the displacement throughout the averaging volume were uniform. The hypothesis is accordingly refined so that the deviations within the averaging volume from the averaged value are associated with differences in the two averages.

Since the displacements are presumed small, linear relations are appropriate, so one can write

$$u_{i,\text{mic}}(\boldsymbol{x}+\boldsymbol{\xi}) = u_i(\boldsymbol{x}) + \sum_j a_{ij}(\boldsymbol{\xi}) \left[u_j(\boldsymbol{x}) - U_j(\boldsymbol{x}) \right]$$

$$+ \quad \sum_{jk} p_{ijk}(\boldsymbol{\xi}) \frac{\partial u_j}{\partial x_k} \quad + \quad \sum_{jk} q_{ijk}(\boldsymbol{\xi}) \frac{\partial U_j}{\partial x_k} \quad (6)$$

$$U_{i,\text{mic}}(\boldsymbol{x}+\boldsymbol{\xi}) = U_i(\boldsymbol{x}) + \sum_j b_{ij}(\boldsymbol{\xi}) \left[u_j(\boldsymbol{x}) - U_j(\boldsymbol{x}) \right]$$

$$+ \quad \sum_{jk} r_{ijk}(\boldsymbol{\xi}) \frac{\partial u_j}{\partial x_k} \quad + \quad \sum_{jk} s_{ijk}(\boldsymbol{\xi}) \frac{\partial U_j}{\partial x_k} \quad (7)$$

Here the coefficients depend on the local geometry and on the choice of origin for the relative position vector $\boldsymbol{\xi}$. They are hypothesized to be independent of time during dynamical disturbances.

The hypothesis is, at best, an approximation. A related statement is that the coefficients, which might be called "local influence functions," actually exist. Whatever they may be, the definitions of the local averages requires that

$$\frac{1}{V_s} \int' a_{ij} dV_s = 0; \qquad \frac{1}{V_f} \int'' b_{ij} dV_f = 0. \quad (8)$$

with analogous null relations for integrals over the remaining coefficients.

The latter terms, involving the derivatives of the averaged displacements, are expected to be considerably smaller than the terms proportional to the averaged displacements. The coefficients may vary with position much more rapidly than do those that multiply the average displacements.

In regard to the nature of the approximation implied by the hypothesis, one can draw a parallel with the Rayleigh-Ritz method in vibration analysis [8]. One has a mechanical system that can vibrate in a large number of discrete natural modes. To simplify the analysis, one assumes at the outset that all the vibrations of interest can be represented as a superposition of a small number of modes, where these "assumed modes" are not necessarily natural modes. A continuous field is accordingly described by a small number of discrete time-dependent functions. In the case of the Biot theory as interpreted here, there are a relatively small number of assumed modes in each local region. These modes are not explicitly known at the outset, but it is asserted that they are unique and are independent of the nature of any dynamical disturbance.

The analogy with the use of the Rayleigh-Ritz approximation in vibration theory leads to the supposition that the two displacement field approximation is inherently a low frequency approximation, independent of any details introduced at other stages of the model development. If one desires a model applicable to higher frequencies, then one must introduce more "assumed modes."

APPLICABILITY TO SUSPENSIONS

A question is raised as to whether Biot's model applies to suspensions, where small solid particles float around in a fluid and rarely touch each other. In such a situation, the particles are prevented from aggregating, such as would be caused by gravity, because of thermal agitation. For sufficiently small amplitude disturbances in the fluid, one would expect the particles to move with the fluid. Viscosity would play a passive role but would have a sufficient effect that the no-slip condition [9] would have to apply at the interface between the grains and the fluid. In such a case, one expects the displacements $\boldsymbol{u}$ and $\boldsymbol{U}$ to be nearly equal, so a single displacement model should be sufficient at low frequencies, and if one desires to allow for the possibility that the two are slightly different, then an appropriate displacement field would be that given by Eq. (5).

The hypothesis stated above, that the averaged fields and their spatial derivatives specify details, does not necessarily rule out suspensions. While just where the individual grains are situated may appear arbitrary, one can argue that once the configuration is given, then the macro-strains can serve as boundary conditions on a combined elastostatic and hydrostatic solution for the details of the detailed displacements on a micro-scale. The argument may appear somewhat circular, in that the equations of mechanics, in the sense of statics, has to be understood before one determines the various coefficients that appear in Eqs. (6) and (7).

MARINE SEDIMENTS

In a realistic marine sediment, gravity causes grains to touch, and the sizes of the contact areas depends on depth h into the sediment and on the acceleration of gravity g roughly as $(gh)^{2/3}$ [10]. The formulations developed by Biot, as best the present authors can determine, make no explicit mention of gravity. However, some of the coefficients in Eqs. (6) and (7) are expected to depend markedly on whether the grains are in contact and on the extent of the contact. Given that this is taken into account, there appears to be no apparent restriction that Biot-related models should not apply to marine sediments. However, some of the parameters, such as those associated with shear stresses, in the Biot models should depend on depth.

Another subtlety associated with sediments is that some of the grains, especially the smaller ones, may actually be floating within the pores that exist in the fluid regions separating the larger grains. Again, this does not seem in itself to be a impediment to the applicability of Biot models.

ENERGY FUNCTIONS

The hypotheses discussed in the previous sections along with some plausible statistical assumptions leads to the Biot expressions for the kinetic energy per unit volume and potential energy (strain energy) per unit volume.

Kinetic energy density

The kinetic energy per unit volume that one would ideally want to approximate, in an analytical sense, is

$$\mathscr{T} = \frac{1}{V_{\text{total}}}\left\{\int' \frac{1}{2}\rho_s \mathrm{v}_{\text{mic}}^2 dV_s + \int'' \frac{1}{2}\rho_f V_{\text{mic}}^2 dV_f\right\} \tag{9}$$

with the abbreviations

$$\mathbf{v}_{\text{mic}} = \frac{\partial \boldsymbol{u}_{\text{mic}}}{\partial t}, \qquad \boldsymbol{V}_{\text{mic}} = \frac{\partial \boldsymbol{U}_{\text{mic}}}{\partial t}, \tag{10}$$

for the microscopic velocities. In the indicated integrals, the integration is over the hypothesized small averaging volume. It is presumed that, for a typical dynamical disturbance, the result is insensitive to the precise size of the averaging volume, and the plausibility of this assumption seems evident.

To express this kinetic energy per unit volume in terms of the two displacement fields, one inserts the expressions of Eqs. (6) and (7) into the above expression for $\mathscr{T}$. For this calculation, one can argue that the terms involving the the derivatives of the average displacements have a negligible influence, so they can be discarded.

As a result of the substitution, one recognizes the following quantities that involve integrals over sums of products of local influence functions

$$I_{aa,jk} = \frac{1}{V_{\text{total}}}\int' \sum_i a_{ij} a_{ik} dV_s; \tag{11}$$

$$I_{bb,jk} = \frac{1}{V_{\text{total}}}\int'' \sum_i b_{ij} b_{ik} dV_f \tag{12}$$

Here, one can argue that both of the above quantities are insensitive to the precise size of the averaging volume. Furthermore, if they medium is locally statistically homogeneous, they are, at worst, slowly varying functions of the center point $\boldsymbol{x}$ of the averaging volume. If the medium is statistically homogeneous in the wider sense, then they are independent of $\boldsymbol{x}$. A further simplification results if the medium is statistically isotropic, so that there is no preferred spatial direction. In this event, one has

$$I_{aa,jk} = I_{aa}\delta_{jk} \tag{13}$$

where δ_{jk} is the Kronecker delta. Analogous diagonalization hold for the other integrals.

With all the assumption just described the kinetic energy per unit volume becomes

$$\begin{aligned}\mathscr{T} &= \frac{1}{2}(1-\chi)\rho_s \mathrm{v}^2 + \frac{1}{2}\chi\rho_f V^2 \\ &+ \frac{1}{2}\rho_s I_{aa}(\mathbf{v}-\boldsymbol{V})\cdot(\mathbf{v}-\boldsymbol{V}) \\ &+ \frac{1}{2}\rho_f I_{bb}(\mathbf{v}-\boldsymbol{V})\cdot(\mathbf{v}-\boldsymbol{V}).\end{aligned} \tag{14}$$

With a suitable interpretation of symbols, this is recognized as Biot's expression for the kinetic energy per unit volume, which is

$$\mathscr{T} = \frac{1}{2}\rho_{11}\frac{\partial \boldsymbol{u}}{\partial t}\cdot\frac{\partial \boldsymbol{u}}{\partial t} + \rho_{12}\frac{\partial \boldsymbol{u}}{\partial t}\cdot\frac{\partial \boldsymbol{U}}{\partial t} + \frac{1}{2}\rho_{22}\frac{\partial \boldsymbol{U}}{\partial t}\cdot\frac{\partial \boldsymbol{U}}{\partial t} \tag{15}$$

The appropriate identifications for the elements of Biot's mass tensor are

$$\rho_{11} = [(1-\chi)+I_{aa}]\,\rho_s + I_{bb}\rho_f; \tag{16}$$

$$\rho_{12} = -\rho_s I_{aa} - \rho_f I_{bb}; \tag{17}$$

$$\rho_{22} = [\chi + I_{bb}]\,\rho_f + I_{aa}\rho_s, \tag{18}$$

so that

$$\rho_{11}+\rho_{12} = (1-\chi)\,\rho_s. \tag{19}$$

$$\rho_{12}+\rho_{22} = \chi\rho_f. \tag{20}$$

Strain-energy per unit volume

From a microscopic viewpoint, the strain-energy per unit volume should properly be expressed in terms of the microscopic strains

$$\varepsilon_{ij,\mathrm{mic}} = \frac{1}{2}\left(\frac{\partial u_{i,\mathrm{mic}}}{\partial x_j} + \frac{\partial u_{j,\mathrm{mic}}}{\partial x_i}\right) \tag{21}$$

$$E_{ij,\mathrm{mic}} = \frac{1}{2}\left(\frac{\partial U_{i,\mathrm{mic}}}{\partial x_j} + \frac{\partial U_{j,\mathrm{mic}}}{\partial x_i}\right) \tag{22}$$

For simplicity, it is here assumed that the solid matter is elastic and isotropic so that the strain energy per unit volume within the solid matter is

$$\mathscr{U}_s = \mu_L \sum_{ij} \varepsilon_{ij,\mathrm{mic}}^2 + \frac{1}{2}\lambda_L \left(\sum_k \varepsilon_{kk,\mathrm{mic}}\right)^2 \tag{23}$$

where

$$\mu_L = \frac{\nu E}{(1+\nu)(1-2\nu)}; \qquad \nu_L = G = \frac{E}{2(1+\nu)} \tag{24}$$

are the two Lamè constants.

For the fluid matter there is no static resistance to shear, and the strain energy per unit volume can be expressed

$$\mathscr{U}_f = \frac{1}{2}B_f \left(\sum_k E_{kk,\mathrm{mic}}\right)^2 \tag{25}$$

where

$$B_f = \rho_f c_f^2 \tag{26}$$

is the bulk modulus of the fluid.

In regard to the calculation of the microscopic strains, one notes that, to (what might seem to be) a good approximation, Eqs. (6) and (7) lead to

$$\frac{\partial u_{i,\mathrm{mic}}}{\partial x_j} = \frac{\partial u_i}{\partial x_j} + \sum_j a_{ik}(\boldsymbol{\xi})\left[\frac{\partial u_k}{\partial x_j} - \frac{\partial U_k}{\partial x_j}\right] + \sum_{kl} \frac{\partial p_{ikl}}{\partial \xi_j}\frac{\partial u_k}{\partial x_l} + \sum_{kl} \frac{\partial q_{ikl}}{\partial \xi_j}\frac{\partial U_k}{\partial x_l} \tag{27}$$

$$\frac{\partial U_{i,\mathrm{mic}}}{\partial x_j} = \frac{\partial U_i}{\partial x_j} + \sum_j b_{ik}(\boldsymbol{\xi})\left[\frac{\partial u_k}{\partial x_j} - \frac{\partial U_k}{\partial x_j}\right] + \sum_{kl} \frac{\partial r_{ikl}}{\partial \xi_j}\frac{\partial u_k}{\partial x_l} + \sum_{kl} \frac{\partial s_{ikl}}{\partial \xi_j}\frac{\partial U_k}{\partial x_l} \tag{28}$$

These are to be inserted into the expression

$$\mathscr{V} = \frac{1}{V_{\mathrm{total}}}\left\{\int' \mathscr{U}_s dV_s + \int'' \mathscr{U}_f dV_f\right\} \tag{29}$$

which represents the strain-energy per unit volume.

While the algebra is cumbersome, one anticipates that the result has to have the general form

$$\mathscr{V} = \sum_{ijkl}\left(A_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l} + B_{ijkl}\frac{\partial u_i}{\partial x_j}\frac{\partial U_k}{\partial x_l} + C_{ijkl}\frac{\partial U_i}{\partial x_j}\frac{\partial U_k}{\partial x_l}\right) \tag{30}$$

where the various coefficients, A_{ijkl}, etc., can be identified an explicit carrying out of the algebraic manipulations. The assumption of statistical local homogeneity requires that these coefficients be insensitive to the size of the averaging volume and also that they should be insensitive to the location of the center of the averaging volume, so they vary slowly with position over distances of the order of many grain diameters.

Biot, in his original discussion, invoked the notion that the medium we statistically isotropic, and made various postulates which in retrospect imply that the potential energy per unit volume is of the form

$$\mathscr{V} = \frac{1}{4}N\sum_{ij}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 + \frac{1}{2}A(\nabla\cdot\boldsymbol{u})^2 + Q(\nabla\cdot\boldsymbol{u})(\nabla\cdot\boldsymbol{U}) + \frac{1}{2}R(\nabla\cdot\boldsymbol{U})^2 \tag{31}$$

Here there are only four constants, these being denoted by N, A, Q, and R.

One can naturally ask if this general form does indeed follow from Eqs. (27), (28), (29), plus the assumption of statistical isotropy. The present authors have not yet succeeded in formally proving this, and there appears to be a possibility of yet one more term, this being one proportional to the square of the difference of the two curls, this term being

$$\Delta\mathscr{V}_1 = S(\nabla\times\boldsymbol{u} - \nabla\times\boldsymbol{U})\cdot(\nabla\times\boldsymbol{u} - \nabla\times\boldsymbol{U}) \tag{32}$$

with yet another constant S. While it may indeed be true that the constant S must be identically zero, the authors see no philosophical reason at present for it being zero.

Another possible additional term is

$$\Delta\mathscr{V}_2 = T(\boldsymbol{u} - \boldsymbol{U})\cdot(\boldsymbol{u} - \boldsymbol{U}) \tag{33}$$

While this is inconsistent with the approximations inherent in Eqs. (27) and (28), one could still conceive of a spring-like term, proportional to the square of the difference of the displacements of the fluid and the solid. Such a term might be relevant for a material containing closed holes containing fluid inside a solid medium. The fluid is compressible, so the center of mass of the fluid in any given hole could move relative to the solid, and this tendency is opposed by the springing caused by the fluid's compressibility.

In the remainder of the paper, the reservations just mentioned will be ignored, and the expression Eq. (31) will be used for the potential energy per unit volume in a porous medium.

LAGRANGE-EULER EQUATIONS

Biot's initial statement of his coupled equations follows directly from Hamilton's principle, and the coupled equations that result can be viewed as Lagrange-Euler equations resulting from a variational principle. The assumption that Hamilton's principle applies is rather innocuous, given all the assumptions concerning the integrand in the Hamilton's principle integral, an integral over space and time.

In retrospect, it is apparent that Biot also, at least initially, assumed that there were internal forces (dissipative forces) within the material which were not implicitly contained in the potential energy density function. His initial assumption was that they could be lumped together as a force per unit volume exerted on the solid portion by the fluid portion and a force per unit volume exerted on the fluid portion by the solid portion. These forces were related by Newton's third law.

The difference of the kinetic and potential energy densities is the Lagrangian density

$$\mathfrak{L} = \mathscr{T} - \mathscr{V} \tag{34}$$

which in accord with Hamilton's principle [11], satisfies the Lagrange-Euler equations,

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{L}}{\partial(\partial u_i/\partial t)}\right) + \sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial \mathfrak{L}}{\partial(\partial u_i/\partial x_j)}\right) = \mathfrak{f}_i \tag{35}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathfrak{L}}{\partial\left(\partial U_i/\partial t\right)}\right)+\sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial \mathfrak{L}}{\partial\left(\partial U_i/\partial x_j\right)}\right)=\mathfrak{F}_i \quad (36)$$

where the quantities on the right sides represent the forces per unit volume exerted on the fluid matter by the solid matter and conversely, these being equal and opposite in accordance with Newton's third law.

DISSIPATION TERMS

Biot, in his original exposition, followed the example of Frenkel [7] and took the force terms not contained in the energy functions to be proportional to the difference of the two velocities, reminiscent of forces associated with dashpots, so that

$$\mathfrak{F}_i=-\mathfrak{f}_i=-b\left(\frac{\partial U_i}{\partial t}-\frac{\partial u_i}{\partial t}\right)=-b\left(V_i-\mathrm{v}_i\right) \quad (37)$$

Here the quantity b can be regarded as the apparent dashpot constant per unit volume. This form is such that the derived equations, in the limit of vanishingly small frequencies, are consistent with Darcy's law for steady fluid flow through a porous medium. The form is also consistent with the notion that dissipation forces should be associated with the viscosity of the fluid.

BIOT'S FIRST MODEL

If the Lagrange-Euler equations are written out explicitly with the terms identified as discussed above, the result is

$$\frac{\partial^2}{\partial t^2}\left(\rho_{11}u_i+\rho_{12}U_i\right)-\sum_j \frac{\partial}{\partial x_j}\left[N\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]$$

$$-\frac{\partial}{\partial x_i}\left(A\nabla\cdot\boldsymbol{u}\right)-\frac{\partial}{\partial x_i}\left(Q\nabla\cdot\boldsymbol{U}\right)=-b\left(\mathrm{v}_i-V_i\right) \quad (38)$$

$$\frac{\partial^2}{\partial t^2}\left(\rho_{12}\boldsymbol{u}+\rho_{22}\boldsymbol{U}\right)-\nabla\cdot\left(Q\nabla\cdot\boldsymbol{u}\right)-\nabla\cdot\left(R\nabla\cdot\boldsymbol{U}\right)$$

$$=-b\left(\boldsymbol{V}-\mathbf{v}\right) \quad (39)$$

Here both equations are written in a form appropriate for an inhomogeneous medium. The first equation is written in an indicial form, while the second is written in a vector form.

ENERGY CONSERVATION COROLLARY

A relevant consequence of the Euler-Lagrange equations is a single equation which can be considered as an energy conservation corollary

$$\frac{\partial \mathscr{E}}{\partial t}+\nabla\cdot\boldsymbol{I}=-\mathscr{D} \quad (40)$$

Here

$$\mathscr{E}=\mathscr{T}+\mathscr{V} \quad (41)$$

is the disturbance energy per unit volume, while

$$I_i=-\sum_j \mathrm{v}_j\frac{\partial \mathscr{V}}{\partial\left(\partial u_j/\partial x_i\right)}-\sum_j V_j\frac{\partial \mathscr{V}}{\partial\left(\partial U_j/\partial x_i\right)} \quad (42)$$

is the i-th component of the energy flux vector $\boldsymbol{I}$ (energy per unit time per unit area, with direction in which the energy is flowing). The quantity on the left,

$$\mathfrak{D}=b\left(\boldsymbol{V}-\boldsymbol{v}\right)\cdot\left(\boldsymbol{V}-\boldsymbol{v}\right), \quad (43)$$

is the time rate at which energy is being dissipated per unit volume.

DISTURBANCES AT LOW FREQUENCIES

Biot's stated intent [1] in the first of his two 1956 papers was that the equations as stated above were to apply for low frequencies. If one accepts this restriction then those equations should be examined primarily in that limit.

If one assumes that the medium is homogeneous, so that quantities such as N, A, Q, R are independent of position, then a simple consequence of the equations in the low frequency limit is that any disturbance can be decomposed into disturbances of three distinct types. One writes

$$\boldsymbol{U}=\boldsymbol{U}_{\mathrm{ac}}+\boldsymbol{U}_{\mathrm{D}}+\boldsymbol{U}_{\mathrm{sh}} \quad (44)$$

$$\boldsymbol{u}=\boldsymbol{u}_{\mathrm{ac}}+\boldsymbol{u}_{\mathrm{D}}+\boldsymbol{u}_{\mathrm{sh}} \quad (45)$$

Here the three subscripts stand for acoustic (ac), Darcy (D), and shear (sh) modes. These modes are referred to in what follows as the acoustic wave mode, the Darcy diffusion mode, and the shear wave mode.

For the acoustic and Darcy modes, the curl of the two displacement fields are identically zero. For the shear wave mode, the divergence of the two displacement fields is zero.

In what follows the basic equations governing these modes are explicitly given. The derivation of these from Biot's initial equations is abbreviated, for brevity.

Acoustic wave mode

The acoustic wave mode is one of two modes where the curls of the displacements are zero, so that

$$\nabla\times\boldsymbol{U}_{\mathrm{ac}}=0; \qquad \nabla\times\boldsymbol{u}_{\mathrm{ac}}=0 \quad (46)$$

In the limit of zero frequency, the two displacement fields are the same for this mode. A natural weighted average that might be used for frequencies slightly different from zero is that given by Eq. (5), which in Biot's notation is written as

$$\boldsymbol{u}_{\mathrm{ac,eff}}=\frac{\left(\rho_{11}+\rho_{12}\right)\boldsymbol{u}_{\mathrm{ac}}+\left(\rho_{12}+\rho_{22}\right)\boldsymbol{U}_{\mathrm{ac}}}{\rho_{11}+2\rho_{12}+\rho_{22}}, \quad (47)$$

so that

$$\boldsymbol{u}_{ac}=\boldsymbol{u}_{\mathrm{ac,eff}}+\frac{\rho_{12}+\rho_{22}}{\rho_{11}+2\rho_{12}+\rho_{22}}\left(\boldsymbol{u}_{\mathrm{ac}}-\boldsymbol{U}_{\mathrm{ac}}\right) \quad (48)$$

$$\boldsymbol{U}_{ac}=\boldsymbol{u}_{\mathrm{ac,eff}}-\frac{\rho_{11}+\rho_{12}}{\rho_{11}+2\rho_{12}+\rho_{22}}\left(\boldsymbol{u}_{\mathrm{ac}}-\boldsymbol{U}_{\mathrm{ac}}\right) \quad (49)$$

If one adds, component by component, Eq. (38) with Eq. (39), one obtains

$$\rho_{\mathrm{ac}}\frac{\partial^2\boldsymbol{u}_{\mathrm{ac,eff}}}{\partial t^2}-\rho_{\mathrm{ac}}c_{\mathrm{ac}}^2\nabla^2\boldsymbol{u}_{\mathrm{ac,eff}}$$

$$-\quad\frac{D}{\rho_{\mathrm{ac}}}\nabla^2\left(\boldsymbol{u}_{\mathrm{ac}}-\boldsymbol{U}_{\mathrm{ac}}\right)=0 \quad (50)$$

with the abbreviations

$$\rho_{\mathrm{ac}}=\rho_{11}+2\rho_{12}+\rho_{22} \quad (51)$$

$$c_{\mathrm{ac}}^2=\frac{2N+A+2Q+R}{\rho_{\mathrm{ac}}} \quad (52)$$

$$D=\left(2N+A+Q\right)\left(\rho_{12}+\rho_{22}\right)-\left(Q+R\right)\left(\rho_{11}+\rho_{12}\right) \quad (53)$$

In Eq. (50), the term involving the difference of the two displacements is assumed small, so a perturbation approach with this recognition applied to either of Eqs. (38) or (39) results in

$$-\frac{D}{\rho_{\mathrm{ac}}c_{\mathrm{ac}}^2}\frac{\partial^2\boldsymbol{u}_{\mathrm{ac,eff}}}{\partial^2 t}=-b\frac{\partial}{\partial t}\left(\boldsymbol{u}_{\mathrm{ac}}-\boldsymbol{U}_{\mathrm{ac}}\right) \quad (54)$$

which integrates, in turn, to

$$\boldsymbol{u}_{\rm ac} - \boldsymbol{U}_{\rm ac} = \frac{D}{b\rho_{\rm ac}c_{\rm ac}^2}\frac{\partial \boldsymbol{u}_{\rm ac,eff}}{\partial t} \tag{55}$$

If this approximate expression is inserted into Eq. (50), the following dissipative wave equation results

$$\rho_{\rm ac}\frac{\partial^2 \boldsymbol{u}_{\rm ac,eff}}{\partial t^2} - \rho_{\rm ac}c_{\rm ac}^2\nabla^2\boldsymbol{u}_{\rm ac,eff} - \frac{D^2}{b\rho_{\rm ac}^2c_{\rm ac}^2}\frac{\partial}{\partial t}\nabla^2(\boldsymbol{u}_{\rm ac} - \boldsymbol{U}_{\rm ac}) = 0 \tag{56}$$

Darcy diffusion mode

For this mode, one has

$$\boldsymbol{\nabla}\times\boldsymbol{U}_{\rm D} = 0; \qquad \boldsymbol{\nabla}\times\boldsymbol{u}_{\rm D} = 0 \tag{57}$$

Also, the inertia terms in Eqs. (38) and (39) are of minor importance. Consistency of the two equations requires that

$$(2N+A)\,\boldsymbol{u}_D + Q\boldsymbol{U}_D \approx -Q\boldsymbol{u}_D - R\boldsymbol{U}_D, \tag{58}$$

so that

$$\boldsymbol{u}_D \approx -\frac{Q+R}{2N+A+2Q+R}(\boldsymbol{U}_D - \boldsymbol{u}_D) \tag{59}$$

$$\boldsymbol{U}_D \approx \frac{2N+A+Q}{2N+A+2Q+R}(\boldsymbol{U}_D - \boldsymbol{u}_D) \tag{60}$$

The partial differential equation that results with the neglect of the inertia terms is

$$\left(\frac{(2N+A)R - Q^2}{2N+A+2Q+R}\right)\nabla^2(\boldsymbol{U}_D - \boldsymbol{u}_D) = b\frac{\partial}{\partial t}(\boldsymbol{U}_D - \boldsymbol{u}_D) \tag{61}$$

which is recognized as the diffusion equation

$$\nabla^2(\boldsymbol{U}_D - \boldsymbol{u}_D) = \kappa\frac{\partial}{\partial t}(\boldsymbol{U}_D - \boldsymbol{u}_D) \tag{62}$$

with

$$\kappa = \frac{b(2N+A+2Q+R)}{(2N+A)R - Q^2} \tag{63}$$

An improved approximation results if the intertial terms are treated as a perturbation. Manipulation of the two primary equations results in the equation

$$\begin{aligned}\frac{\partial^2}{\partial t^2}&\left[(\rho_{11}+\rho_{12})(Q+R)\,\boldsymbol{u}_D - (\rho_{12}+\rho_{22})(2N+A+Q)\,\boldsymbol{U}_D\right]\\ &+ \left[(2N+A)R - Q^2\right]\nabla^2(\boldsymbol{U}_D - \boldsymbol{u}_D)\\ &= b(2N+A+2Q+R)\frac{\partial}{\partial t}(\boldsymbol{U}_D - \boldsymbol{u}_D)\end{aligned} \tag{64}$$

To carry out the perturbation, one inserts Eqs. (59) and (60) and thereby obtains

$$\begin{aligned}-G\frac{\partial^2}{\partial^2 t}(\boldsymbol{U}_D - \boldsymbol{u}_D) &+ \left[(2N+A)R - Q^2\right]\nabla^2(\boldsymbol{U}_D - \boldsymbol{u}_D)\\ &= b(2N+A+2Q+R)\frac{\partial}{\partial t}(\boldsymbol{U}_D - \boldsymbol{u}_D)\end{aligned} \tag{65}$$

where

$$G = \frac{(\rho_{11}+\rho_{12})(Q+R)^2 + (\rho_{12}+\rho_{22})(2N+A+Q)^2}{2N+A+2Q+R} \tag{66}$$

At low frequencies the term involving the second derivative with respect to time should have minor effect.

Shear wave mode

The shear wave mode is characterized by displacement fields which have zero divergence, so that

$$\boldsymbol{\nabla}\cdot\boldsymbol{U}_{\rm sh} = 0; \qquad \boldsymbol{\nabla}\cdot\boldsymbol{u}_{\rm sh} = 0 \tag{67}$$

With this restriction, the Eqs. (38) and (39) reduce to

$$\frac{\partial^2}{\partial t^2}(\rho_{11}\boldsymbol{u}_{\rm sh} + \rho_{12}\boldsymbol{U}_{\rm sh}) - N\nabla^2\boldsymbol{u}_{\rm sh} = -b\frac{\partial}{\partial t}(\boldsymbol{u}_{\rm sh} - \boldsymbol{U}_{\rm sh}) \tag{68}$$

$$\frac{\partial^2}{\partial t^2}(\rho_{21}\boldsymbol{u}_{\rm sh} + \rho_{22}\boldsymbol{U}_{\rm sh}) = b\frac{\partial}{\partial t}(\boldsymbol{u}_{\rm sh} - \boldsymbol{U}_{\rm sh}) \tag{69}$$

Given that the frequency is sufficiently low, the “dissipation” terms on the right dominate, and these require that the two displacements fields be appropriately, just as was the case for the acoustic mode.

Addition of the two equations above yields

$$\rho_{\rm sh}\frac{\partial^2\boldsymbol{u}_{\rm sh,eff}}{\partial t^2} - N\nabla^2\boldsymbol{u}_{\rm sh,eff} - N\frac{\rho_{12}+\rho_{22}}{\rho_{11}+2\rho_{12}+\rho 22}\nabla^2(\boldsymbol{u}_{\rm sh} - \boldsymbol{U}_{\rm sh}) = 0 \tag{70}$$

with the abbreviations

$$\rho_{\rm sh} = \rho_{11} + 2\rho_{12} + \rho_{22} \tag{71}$$

$$\boldsymbol{u}_{\rm sh,eff} = \frac{(\rho_{11}+\rho_{12})\,\boldsymbol{u}_{\rm sh} + (\rho_{12}+\rho_{22})\,\boldsymbol{U}_{\rm sh}}{\rho_{11}+2\rho_{12}+\rho_{22}}, \tag{72}$$

so that

$$\boldsymbol{u}_{sh} = \boldsymbol{u}_{\rm sh,eff} + \frac{\rho_{12}+\rho_{22}}{\rho_{11}+2\rho_{12}+\rho_{22}}(\boldsymbol{u}_{\rm sh} - \boldsymbol{U}_{\rm sh}) \tag{73}$$

$$\boldsymbol{U}_{sh} = \boldsymbol{u}_{\rm sh,eff} - \frac{\rho_{11}+\rho_{12}}{\rho_{11}+2\rho_{12}+\rho_{22}}(\boldsymbol{u}_{\rm sh} - \boldsymbol{U}_{\rm sh}) \tag{74}$$

The third term in Eq. (70) is of minor importance and can be replaced by a suitable expression using a perturbation technique. To this purpose one inserts $\boldsymbol{u}_{\rm sh} \to \boldsymbol{u}_{\rm sh,eff}$, $\boldsymbol{U}_{\rm sh} \to \boldsymbol{u}_{\rm sh,eff}$ into Eq. (69 and finds

$$\boldsymbol{u}_{\rm sh} - \boldsymbol{U}_{\rm sh} = \frac{\rho_{21}+\rho_{22}}{b}\frac{\partial}{\partial t}\boldsymbol{u}_{\rm sh,eff} \tag{75}$$

This, in turn, when inserted into Eq. (70) yields the dissipative wave equation

$$\rho_{\rm sh}\frac{\partial^2\boldsymbol{u}_{\rm sh,eff}}{\partial t^2} - \rho_{\rm sh}c_{\rm sh}^2\nabla^2\boldsymbol{u}_{\rm sh,eff} - \frac{N(\rho_{12}+\rho_{22})^2}{b\rho_{\rm sh}}\nabla^2\frac{\partial}{\partial t}\boldsymbol{u}_{\rm sh,eff} = 0 \tag{76}$$

with the abbreviation

$$c_{\rm sh}^2 = \frac{N}{\rho_{\rm sh}} \tag{77}$$

The quantity $c_{\rm sh}$ is recognized as the appropriate wave speed for shear waves.

Plane shear waves are transverse waves in accordance with the requirement that the divergences of the of the two displacement fields are zero.

ATTENUATION COEFFICIENTS

The Biot low-frequency predicts the low-frequency attenuation coefficients for both the acoustic wave mode and the shear wave mode. To derive the attenuation coefficients, one lets the relevant amplitudes have time and spatial dependence as

$$\psi(x,t) = \mathrm{Re}\left\{\psi_o e^{-i\omega t} e^{ikx}\right\} \tag{78}$$

where the wave number k is a complex number that depends on the angular frequency ω.

Insertion of the above expression into the dissipative wave equation for the acoustic mode yields

$$-\omega^2\rho_{\mathrm{ac}} + \rho_{\mathrm{ac}}c_{\mathrm{ac}}^2 k^2 - i\frac{D^2}{b\rho_{\mathrm{ac}}^2 c_{\mathrm{ac}}^2}\omega k^2 \tag{79}$$

If this is solved for k, with the root with positive real part selected, and if the expression is expanded in a power series in ω, the leading two terms

$$k = \frac{\omega}{c_{\mathrm{ac}}} + i\frac{D^2}{2bc_{\mathrm{ac}}^5\rho_{\mathrm{ac}}^3}\omega^2 \tag{80}$$

From this one identifies

$$\alpha_{\mathrm{ac}} = \frac{D^2}{2bc_{\mathrm{ac}}^5\rho_{\mathrm{ac}}^3}\omega^2 \tag{81}$$

A similar calculation carried out for the dissipative wave equation for the shear wave mode yields

$$\alpha_{\mathrm{sh}} = \frac{N\left(\rho_{12}+\rho_{22}\right)^2}{2bc_{\mathrm{sh}}^3\rho_{\mathrm{sh}}^2}\omega^2 \tag{82}$$

Both attenuation coefficients, in the considered low frequency limit, vary with the frequency as the square of the frequency.

EXTENT OF VALIDITY

Allowing all possible choices for the seven coefficients that appear in the Biot low frequency model, one naturally wonders if the predictions described are consistent. One may argue that there should indeed be three modes, that there be two propagating modes and a diffusion mode, and that the attenuation coefficients should be proportional to frequency squared. This is all a plausible confirmation, but there are quantitative interconnections that may or may not be consistent with the model.

Let us assume that the following parameters can be independently measured:

- $\rho_{11}+\rho_{12}$
- $\rho_{12}+\rho_{22}$
- c_{ac}
- c_{sh}
- κ
- $\alpha_{\mathrm{ac}}/\omega^2$
- $\alpha_{\mathrm{sh}}/\omega^2$

The acoustic mode wave speed, if measured, gives an identification for

- $a_1 = 2N + A + 2Q + R$

The shear mode wave speed, if measured, gives an identification for

- $a_2 = N$

Note also that the quantities

- $a_3 = \kappa\omega^2/\alpha_{\mathrm{ac}}$
- $a_4 = \kappa\omega^2/\alpha_{\mathrm{sh}}$

are independent of the parameter b and consequently can be regarded as functions of the four modulus constants.

Consequently, presuming that a_1, a_2, a_3, a_4 are all determined, one has four equations in four unknowns, the unknowns being N, A, Q, and R. The latter two equations are not linear equations, but still one presumes that a solution exists for the four modulus constants. Once those quantities are determined, the determination of the parameter b follows from the presumption that κ has been independently measured.

However, if the Biot equations are to be used to predict the parameter G that appears in Eq. (64), or to predict higher order terms in power expansions in ω of phase velocities and attenuation coefficients, the number of adjustable constants is insufficient. One suspects that it would be fortuitous if the low frequency equations would simultaneously predict such parameters along with all those in the above bulleted list.

One disturbing feature of the model is that all of the presumably measurable (in the low frequency limit) quantities depend on the three density constants ρ_{11}, ρ_{12} and ρ_{22} only in the combinations $\rho_{11}+\rho_{12} = (1-\chi)\rho_s$ and $\rho_{12}+\rho_{22} = \chi\rho_f$. Here one has three unknowns and only two equations.

EMERGENCE OF THE SLOW WAVE

In his first paper [1], Biot began with analyzing the case when the parameter b was identically zero and discovered that there were two (not just one) propagating waves for which the curl of the two displacements was zero. In retrospect, this case was inconsistent with the claim that the model was to be for low frequencies. At low frequencies, the terms involving b (one time derivative) dominate the intertial terms (two time derivatives), except for the circumstances when the displacements are very nearly the same. This is so for the acoustic wave mode, but not so for the Darcy diffusion mode.

Evidently, if one does assume that the low frequency model does actually apply at high frequencies, then the Darcy mode must eventually morph into a propagating mode at high frequencies. To investigate this possibility, one can derive two coupled equations for the divergences $\nabla\cdot\boldsymbol{u}$ and $\nabla\cdot\boldsymbol{U}$, and then examine their circumstances. One sets

$$\nabla\cdot\boldsymbol{u} = \mathrm{Re}\left\{A_u e^{-i\omega t} e^{ikx}\right\} \tag{83}$$

$$\nabla\cdot\boldsymbol{U} = \mathrm{Re}\left\{A_U e^{-i\omega t} e^{ikx}\right\} \tag{84}$$

and then obtains the equations, in matrix form

$$-\omega^2[\rho]\{A\} + k^2[B]\{A\} = i\omega b[J]\{A\} \tag{85}$$

where

$$[\rho] = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{bmatrix} \tag{86}$$

is the density matrix,

$$[B] = \begin{bmatrix} 2N+A & Q \\ Q & R \end{bmatrix} \tag{87}$$

is the modulus matrix, and

$$[J] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{88}$$

The column vector

$$\{A\} = \begin{Bmatrix} A_u \\ A_U \end{Bmatrix} \tag{89}$$

contains the wave amplitudes.

Because the amplitude vector has two components there must be two "modes." Dispersion relations for the two modes are obtained by setting the determinant of the coefficients to zero, so that

$$\det[M] = 0 \quad (90)$$

$$[M] = \begin{bmatrix} -\omega^2\rho_{11} + k^2(2N+A) - i\omega b & -\omega^2\rho_{12} + k^2 Q + i\omega b \\ -\omega^2\rho_{12} + k^2 Q + i\omega b & -\omega^2\rho_{22} + k^2 R - i\omega b \end{bmatrix} \quad (91)$$

The resulting equation for k^2 can be expressed

$$\alpha k^4 - \left(\beta\omega^2 + i\varepsilon b\omega\right)k^2 + \left(\gamma\omega^4 + i\delta b\omega^3\right) = 0 \quad (92)$$

where

$$\alpha = (2N+A)R - Q^2 = \det[B] \quad (93)$$

$$\beta = R\rho_{11} + (2N+A)\rho_{22} - 2Q\rho_{12} \quad (94)$$

$$\gamma = \rho_{11}\rho_{22} - \rho_{12}^2 = \det[\rho] \quad (95)$$

$$\varepsilon = 2N + A + 2Q + R \quad (96)$$

$$\delta = \rho_{11} + 2\rho_{12} + \rho_{22} \quad (97)$$

In the low frequency limit, one root for k^2 is proportional to ω and the other is proportional to ω^2. For frequencies slightly different from zero, the power series expressions can be developed by iteration of the equations

$$k^2 = i\frac{\varepsilon b}{\alpha}\omega + \frac{\beta}{\alpha}\omega^2 - \frac{\omega^3}{k^2}(i\delta b + \gamma\omega) \quad (98)$$

$$k^2 = \omega^2\frac{\delta}{\varepsilon}\frac{(1 - i[\gamma/(\delta b)]\omega)}{(1 - i[\beta/(\varepsilon b)]\omega)} - i\frac{\alpha k^4}{\varepsilon b\omega}\frac{1}{(1 - i[\beta/(\varepsilon b)]\omega)} \quad (99)$$

The first solution corresponds to the Darcy diffusion mode, and the second corresponds to the acoustic wave mode.

In the other limit, in the limit of high frequencies, both roots for k^2 are approximately proportional to ω^2. One defines r_1 and r_2 as the two roots of the equation

$$\alpha r^2 - \beta r + \gamma = 0 \quad (100)$$

so that

$$r_{1,2} = \frac{\beta}{2\alpha} \mp \left[\left(\frac{\beta}{2\alpha}\right)^2 - \frac{\gamma}{\alpha}\right]^{1/2} \quad (101)$$

Then the quadratic equation becomes

$$\left(k^2 - \omega^2 r_1\right)\left(k^2 - \omega^2 r_2\right) - i\left(\frac{b\omega}{\alpha}\right)\left(\varepsilon k^2 - \delta\omega^2\right) = 0 \quad (102)$$

Series expansions for the two roots for k^2 can be developed by iteration of the equations

$$k_1^2 = \omega^2 r_1 + i\frac{b\omega}{\alpha}\varepsilon\left[\frac{(k^2/\omega^2) - (\delta/\varepsilon)}{(k^2/\omega^2) - r_2}\right] \quad (103)$$

$$k_2^2 = \omega^2 r_2 + i\frac{b\omega}{\alpha}\varepsilon\left[\frac{(k^2/\omega^2) - (\delta/\varepsilon)}{(k^2/\omega^2) - r_1}\right] \quad (104)$$

and the first iteration yields

$$k_1^2 = \omega^2 r_1 + i\frac{b\omega}{\alpha}\varepsilon\left[\frac{r_1 - (\delta/\varepsilon)}{r_1 - r_2}\right] \quad (105)$$

$$k_2^2 = \omega^2 r_2 + i\frac{b\omega}{\alpha}\varepsilon\left[\frac{r_2 - (\delta/\varepsilon)}{r_2 - r_1}\right] \quad (106)$$

Both r_1 and r_2 are positive and one can conjecture that their magnitudes are such that

$$r_1 < \frac{\delta}{\varepsilon} < r_2 \quad (107)$$

If such is so, then the right sides of the first-iterated equation have positive imaginary parts.

So one determines that the wave speeds and attenuation coefficients of the two modes have wave speeds of

$$c_1 = 1/\sqrt{r_1}; \qquad c_2 = 1/\sqrt{r_2} \quad (108)$$

and attenuation coefficients of

$$\alpha_1 = \frac{b\varepsilon c_1}{2\alpha}\left[\frac{(\delta/\varepsilon) - r_1}{r_2 - r_1}\right] \quad (109)$$

$$\alpha_2 = \frac{b\varepsilon c_2}{2\alpha}\left[\frac{r_2 - (\delta/\varepsilon)}{r_2 - r_1}\right] \quad (110)$$

Note that the two attenuation coefficients in this limit are independent of frequency. (Biot's original analysis assumed that b was zero, but such an assumption was inconsistent with the assertion that the model was to be applicable at low frequencies. It possibly could be applicable at high frequencies also, but one must nevertheless carry through the derivation as if the quantity b were nonzero.)

One can term the solution with the higher wave speed the fast wave, and presumably associate it with the acoustic mode that exists at low frequencies. One possibly disturbing feature is that his wave, while also nearly nondispersive, has a phase velocity considerably different than that of the acoustic mode wave at low frequencies.

The other wave, the one with lower speed, is presumably what the Darcy diffusion mode morphs into at high frequency. Because the attenuation is independent of frequency, the attenuation per wavelength becomes smaller and smaller as the frequency increases. One can conjecture that the order of magnitude of the frequency at which the transition from diffusion to propagating wave is where the real and imaginary parts of the wave number, according to the high frequency approximation, first become equal. Such leads to

$$\omega_{\text{tran}} = \alpha_1 c_1 = \frac{b\varepsilon}{2\alpha r_1}\left[\frac{(\delta/\varepsilon) - r_1}{r_2 - r_1}\right] \quad (111)$$

[The authors have not attempted to estimate the numerical value of this quantity.]

MODIFICATIONS IN LATER PAPERS

In later papers, Biot replaced his constant parameter b by a frequency dependent operator. What he did in his second paper [2] can be roughly characterized as a "patch job," and Biot gave two distinct formulas, one assuming that the fluid flow in the pores was like fluid flow between parallel plates and the other assuming that it was like fluid flow in circular ducts.

In the 1962 paper [3], viewed in retrospect, it appears that the modification can be regarded as a replacement of b by a frequency-dependent function, where the actual function could be determined by fitting the overall model to experimental data.

Nevertheless, it does appear that, if all that is done is to replace b as a frequency dependent quantity $b(\omega)$, and if $b(\omega)$ is bounded at high frequencies, then the conclusion that there be two propagating waves at high frequencies has to remain a consequence of the overall model.

The authors suspect that later modifications to the Biot model, such as were introduced by Stoll [4] and by Chotiros and Isakson [12], will not change this prediction.

THE BIOT-STOLL MODEL

A commonly used hybrid model, frequently termed the Biot-Stoll model, dates back to a 1970 paper by Stoll and Bryan [4]. In essence, the model presumes that, for constant frequency waves, the Biot model described above is correct, but only if all the elastic constants are taken as complex constants, so that

$$A \to A_R + iA_I; \qquad Q \to Q_R + iQ_I, \qquad (112)$$

etc., where the imaginary parts are presumed small compared to the real parts. Given that one is using $\exp(-i\omega t)$ time dependence, these imaginary parts are all expected to be negative numbers.

What such a substitution does at low frequencies is to cause the wave speeds to have a constant negative imaginary part. This causes the attenuation at low frequencies to have a term proportional to frequency in addition to one proportional to frequency squared. There is some feeling that such a term in the zero frequency would violate considerations of causality [13].

EXPERIMENTALLY OBSERVED SLOW WAVES

Experimental observation of a second propagating wave (a slow wave) has been reported in papers by Plona [14] and by Johnson and Plona [15]. In the text by Bourbie, Coussy, and Zinszner [16], it is implied that these experiments confirm the validity of the general Biot model.

NONEXISTENCE OF SLOW WAVE

A principal contention of the present paper is that the experimental observations of the slow wave were for cases where the simulated porous medium was perfectly periodic. One can show that, unless the medium is perfectly periodic (or near perfectly periodic) a slow wave cannot be produced.

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