Modelling coherent elastic wave propagation through creep damaged material

Mihai Caleap, Bruce W. Drinkwater and Paul D. Wilcox

Department of Mechanical Engineering, University of Bristol, Queens Building, University Walk, Bristol BS8 1TR, UK

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ABSTRACT

In this paper, a creep-damaged material is modelled as a two-phase composite material comprising a matrix and a distribution of clustered spherical voids. The voids are dispersed uniformly within oblate ellipsoidal regions that represent preferred regions of voiding close to grain boundaries. In turn, the ellipsoidal regions are distributed randomly in the matrix. A double composite model based on coherent elastic wave propagation is used to determine the effective stiffness and the overall density of the two-phase material. As the creep progresses, the ellipsoidal elements are sparsely scattered in the matrix, but they continue to grow in volume, containing more and more voids within them. This evolution results in an anisotropy increase due to the preferential void formation within the ellipsoid elements. Velocity estimates can be used to predict the elastic softening and the development of anisotropy, providing bulk-average information pertinent to the assessment of creep damage.

INTRODUCTION

Problems involving the modelling of wave propagation in particulate composites are generally recognized as being impossible to solve analytically. This is because infinite orders of rescattering have to be considered along with the need to satisfy the continuity conditions for stresses and displacements across each of the cavity boundaries when, in many cases, the exact location of each of the cavities (with respect to some chosen origin) is unknown.

Often, however, the analysis can be simplified by replacing the composite medium by a homogeneous effective medium that can be characterized by an appropriate choice of effective static and dynamic parameters (e.g., wavenumber, elastic moduli, density, etc.). For a low concentration of sparsely distributed scatterers, when multiple scattering effects are negligible, single scattering approaches can often be used reliably. If the dimensions of the cavities are sufficiently small so that their resonant frequencies lie outside the frequency band of interest, then models, such as that of Kuster and Toksöz [1], can be employed. Many other homogenisation schemes have been proposed to obtain the effective elastic properties of particulate composites, see e.g. [2-5]. In particular, optimal bounds in which the effective moduli must reside were developed [6]. For higher concentrations of scatterers, multiple scattering approaches can also be used, for example, that of Waterman and Truell [7] which accounts for the first order of rescattering but not higher orders, or that developed more exactly by Varadan et al. [8] in which the effective wavenumbers in the low frequency limit are obtained from simple dispersion relations.

In effect, all of these models provide a means of obtaining a set of frequency-dependent estimates for effective parameters (e.g., wavenumber, elastic moduli, density, etc.) as a function of the material properties of the constituent media, the dimensions of the cavities, and the void fraction.

The present paper is concerned only with the modelling of elastic wave propagation through the various types of void-filled media in the low-frequency limit (long wavelength compared with the lengthscale of the microstructure of the composite). Enhancements to allow modelling in higher frequency regions is the subject of on-going research. Further, since effective medium techniques are now well established for simple shaped and uniformly distributed cavities and elastic inclusions, the work concentrates on their application to creep-damaged materials.

We consider creep damage in a metal to result in a two-phase composite material comprising an isotropic matrix and spherical voids. The voids are assumed to be dispersed uniformly within oblate ellipsoidal regions (or volume elements). The volume elements are randomly distributed in the matrix. A two-dimensional sketch of this double composite material is illustrated in Figure 1. The calculation procedure with this model consists of two steps. First, the spherical voids are assumed to be randomly and uniformly distributed in the matrix. We calculate the effective stiffness
and the overall density of the porous material. Second, the oblate ellipsoidal inclusions, whose density and stiffness are known from the previous step, are distributed randomly in the matrix with the minor axes parallel to the stress direction, and the new effective stiffness is obtained. This model allows us to determine the effective velocities of the coherent elastic waves propagating and polarized in the principal directions of the transverse isotropy. The present treatment draws much on previous work by many researchers, especially of Ledbetter et al. [9] and Jeong and Kim [10]. For simplicity, we assume that the effective velocities are affected only by the creep voids.

**POROUS MATERIAL**

Suppose that identical spherical cavities of radius \( a \) are distributed randomly and uniformly in an isotropic homogeneous elastic solid. The concentration of cavities is denoted by \( \phi_c \). In view of the coherent elastic wave propagation with frequency \( \omega \), the given medium may be seen as isotropic homogeneous made up of some effective material. Let \( \rho_c \), \( \kappa_c \), and \( \mu_c \) denote respectively, the mass density, the bulk modulus, and the shear stiffness of the matrix material. The density \( \rho \), the bulk modulus \( \kappa \), and the shear stiffness \( \mu \) of the effective material depend on the concentration \( \phi_i \) and on the scattering dispersion parameter

\[
\tilde{\omega}_c = k_c a, \quad w = L, T, 
\]

where

\[
k_c = \omega \sqrt{\frac{\rho_c}{\kappa_c + 4/3 \mu_c}} \quad \text{and} \quad k_c = \omega \sqrt{\frac{\rho_c}{\mu_c}} 
\]

are the longitudinal and shear wavenumbers in the host material, respectively. According to [8, 11], this dependence is as follows (see Appendix A for more details)

\[
\frac{\rho_c(\tilde{\omega})}{\rho_c} = (1 - \phi_c)(1 + r(\tilde{\omega}))], 
\]

\[
\frac{\kappa_c(\tilde{\omega})}{\kappa_c} = \frac{4\mu_c(1 - \phi_c)}{3\kappa_c\phi_c + 4\mu_c}[1 + k(\tilde{\omega})]], 
\]

\[
\frac{\mu_c(\tilde{\omega})}{\mu_c} = \frac{1 - \phi_c}{1 + \frac{2}{3}\phi_c}[1 + m(\tilde{\omega})]], 
\]

where \( r(\tilde{\omega}) \), \( k(\tilde{\omega}) \), and \( m(\tilde{\omega}) \) are certain complex-valued functions, which tend to zero along with their first derivatives at \( \tilde{\omega} = 0 \). By a different procedure, the effective material properties were also found by Kuster and Toksöz [1] and are applicable to a dense concentration of voids. The effective bulk and shear moduli, (4) and (5), coincide with either the upper or the lower Hashin-Strikman bounds [6].

**DOUBLE COMPOSITE MATERIAL**

Suppose now that identical ellipsoidal inclusions are distributed randomly and uniformly in the isotropic matrix as pictured in Figure 1. The shape of the ellipsoids is represented by an aspect ratio \( \tau \) defined by \( \tau = a_3 / a_1 \), where \( a_1 \) and \( a_3 \) are the radii along the semi-axes of ellipsoids. Let \( \phi_i \) denote the volume fraction of the ellipsoids. The mass density \( \rho, \) the bulk modulus \( \kappa, \) and

\[
\text{the shear stiffness } \mu, \text{ of the inclusion material are given by Eqs. (3)-(5) in the previous section.}
\]

When the ellipsoids are aligned, the wave velocities depend on propagation direction and the coherent waves are in general quasi-longitudinal or quasi-shear. Pure longitudinal waves or pure shear waves can propagate only along directions parallel to the principal axes of ellipsoids. For simplicity we consider here propagation parallel to the principal axes.

Now, if the wavelength of the incident waves is large compared to the size of the ellipsoid, then the field both near the ellipsoid and inside will be essentially quasi-static and uniform.

As discussed in the Introduction, there are various techniques for constructing a homogenized medium to represent the matrix-plus-inclusions composite. The self-consistent method developed in [12] is used to estimate the effective macroscopic elastic constants for the composite material with ellipsoidal inclusions. The predictions of the effective elastic properties are shown to be exactly the same as those derived via either Foldy [13] or the Waterman-Truel [7] theories. Links with other homogenisation schemes and multiple scattering theories (in the low-frequency limit) can be found in [12].

In the following we shall use the tilde sign to non-dimensionalise effective parameters.

We note that the effective inertia of the composite is just the spatial average of the density, i.e.,

\[
\tilde{\rho} = 1 + (\rho_c - 1)\phi_i = 1 - \phi_c, 
\]

where \( \rho \) and \( \rho_c \) are scaled on \( \rho_i \) (i.e., \( \rho = \rho_i \tilde{\rho} \), etc.), and \( \phi \) is the porosity determined by

\[
\phi = \phi_i \phi_c. 
\]

According to [12], for the case of longitudinal waves propagating in the \( x_3 \)-direction, a homogeneous material with wavenumber \( \gamma_{11} \) (scaled on \( k_{11} \) ) would have the form

\[
\gamma_{11}^2 = \tilde{\rho} / \tilde{C}_{11} = \rho(1 - \phi_i \beta_{11}), 
\]

where \( \beta_{11} \) is defined in Eq. (B.9) and

\[
\tilde{C}_{11} = (1 - \phi_i \beta_{11})^{-1}, 
\]

is the effective modulus. Similarly, considering propagation in the \( x_1 \)-direction, we find that

\[
\tilde{C}_{13} = (1 - \phi_i \beta_{13})^{-1}, 
\]

where \( \beta_{13} \) is defined in Eq. (B.10). Note that the effective moduli \( C_{11} \) and \( C_{13} \) are scaled on \( \kappa_i + \frac{4}{3} \mu_i \).

For the case of shear waves polarised in the \( x_2 \) - and \( x_3 \) -directions, but propagating in the \( x_1 \) -direction with effective wavenumbers (scaled on \( k_{11} \) )

\[
\gamma_{12}^2 = \tilde{\rho} / \tilde{C}_{12}, \quad \gamma_{13}^2 = \tilde{\rho} / \tilde{C}_{13}, 
\]

respectively, we find that

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\( C_{\mu} = (1 - \phi \beta_3)^{-1}, \quad C_{\nu} = (1 - \phi \beta_3) \),

where these effective moduli are scaled by \( \mu_0 \). Coefficients \( \beta_2 \) and \( \beta_3 \) are defined in Eq. (B.11). Since the material is transversely isotropic, we can find \( \tilde{C}_{12} \) from the expression

\[ \tilde{C}_{12} = \frac{1}{2} \left( \tilde{C}_{11} - \tilde{C}_{22} \right). \tag{13} \]

Thus, for the transversely isotropic case we have recovered four of the five independent elastic constants in explicit form. In order to obtain \( \tilde{C}_{11} \), we require non-normal incidence, which is beyond the scope of this paper.

If \( \tau, \phi, \) and \( \phi \) were all known, it would be straightforward to calculate the velocities of the elastic waves propagating and polarized in the principal directions of the transverse isotropy, i.e.,

\[ \tilde{v}_{11} = \sqrt{\frac{C_{11}}{\rho}}, \quad \tilde{v}_{22} = \sqrt{\frac{C_{22}}{\rho}}, \quad \tilde{v}_{12} = \sqrt{\frac{C_{12}}{\rho}}, \quad \tilde{v}_{13} = \sqrt{\frac{C_{13}}{\rho}}. \tag{14} \]

Both the density and the stiffness are reduced by the presence of voids. The stiffness decreases at a greater rate than the density, leading to a more significant (and potentially measurable) velocity decrease.

In the following, we apply the double composite model to a notional copper-like material subjected to an intergranular creep process and calculate the effective velocities.

**CREEP DAMAGED MATERIAL**

Photomicrographic observations of pure copper samples subjected to the intergranular creep process suggest that voids are not randomly positioned [10]. They tend to gather preferentially on the grain boundaries perpendicular to the creep axis. Typically, the void diameter is approximately 10 \( \mu m \) [9, 10]. The grain shape is nearly equiaxial and the grain size is approximately 0.1 mm [9, 10]. The oriented growth of voids makes the material transversely isotropic around the stress axis.

To predict the effective velocities of materials that have undergone creep, one has to account for the transversely isotropic effective stiffness. We could model any of the following features or a combination thereof: the void shape distribution and orientation.

The damage morphology observed by photomicrograph motivates us to consider a composite modelling relying on the non-random positions of spherical voids. A sketch of this idea is illustrated in Figure 1. Based on these observations, the overall progression of the creep damage can be described as follows. The spherical voids nucleate and grow preferentially on the grain boundaries mostly in areas perpendicular to the applied stress direction. As the creep progresses, the voids form into clusters and the void volume fraction increases. In this paper, the clustered regions are modelled as oblate ellipsoids.

We shall now compare the effective velocities predicted by the double composite model with the experimental results obtained in [14]. This allows us to assess the efficacy of the model.

![Figure 2](image)

**Figure 2.** Variation of longitudinal and transverse effective velocities in creep-damaged copper. Solid lines show the predicted velocities for various aspect ratios \( \tau \) and a constant volume fraction \( \phi = 0.07 \) of oblate-ellipsoidal clusters. The circles represent the results based on the fitting curves of measured velocities presented in [14].

Morishita and Hirao [14] have measured the ultrasonic velocities for pure copper samples subjected to the intergranular creep process. The elastic properties and the density of the polycrystalline copper used for the experimental study [14] are given in Table 1.

<table>
<thead>
<tr>
<th>Material properties of polycrystalline copper.*</th>
<th>( \rho_0 ) (g/cm(^3))</th>
<th>( \mu_0 ) (GPa)</th>
<th>( \kappa_0 ) (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polycrystalline copper</td>
<td>8.98</td>
<td>46.7</td>
<td>136</td>
</tr>
</tbody>
</table>

*Source: (Morishita and Hirao, 1996)

Table 1

Figure 2 shows the variation of longitudinal and transverse velocities, for both the axial direction \( x_1 \) and the radial direction \( x_2 \). Circles represent least-square fits of measurement points obtained in [14]. The effective velocities decrease monotonically with increasing porosity \( \phi \). The agreement between the predictions and measurements is best when the oblate-ellipsoid volume fraction \( \phi \) is about 0.07. Except for longitudinal velocity \( \tilde{v}_{11} \), all other measured velocities lie between predictions corresponding to oblate ellipsoids of aspect ratio \( \tau = 0.01 \) and \( \tau = 0.1 \).

From Figure 2, we see a satisfactory agreement for the four velocities over the whole creep life (i.e. wide range of different volume fractions). Therefore, the double composite model is appropriate and is capable of explaining the void-velocity relationships.

**CONCLUSIONS**

We considered a double composite model to explain the void-velocity relationships of materials subjected to the intergranular creep process. As creep progresses spherical voids nucleate and grow preferentially on the grain boundaries perpendicular to the applied stress direction. In our model the voids are clustered and the void volume fraction increased. The clustered regions are modelled by oblate ellip-
The ellipse volume elements are sparsely distributed in the matrix, but they continue to grow in volume, containing more and more voids. This evolution produces an anisotropy increase due to the preferential void formation direction. The double composite model, in which the aspect ratio of oblate-ellipsoidal clusters decreases with the increase of creep damage, gave a satisfactory agreement between the predicted and measured velocities. From the model-measurement comparison, one can estimate the oblate-ellipsoid aspect ratio.

REFERENCES

\begin{align*}
\frac{\gamma^2}{k_T^2} &= \frac{(1 - 3i\phi_0A)\left[1 - 9i\phi_0A\left(\frac{\gamma}{2\mu_0} + 1\right)\right]}{1 + 15i\phi_0A - 9i\phi_0A}\left(\frac{\gamma}{2\mu_0} + 1\right), \\
\frac{\gamma^2}{k_T^2} &= \frac{(9 - 9i\phi_0A)\left[1 - 9i\phi_0A\left(\frac{\gamma}{2\mu_0} + 1\right)\right]}{1 + 6i\phi_0A\left(\frac{9\gamma}{8\mu_0} + 1\right)},
\end{align*}
\tag{A.10, A.11}

which are identical to the results obtained by Caleap et al. [16] if the correlation between scatterers is neglected. Therefore, if the effective mass density is defined as in Eq. (3) and the scattering coefficients \( A \) given by

\begin{align*}
\tilde{A}_i &= \frac{\kappa_0}{\mu_0}, \quad \tilde{A}_j = \frac{1}{9}, \\
\tilde{A}_k &= \frac{4}{3}\frac{\mu_0}{9\kappa_0 + 8\mu_0},
\end{align*}
\tag{A.12}

then the effective bulk and shear moduli are obtained from the relations

\begin{align*}
\kappa_0 &= \frac{4}{3}\mu_0 , \\
\mu_0 &= \frac{\rho_0\omega^2}{\gamma^2},
\end{align*}
\tag{A.13}

as reported in Eqs. (4) and (5).

Note that the effective wavenumbers (A.10) and (A.11) are the same to \( O(\phi_0) \) as those predicted via the Foldy theory (see [17] for an application of this theory). Indeed, the effective wavenumbers are given by

\begin{align*}
\frac{\gamma^2}{k_T^2} &= 1 + \frac{3\phi_0}{\omega^2} A_0^T \left(\frac{\gamma}{\sqrt{\omega^2}}\right), \\
\frac{\gamma^2}{k_T^2} &= 1 + \frac{3\phi_0}{\omega^2} A_0^L \left(\frac{\gamma}{\sqrt{\omega^2}}\right),
\end{align*}
\tag{A.14}

Upon using Eqs. (A.2) and (A.6) together with the fact that

\begin{align*}
A^T \left(\theta\right) &= A^L \left(\theta, 0\right) \cdot e_x, \\
A^T \left(\theta\right) &= A^L \left(\theta, 0\right) \cdot e_z,
\end{align*}

in Eq. (A.14), we find that

\begin{align*}
\frac{\gamma^2}{k_T^2} &= 1 + \phi_0 \left(\sigma + \alpha - 1\right), \\
\frac{\gamma^2}{k_T^2} &= 1 + \phi_0 \left(\sigma - 1\right),
\end{align*}
\tag{A.15}

which are identical to those derived using a pair-correlation function of hole-correction type in Eqs. (A.10) and (A.11), correct to \( O(\phi_0^2) \).

**APPENDIX B**

**Eshelby tensor**

In the case of ellipsoidal inclusions with semi-axes \( a_1 = a_2 = a_3 \), and aspect ratio \( \tau = a_1 / a_3 \), the Eshelby tensor is transversely isotropic, defined by the five components (in Voigt’s notation) \( S_{11}, S_{12}, S_{13}, S_{13} \) and \( S_{15} \) (see [18]). They are given explicitly by

\begin{align*}
S_{11} &= 3A_{11} + B_{11}, \\
S_{12} &= A_{12} + B_{12}, \\
S_{13} &= A_{13} + B_{13}, \\
S_{13} &= \frac{1}{8\pi} \left[\left(1 + \tau^{-2}\right)A_{13} + B\left(I_{13} + I_{13}\right)\right],
\end{align*}
\tag{B.1, B.2, B.3, B.4}

where

\begin{align*}
A &= \frac{1}{8\pi(1 - \nu_0)}, \\
B &= \left(1 - 2\nu_0\right)A,
\end{align*}
\tag{B.4}

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and \( \nu_0 \) is the Poisson ratio of the host medium. We have also defined

\begin{align*}
I_1 &= \frac{2\pi}{1 - \tau^2} \left\{ \begin{array}{ll}
\cos^{-1}\tau, & \tau < 1, \\
\sqrt{1 - \tau^2}, & \tau > 1,
\end{array} \right.
\tag{B.5}
\end{align*}

\begin{align*}
I_3 &= 4\pi - 2I_1, \\
I_{33} &= \frac{4\pi}{3} - \frac{2\tau^2}{3} \left(I_{13} - I_{13}\right) - \frac{2\tau^2}{3} \left(\tau^2 - 1\right),
\tag{B.6}
\end{align*}

\begin{align*}
I_{11} &= \pi - \frac{I_{13} - I_{13}}{4\tau^2 - 1}, \\
I_3 &= \frac{\tau^2}{1 - \tau^2},
\tag{B.7}
\end{align*}

Note that when \( \tau \rightarrow 1 \) (spheres), Eqs. (B.5)-(B.7) reduce to

\begin{align*}
I_1 &= I_3 = \frac{4\pi}{3} \quad \text{and} \quad I_{11} = I_{33} = I_3 = \frac{4\pi}{5},
\tag{B.8}
\end{align*}

**Coefficients \( \beta \)**

We can now define the coefficients \( \beta \) appearing in Eqs. (8)-(10) and (12). One has [12]

\begin{align*}
\beta_{11} &= c_{11}^2(2z_1 + z_4) + c_{11}^2(2z_4 + z_2), \\
\beta_{44} &= c_{11}^2(2z_3 + z_4) + 2c_{14}z_5, \\
\beta_{12} &= c_{12}^2z_2, \quad \beta_{13} = \frac{1}{2}c_{13}^2,
\end{align*}
\tag{B.9, B.10, B.11}

where

\begin{align*}
z_1 &= \frac{z_6}{\Delta}, \quad z_2 = \frac{1}{\Delta}, \quad z_3 = -\frac{z_1}{\Delta}, \quad z_4 = -\frac{z_2}{\Delta}, \\
z_5 &= \frac{4}{\Delta}, \quad z_6 = \frac{2z_3}{\Delta}, \quad \Delta = 2(z_2z_6 - z_3z_5),
\end{align*}
\tag{B.12, B.13}

and

\begin{align*}
c_{i1}^2 &= \frac{(\mu_1 - \mu_0)k_1^2}{\rho_0\omega^2 - \frac{2}{3}c_{i}^2}, \\
c_i^2 &= \frac{(\kappa_i - \kappa_0)k_2^2}{\rho_0\omega^2 - \frac{2}{3}c_i^2}.
\tag{B.14}
\end{align*}

The subscript \( w \) in Eq. (B.14) may refer to either \( L \) or \( T \), corresponding to the longitudinal or transverse waves.

The constants \( Z_j \) in Eqs. (B.12) and (B.13) are given by

\begin{align*}
Z_1 &= \frac{1}{2} \left(Z_{11} + Z_{12} + 2aS_{13}\right), \\
Z_2 &= \frac{1}{2} \left(1 + (a + b)\right) \left[S_{12} + S_{13} + 2aS_{12}\right],
\tag{B.15, B.16}
\end{align*}

\begin{align*}
Z_3 &= \frac{1}{2} \left(1 + (a + b)\right) \left[S_{12} + S_{13} + 2aS_{12}\right],
\tag{B.17}
\end{align*}

\begin{align*}
Z_4 &= \frac{1}{2} \left(1 + (a + b)\right) \left[S_{12} + S_{13} + 2aS_{12}\right],
\tag{B.18}
\end{align*}

where

\begin{align*}
\alpha &= \frac{\Delta S_{11} - \Delta S_{13}}{2}, \quad \beta = \frac{\mu_1 - \mu_0}{\mu_0}.
\tag{B.19}
\end{align*}