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Nonlinear propagation with frequency-independent damping: input-output simulation of entropic solutions

Thomas Hélie (1) and Christophe Vergez (2)

(1) IRCAM, CNRS UMR9912, 1 Pl Igor Stravinsky, 75004 Paris, France
(2) Laboratory of Mechanics and Acoustics, CNRS UPR7051, 31 Ch. Joseph Aiguier, 13402 Marseille Cedex 20, France

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ABSTRACT

We present an exact method to solve a one-dimensional nonlinear transport equation in a dissipative non homogeneous media when the damping is frequency-independent. This work was motivated by the case of brass musical instruments whose functioning at high sound levels implies nonlinear propagation. Though in that latter case, the medium is homogeneous, our approach is more general.

Usually, the wave propagation in musical wind instruments is justifiably considered to be linear. A well-known counter-example is the case of brass instruments at high sound level. In this case, the nonlinear effects become dominant. They account for the graduated waveshape distortion due to their cumulative nature which eventually leads to the arrival of shock-waves.

For the class of propagation models under study in this paper, we derive an exact method which allows to recover an input-output formalism and an efficient algorithm in the time domain. The method is based on three key points: (1) a change of function which turns the original problem into a conservative problem of hyperbolic type, (2) the adaptation of the standard "characteristics method" from which all possible solutions can be deduced, and (3) the introduction of an easily computable criterion which naturally selects the "physically meaningful" solution (this latter point provides a generalization of the "potential function" proposed by Hayes ([(1)], see also [(2)]). This approach operates for regular and continuous solutions as well as shocks and multiple shocks. Finally, a fast algorithm is deduced and proposed for real-time sound synthesis issues.

INTRODUCTION

In practice, input-output representations used in system theory and control engineering techniques are well-suited to real-time simulations and sound synthesis purposes (e.g. as digital waveguide techniques): in the linear case, efficient algorithms in the time domain can be deduced from the study of transfer functions (and possibly basic approximations or using standard model order reduction techniques). Deriving such representations prove to be difficult in nonlinear cases (for which the solution existence is even not always guaranteed).

It is shown in this paper that it is possible to find such a formulation when the original problem is a 1D nonlinear transport equation with frequency independent losses in a non homogeneous medium. After problem statement (section "problem statement"), it is shown how the model can be reformulated through a change of variable as a conservative equation. Strong solutions are obtained through an adapted version of the method of characteristics (section "strong solutions"). As soon as strong solutions become multivalued, weak solutions must be considered, but they are not unique. A criterion is introduced to select a unique solution (section "weak solutions"). In the example considered here (the nonlinear propagation of acoustic waves), it corresponds precisely to the potential of Hayes (1, 2)) that identifies the "physically meaningful" solution branches (in the sense of an entropy criterion). This method allows to construct a single-valued solution, even in the case of multiple shocks. This is examplified for simulation and sound synthesis purposes.

PROBLEM STATEMENT

Consider the 1D propagation in the domain $\Omega =]0, X[$ where $X > 0$ described by, for all $(x,t) \in \Omega \times \mathbb{R}_+^*$,

$$\partial_x p(x,t) + \frac{1}{c\big(x, p(x,t)\big)} \partial_t p(x,t) + \alpha(x)\, p(x,t) = 0, \qquad (1)$$

with null initial conditions

$$\forall x \in \Omega, \quad p(x,0) = 0,$$

and a Dirichlet boundary condition at $x = 0$

$$\forall t \in \mathbb{R}, \quad p(0,t) = p_0(t),$$

where $p_0 \in \mathscr{C}^1(\mathbb{R})$ is supposed to be zero on $\mathbb{R}^-$.

Function $(x,p) \mapsto c(x,p)$ is assumed to be continuous on $\overline{\Omega} \times \mathbb{R}$, to have a continuous derivative w.r.t. p and to be such that $c(0,0) > 0$. Function $x \mapsto \alpha(x)$ is supposed to be continuous and positive on $\overline{\Omega}$.

This model governs a traveling wave which propagates at local celerity $c\big(x, p(x,t)\big)$ and is subjected to a damping depending on the position x but not the frequency.

In this article, we seek solutions of smoothness $\mathscr{C}^1$ (sec. STRONG SOLUTIONS) and weak entropic solutions when shocks occur (sec. WEAK SOLUTIONS), under the following condition

$$\exists c^* > 0 \,\big|\, c\big(x, p(x,t)\big) \geq c^* \text{ nearly everywhere in } \overline{\Omega} \times \mathbb{R}. \qquad (2)$$

which guarantees that waves propagate in the direction of positive x at a celerity larger than c^* and that the causality principle is satisfied. The intention is then to construct an efficient numerical solver that is compatible with real-time applications and that computes signal $p(x,t)$ from the input signal $p_0(t)$, at a fixed location x.

Nondimensionalization A dimensionless version of this problem is obtained by applying the change of variable below:

$$(t,x) \to (\widetilde{x},\widetilde{t}) = \big(x/X,\, c(0,0)t/X\big)$$

The problem is still described by previous equations, replacing quantities by their versions denoted with a tilde given by:

$$\begin{aligned}
\widetilde{p}(\widetilde{x},\widetilde{t}) &= p\big(X\widetilde{x}, \widetilde{t}X/c(0,0)\big),\\
\widetilde{\alpha}(\widetilde{x}) &= X\alpha(X\widetilde{x}),\\
\widetilde{c}\big(\widetilde{x},P\big) &= c\big(X\widetilde{x},P\big)/c(0,0)\\
\widetilde{\Omega} &=]0,1[.
\end{aligned}$$

In the sequel, symbols "tilde" are omitted ($X=1$, $c(0,0)=1$).

STRONG SOLUTIONS

This section provides basic results in the case where the solution is $\mathscr{C}^1$-regular ($\mathscr{C}^1(\mathbb{E},\mathbb{F})$ denotes the standard set of continuous functions with a continuous derivative, from $\mathbb{E}$ to $\mathbb{F}$). First, a change of function is introduced so that the problem becomes conservative (Theorem 1). Second, the standard method of characteristics is adapted to the case of problems with coefficients varying w.r.t. the space variable (Theorem 2).

Introduce the decreasing function $A \in \mathscr{C}^1\big(\Omega,]0,1]\big)$

$$A(x) = \exp -\int_0^x \alpha(\xi)\,\mathrm{d}\xi. \tag{3}$$

Let $c^* \in \mathbb{R}_+^*$ and $p_0 \in \mathscr{C}^1\big(\mathbb{R},\mathbb{R}\big)$ be such that $\forall t \in \mathbb{R}_-$, $p_0(t) = 0$ and $\forall (x,t)\overline{\Omega}\times\mathbb{R}$, $c\big(x,A(x)p_0(t)\big) \geq c^*$. Then, the following result holds.

Theorem 1 *If $q \in \mathscr{C}^1\big(\overline{\Omega}\times\mathbb{R}\big)$ is a solution of*

$$\forall (x,t) \in \Omega\times\mathbb{R}, \quad \partial_x q(x,t) + \frac{1}{c\big(x,A(x)q(x,t)\big)}\partial_t q(x,t) = 0, \tag{4}$$

$$\forall t \in \mathbb{R}, \quad q(0,t) = p_0(t), \tag{5}$$

then $p : (x,t) \in \overline{\Omega}\times\mathbb{R}_+ \mapsto A(x)q(x,t)$ belongs to $\mathscr{C}^1\big(\overline{\Omega}\times\mathbb{R},\mathbb{R}\big)$ and is a solution of the original problem. The converse is also true.

The proof is straightforward.

Definition 1 (Characteristics) *Let K be defined by*

$$\begin{array}{rccc} K: & \overline{\Omega}\times\mathbb{R} & \longrightarrow & \overline{\Omega}\times\mathbb{R}\\ & (x,t) & \longmapsto & \big(x,T(x,t)\big)\end{array} \tag{6}$$

where, for all $(x,t) \in \overline{\Omega}\times\mathbb{R}$,

$$T(x,t) = t + \int_0^x \frac{1}{c\big(y,A(y)p_0(t)\big)}\,\mathrm{d}y \tag{7}$$

Properties 1 *Functions T and K are such that:*

(i) *T and K are $\mathscr{C}^1$-regular functions;*
(ii) *$\forall (x,t) \in \overline{\Omega}\times\mathbb{R}$, $T(x,t) \geq t$;*
(iii) *K is surjective;*
(iv) *If $\forall (x,t) \in \overline{\Omega}\times\mathbb{R}$, $\partial_t T(x,t) > 0$, then K is a $\mathscr{C}^1$-regular diffeomorphism.*

Proof (i) is straightforward since p_0 is $\mathscr{C}^1$-regular, A is continuous and $(x,t) \mapsto 1/c(x,A(x)p_0(t))$ is bounded and continuous ($c^* > 0$).
(ii) is a straightforward consequence of the positivity of $c\big(y,A(y)p_0(t)\big)$ in (7).
Concerning (iii), for all $t \leq 0$, $p_0(t) = 0$ so that for all $x \in \overline{\Omega}$, $T(x,t) = t + T(x,0) \underset{t\to-\infty}{\longrightarrow} -\infty$. Moreover, from (ii), $T(x,t) \underset{t\to+\infty}{\longrightarrow} +\infty$ for all $x \in \overline{\Omega}$. Therefore, (iii) is satisfied because T is continuous.
Concerning (iv), if $\forall (x,t) \in \overline{\Omega}\times\mathbb{R}$, $\partial_t T(x,t) > 0$, then for all $x \in \overline{\Omega}$, $t \in \mathbb{R} \mapsto T(x,t)$ is injective since it is a strictly increasing function. So in subsequent to (i) and (iii), K is $\mathscr{C}^1$-regular and bijective. The Jacobian matrix $J(x,t) = \begin{pmatrix} 1 & 0\\ \partial_x T(x,t) & \partial_t T(x,t)\end{pmatrix}$ of K is continuous and its determinant $\det J(x,t) = \partial_t T(x,t) > 0$ is invertible, that concludes the proof. ◇

Theorem 2 (Strong solution) *Suppose that $\forall (x,t) \in \overline{\Omega}\times\mathbb{R}$, $\partial_t T(x,t) > 0$. Define the $\mathscr{C}^1$-regular function $\tau : (x,\theta) \in \overline{\Omega}\times\mathbb{R} \mapsto [K^{-1}(x,\theta)]_2 \in \mathbb{R}$. Then,*

$$Q = p_0 \circ \tau \in \mathscr{C}^1\big(\overline{\Omega}\times\mathbb{R},\mathbb{R}\big)$$

is a strong solution of (4-5).

Proof From property 1 (iv), τ and Q are $\mathscr{C}^1$-regular and

$$\forall (x,t) \in \overline{\Omega}\times\mathbb{R},\ T\big(x,\tau(x,t)\big) = t. \tag{8}$$

Moreover, computing the left and the right hand sides of $\dfrac{\partial_x\tau(x,t)\,\partial_t\big[(8)\big] - \partial_t\tau(x,t)\partial_x\big[(8)\big]}{D_2T\big(x,\tau(x,t)\big)}$ where $D_2T\big(x,\tau(x,t)\big) > 0$ yields

$$-D_1T\big(x,\tau(x,t)\big)\partial_t\tau(x,t) = \partial_x\tau(x,t). \tag{9}$$

Now, $\partial_x Q(x,t) + \frac{1}{c\left(x,A(x)\,Q(x,t)\right)}\partial_t Q(x,t) = [p_0'\circ\tau](x,t)\Big(\partial_x\tau(x,t) + \frac{1}{c\left(x,A(x)\,Q(x,t)\right)}\partial_t\tau(x,t)\Big)$ is zero from (9) and since, from (7), $D_1T\big(x,\tau(x,t)\big) = \frac{1}{c\left(x,A(x)\,Q(x,t)\right)}$. Finally, Q is a solution of (4), that concludes the proof. ◇

Therefore, in the case where the solution is strong, theorems 1 and 2 provide a solution to the original problem, given by

$$p(x,t) = A(x)\,p_0\big(\tau(x,t)\big) \tag{10}$$

The existence of such a solution is conditioned by that of τ, that is, by the fact that $\partial_t T > 0$ to ensure the bijectivity of K.

APPLICATION: FAST INPUT-OUTPUT ALGORITHM

Consider the simple case for which both the damping and the celerity do not depend on the space variable (homogeneous medium) and are characterized by

$$\alpha(x) = \alpha, \qquad c(x,p(x,t)) = 1/(1-p(x,t)). \tag{11}$$

This celerity corresponds to a nonlinear acoustic model used to represent the wave propagation in some musical wind instruments at fortissimo nuances [(3)]. For this application, $A(x) = \exp(-\alpha x)$ (see (3)) and function T is given by (see (7))

$$T(x,t) = t + x - E_\alpha(x)\,q(0,t) \text{ with } E_\alpha(x) = \frac{1-e^{-\alpha x}}{\alpha}. \tag{12}$$

These characteristics are plotted in figure 1 for input signal $q(0,t) = p_0(t) = -0.3\sin(8\pi t)$ and $\alpha = 1$. Because of the damping, in accordance with (12), characteristics in the (x,T)-plane are not straight lines. Strong solutions are then constructed

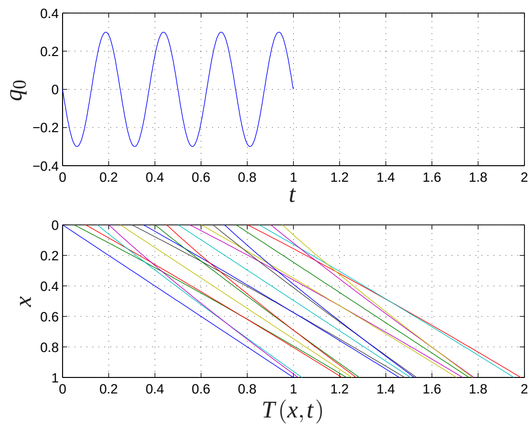

Figure 1: Top: input signal (at $x = 0$). Bottom: characteristics in the $(T(x,t),x)$ plane, given by (6) and (7).

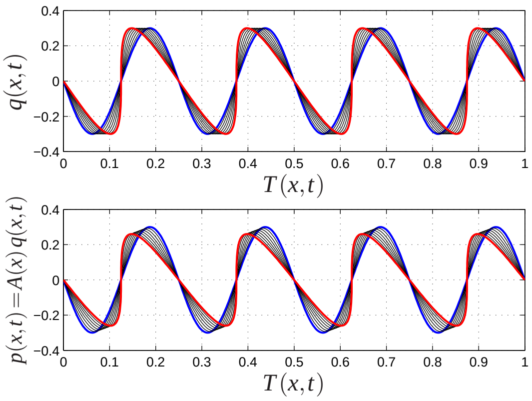

Figure 2: Strong solutions computed at $x_n = \frac{n}{N}x^\star$ for $n = 0$ (blue), $1 \leq n \leq N-1$ (black) and $n = N = 7$ (red), where $x^\star \approx 0.1414$ is the critical shock distance. Top: $q(x,t)$ solution of (4-5). Bottom: $p(x,t) = A(x)q(x,t)$ solution of (1).

using (10) and theorem 2. They are presented in figure 2 for various distances x.

The top picture exhibits the distortion of the sinusoidal waveform $q(x,t)$ that is expected for this type of non-linearity [(4, 6)]. In the bottom picture, the damping effect is included, according to theorem 1 and equation (10).

From theorem 2, the validity of this construction is conditioned by $\partial_t T > 0$ which ensures bijectivity of K. Graphically, the validity limit is reached as soon as characteristics intersect (figure 1, bottom).

A precise analysis of characteristics and figure 2 reveals that the first intersection corresponds to $x = x^\star \approx 0.1414$. Beyond this limit, no strong solutions are available: the use of characteristics leads to a multi-valued solution $q(x,t)$.

WEAK SOLUTIONS

Intersections of characteristics means that distinct quantities are carried in the same place at the same time (multi-valued solution). It corresponds to the appearance of a shock (discontinuous solution, called weak solution). In this case, from the mathematical point of view, the problem must be solved in the sense of distributions. It provides several (mono-valued but discontinuous) solutions. Only one of them is compatible with the entropy principle [(5)].

A worthwhile solution proposed by Hayes [(1)] (without proof) has been recently studied by Coulouvrat [(2)]. The method is based on an *ad hoc* functional called "potential" in [(1, 2)]. A generalization of this method, adapted to problem (4), is proposed below.

Preliminary Results

Definition 2 (Functions ϕ and Φ^+) *For all $x > 0$, $t \in \mathbb{R}$, denote $T_0(x,t) = T(x,t)\big|_{p_0=0}$ and define*

$$\phi(x,t) = \int_{-\infty}^{t} \frac{T_0(x,t') - T(x,t')}{x} \partial_{t'} T(x,t')\,\mathrm{d}t', \quad (13)$$

$$\Phi^+(x,\theta) = \sup_{t \in \mathbb{T}(x,\theta)} \phi(x,t), \quad (14)$$

where $\mathbb{T}(x,\theta) = \{t \in \mathbb{R} \,|\, T(x,t) = \theta\}$.

Note that, for all $t \leq 0$, $T_0(x,t) = T(x,t)$ so that $\phi(x,t)$ is zero if $t \leq 0$.

Properties 2 *The following results hold:*

(i) *ϕ is $\mathscr{C}^1$-regular.*
(ii) *Sets $\mathbb{T}(x,\theta)$ are non empty and closed.*
(iii) *If a and b belong to $\mathbb{T}(x,\theta)$, then*

$$x\big[\phi(x,b) - \phi(x,a)\big] + \int_a^b T(x,t)\,\mathrm{d}t = \theta\,(b-a),$$

(iv) *Function Φ^+ is continuous w.r.t. θ.*

Proof (i) follows from property 1(i).
(ii) $\mathbb{T}(x,\theta) = \widetilde{T}_x^{-1} < \{\theta\} >$ is closed because it is the inverse image of the closed set $\{\theta\}$ for the continuous function $\widetilde{T}_x : t \mapsto T(x,t)$. It is non empty because $\widetilde{T} < \mathbb{R} >= \mathbb{R}$ (See proof of property 1 (ii)).
(iii) is a consequence of (13).
(iv) The proof sketched below is based on the construction of the parametrized curve $t \mapsto \big(T(x,t), \phi(x,t)\big)$ for a fixed x (see the curves in top right of figures 3 and 4).

In areas where $\mathbb{T}$ is a singleton, the continuity of Φ^+ is ensured by that of ϕ and T. For complementary areas, the first step consists of proving that function $\psi_t^+ : \Omega \times]-\infty, \theta^+(t)] \to \mathbb{R}$ where

$$\psi_t^+(x,\theta) = \sup_{a \in \mathbb{T}_t(x,\theta)} \phi(a) \text{ with } \mathbb{T}_t(x,\theta) = \mathbb{T}(x,\theta) \cap]-\infty, t],$$

and $\theta^+(t) = \sup_{a \leq t} T(x,a)$, is continuous on $]-\infty, T(x,t)]$. This result comes from (iii) which yields the following properties: for all $x > 0$, $(a,b) \in \big[\mathbb{T}(x,\theta)\big]^2$ such that $a < b$,

(A) if $T(x,t) < \theta$ for all $t \in]a,b[$, then $\phi(x,b) > \phi(x,a)$,
(B) if $T(x,t) = \theta$ for all $t \in [a,b]$ then $\phi(x,b) = \phi(x,a) \Big(= \phi(x,t)\Big)$,
(C) if $T(x,t) > \theta$ for all $t \in]a,b[$ then $\phi(x,b) < \phi(x,a)$.

Then, we partition each connected sets into ordered sub-intervals $[a_k, b_k]$ of type (A,B,C), possibly completed by intervals $[b_k, a_{k+1}]$ if $a_{k+1} \neq b_k$. The conclusion is obtained by remarking that $T(x,t) \geq t$ and $\theta^+(x,t) \geq t$ so that ψ_t^+ and Φ^+ coincide on $]-\infty, t]$. Thus, the continuous function $\psi_t^+\big|_{]-\infty,t]}$ reconstructs Φ^+ when $t \to +\infty$. $\diamond$

Remark 1 *The type (A) corresponds to a "negative branch" ($\partial_t T \leq 0$) first, and then, a "positive branch". The type (C) corresponds to a "positive branch" first, and then, a "negative branch".*

Theorem 3 (Selection of a weak solution) *The function defined by $\hat{q} = p_0 \circ \hat{\tau}$ with*

$$\hat{\tau}(x,\theta) = \sup\{t \in \mathbb{T}(x,\theta) \,|\, \phi(x,t) = \Phi^+(x,\theta)\}$$

provides a unique solution of (4-5) in the sense of distributions.

The uniqueness of the selection made by $\hat{\tau}$ is obvious and it can be easily checked that q satisfies the Rankine-Hugoniot's condition at the locations of discontinuities.

Application and link with the Hayes method

For the application presented in section APPLICATION, we find that

$$\phi(x,t) = \frac{E_\alpha(x)}{x}\left[\int_0^t p_0(t)\,\mathrm{d}t - \frac{1}{2}E_\alpha(x)\big(p_0(t)\big)^2\right]. \qquad (15)$$

In the case where $\alpha \to 0^+$ (no damping), then $E_\alpha(x) \to x$ and (15) coincides with the "Hayes potential" [(1)] (see also (9) in [(2)] where the role of the space y and time t are exchanged):

$$\phi_H(x,t) = \int_{-\infty}^t p_0(t')\partial_{t'}T(x,t')\,\mathrm{d}t'. \qquad (16)$$

The solutions which are selected by $\hat{\tau}$ and by the Hayes potential are identical. But although ϕ_H and ϕ coincide in this case, this correspondence is lost in general. Moreover, equations (13) and (16) cannot be interpreted in the same way.

Thus, in (13), $(T_0 - T)/x$ represents the difference (averaged per unit length) between the travel duration (from 0 to x) of a zero quantity (reference) and a non zero one (p_0). This difference is is integrated w.r.t. $\partial_{t'}T(x,t')\,\mathrm{d}t'$, that is, the measure of the "output time" T (for a fixed x and as far as T is bijective). The selection made by Φ^+ and $\hat{\tau}$ corresponds to retain maximum parts of a quantity exclusively derived from time transport information. On the contrary, in (16), the interpretation is no longer exclusively based on time information and property 2(iii) is also lost in general.

It has been proved that the Hayes potential provides the entropic solution for this particular application but not in the general case [(2)]. The question remains open for selection by Φ^+ and $\hat{\tau}$.

RESULTS

Figure 3 represents the wave signal at $x = x^\star$. The left bottom picture exhibits the time derivative of the original signal $q(0,t)$ (recalled in top left) and the vertical limit (red) marking the shock onset ($dq(0,t)/dt = \alpha/(1-e^{-\alpha x})$). It shows that $x = x^\star \approx 0.1414$ corresponds to the validity limit of the strong solutions, as suggested by 1. This information is also displayed in the bottom right picture which shows that at certain points $\partial_t T$ approaches 0 (but does not reach). The top right picture represents ϕ defined in (13) plotted w.r.t. $T(x,t)$. Again, the fact that $\phi(x,T(x,t))$ is mono-valued shows that the solution is strong. Thus, subsequent to (13):

$$\Phi^+(x,\theta) = \phi(x,t) \qquad \forall t \in \mathbb{T}(x,\theta), \qquad (17)$$

and each point of the input signal $q(0,t)$ appears in the signal $q(x,T(x,t))$. Makers $\circ$ (green) in the subplots are in correspondence.

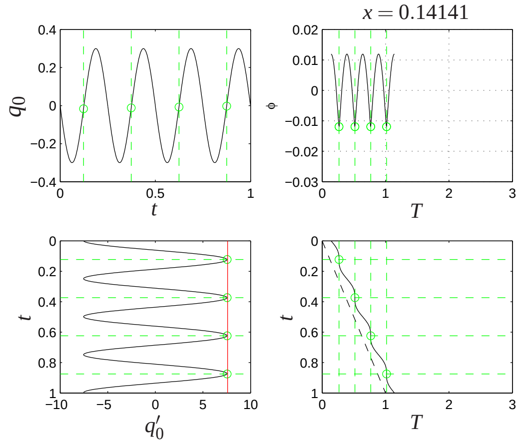

Figure 3: Top left picture: input signal $p_0(t) = q(0,t)$. Top right picture: $\Phi(x^\star,t)$ defined by (13). Bottom left picture: time derivative of the input signal $q(0,t)$ and limit (vertical red line), the non crossing of which signs the absence of shock before $x = x^\star$. Bottom right picture: influence of nonlinear propagation on arrival times at $x = x^\star$ (the dotted black line corresponds to the reference $t = T$).

The situation is different for a propagation on a longer distance. This is illustrated in figure 4, which is comparable to figure 3 but at $x = 1$. On the bottom left picture $\frac{dq(0,t)}{dt}$ exceeds the validity limit of the strong solutions. Non-admissible points (in red) are those for which $dq(0,t)/dt > \alpha/(1-e^{-\alpha x})$ or, equivalently, $\partial_t T < 0$. The top right picture exhibits a multi-valued

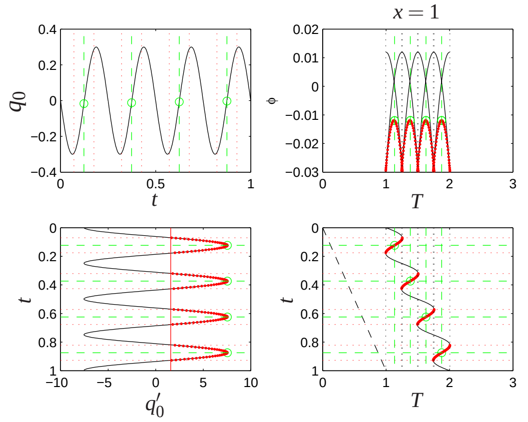

Figure 4: Same quantities as in figure 3, but at $x = 1$. Non-admissible points (in red) are those for which $dq(0,t)/dt > \alpha/(1-e^{-\alpha x})$ or, equivalently, $\partial_t T < 0$ so that they are involved in one shock at least.

potential and shows that $\Phi^+ \neq \phi$. The (weak) solution at $x = 1$ is built according to theorem 3.

The result is presented in figure 5, for $x = nx^\star$, $0 \leq n \leq 7$. On the top picture, the plots of $q(x,t)$ show more and more sheer shocks, until a N-wave appear (see e.g. [(4)]). On the bottom picture, the plots of $p(x,t) = A(x)q(x,t)$ show the solution of the original problem for which the amplitude decreases because of the damping. The waveform is preserved since the damping does not depend on the frequency.

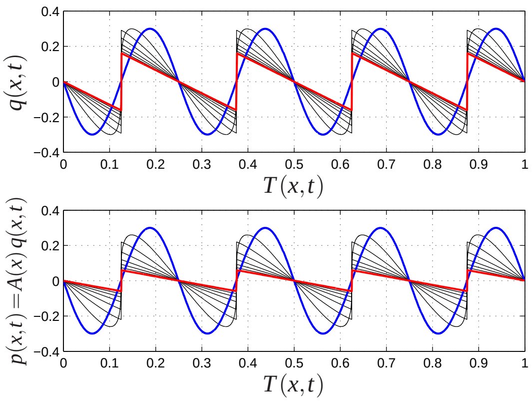

Figure 5: Strong and weak solutions for several distances $x = nx^{\star}$, $0 \leq n \leq 7$. Top picture: $q(x,t)$ solution of (4-5). Bottom picture: $p(x,t) = A(x)\,q(x,t)$ solution of (1).

CONCLUSION

A first advantage of the approach presented in this paper is to separate during the propagation, effects related to damping (independent of frequency) and those related to the nonlinear transport. Moreover, the introduction of a functional allows to treat the problem as a nonlinear input/output problem, even when shocks occur. This functional makes the method of characteristics still usable even for weak solutions. Moreover, the output signal processing is not more expensive in the case of multiple shocks than in the case of single shocks. This property is interesting to built real-time simulations of brass instrument sounds at high sound level. It will be exploited to extend some known sound synthesis algorithms which are currently limited to a lossless propagation and strong solutions as in [(8)], or to weak solutions corresponding to symmetric shocks as in [(7)], or to strong solutions with visco-thermal losses as in [(9)].

The work presented in this article has focused on a particular model of nonlinear transport, but the objective is to prove that any type of equation (1) can be treated in the same formalism.

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