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Theoretical Framework for Initial Transient and Steady-State Frequency Amplitudes of Musical Instruments as Coupled Subsystems

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ABSTRACT

A theoretical framework for initial transients of musical instruments is presented for string-body, reed-air column and bow-string interactions. The time dependent amplitudes of the frequencies present in the two coupled systems are written as sums of phase relations of complex amplitudes of both, the system itself and the coupled system. For the steady-state parts some terms cancel because of their limits summing to zero explaining the take-over of one system by the other, like the forced oscillation of a violin body by its string frequencies. For the initial transients the equation systems consist of additional terms explaining the complex time-dependent behaviour of the initial transients. These terms formulate the temporal development of the amplitudes of additional frequencies during the transients. This enables a simple and real-time simulation of the transient time series. Additionally, two main reasons are found for the slaving of one system by the other in the discussion of zero limit sums of the steady-state phase, which are the difference in damping of the two systems and the difference in dimensionality. So e.g. the one-dimensional string with only two reflection points and low damping is much more capable to force the body to the strings frequencies as the three-dimensional body with strong damping and a complex geometry with multiple wave reflection points.

INTRODUCTION

Many musical instruments consist of at least two coupled oscillating systems, e.g. string - body, bow - string, or reed - air column interactions. Under normal playing condition, one system takes over the other, with guitars the string tells the body how to sound, with reed instruments it is the air column determining the reed frequency ect. This is normally seen as a generator - resonator coupling. Still, with some instruments the generator tells the resonator how to vibrate (like with guitars) while with others, the resonator dominates the generator (like with reed instruments). Also, the important initial transient phase of musical sounds is determined by the initial struggling of the subsystems until one wins the game. This paper tries to develop a theoretical framework for this coupling to aim at a description of the initial transient phase as well as explaining the steady-state take over of one system by the other.

The basic idea of this framework is taken from Synergetics (Haken 1983). In its most simple mathematical formulation

$$\dot{f}_1 = \alpha f_1 + k_1 f_2^3 \qquad (1)$$

$$\dot{f}_2 = \beta f_2 + k_2 f_1 \qquad (2)$$

the temporal change $\dot{f}_{1,2}$ of the time series $f_{1,2}$ depend on the time series themselves with dampings α and β and on the other time series linear or nonlinearly with coupling strengths $k_{1,2}$. The astonishing emergent behaviour of systems described by a temporal development of only first order derivative is explained there by both, the nonlinear coupling and the different damping. So when $\alpha \gg \beta$ then

$$\dot{f}_1 = \alpha f_1 + k_1 f_2^3 \qquad (3)$$

$$\dot{f}_2 = k_2 f_1 \ , \qquad (4)$$

and f_2 only depends on f_1. Note, that the nonlinearity is not necessary for the reasoning of taking over (or synchronizing) of one system by the other. Still, nonlinearities can play a crucial role in other cases.

As vibrating systems are normally described by the wave equation of second order or through bending motion of fourth order derivatives, the Synergetic formulation above was not often used in the field of vibrational theory. Indeed, a second order derivative shows complex behaviour easily, while a first-order one as a reaction-diffusion type normally only shows exponential decay.

Still, musical sounds consist of partials (steady or changing) with amplitudes changing in time. This temporal change of the amplitude of one paratial depends upon itself with a certain damping and upon energy support from another system with a coupling strength. From this standpoint, the Synergetic equation system perfectly fits the problem and we want to discuss it in the sections below with three examples, a string - body, a bow - string and a reed - air column interaction.

In the literature the problem has been discussed from several standpoints. Aschoff was wondering, why with the saxophone the resonator dominates the generator (Aschoff 1936) and finds the reason in the damping of the reed being much stronger than that of the air. Woodhouse formulates the bow-string interaction as an example of chaos theory with the strong nonlinearity of the stick - slip interaction (Woodhouse 1995). Fletcher discusses mode locking of harmonic musical systems (air column, strings), where because of a complex geometry slight deviations from a perfect harmonicity would be expected but where instead a perfect harmonic spectrum is found. Abel discusses locking of close frequencies through air with organ pipes (Abel et al. 2006), as Trendelenburg had found before (Trendeleburg et al. 1938, 'Mitnahmeeffekt'). A huge body of literature about nonlinear behaviour of musical instruments is known, too, which is beyond the scope of this paper. Indeed nonlinearities are not always of crucial importance when it comes to the slaving of one subsystem by the other (Aschoffs explanation is linear, the organ pipe 'Mitnahmeeffekt' is linear, too). So the theory of forced oscillation (see e.g. Fahy 2007) will serve in some cases. So this paper tries to examine how far we can get with a synergetic formulation, especially in terms of initial transients.

STRING - BODY INTERACTION

Stringed instruments like guitars, violins, pianos, ect. are built of a generator, the string(s) and a resonator, a soundboard or instrument body. Indeed, this naming convention is only the result of a complex interaction between those two parts resulting in the string taking over the body to vibrate with the strings frequencies. So the string takes the body over, which is normally denoted as a resonance phaenomenon. But still the body does also act on the string. If a Flamenco guitarist is playing a toques, a knocking on the guitar body, the body would be the generator and the strings could be the resonators. Indeed, after the knocking sound is over, we hear, that the strings have been driven by the knocking which we can clearly hear. But this means that again the strings, driven by the body are then vibrating with their eigenfrequencies. Furthermore, they are then forcing the body to move with their frequencies, otherwise we would not hear the strings, they themselves do not have enough radiation area to be heard, the sound we hear is the vibrating body.

So why is the string always taking the body over, forcing it to go with the strings' frequency? To answer this question, we first need to look at the details of the string - body interaction.

- We assume a linear force - force interaction between the string and the body
- Both linear systems are coupled via a point on the body and a point on the string.
- The string has a damping α which is much less than the damping β of the wooden body.
- The string is one-dimensional.
- The body is three dimensional.

When formulating the system in terms of a simplified equation system we can write

$$\dot{A}_s = \alpha A_s + k_1 A_b$$
$$\dot{A}_b = \beta A_b + k_2 A_s \ .$$

Here, $A_s(f)$ and $A_b(f)$ are the amplitudes (or energies) of one frequency f on the string and body respectively. This is changed in time, denoted by a dot representing a first order derivative in time, because of a damping α and β within each system and because of the other system via the point - point connection of the string on the body with coupling constants k_1 and k_2. Note, that the damping constants α and β are used as the strength of the system to act upon itself, so a high damping means small values of these parameters.

Indeed, to make the system complete we would need to include the energy transfer to the other system on the causing system as energy loss like

$$\dot{A}_s = \alpha A_s + k_1 A_b - k_2 A_s$$
$$\dot{A}_b = \beta A_b + k_2 A_s - k_1 A_b \ .$$

To avoid confusion with too many terms we avoid this in the following. It is not effecting the discussion about the coupling. Still we need to insert it in the end again for completion.

The most easiest way to make one system take over the other is if

$$\alpha \gg \beta \ . \tag{5}$$

Then we have

$$\dot{A}_s = \alpha A_s + k_1 A_b$$
$$\dot{A}_b = k_2 A_s \ ,$$

and the change of the amplitude of the body is only determined by the string. Therefore, if one frequency is not present in the string than it is not in the body. On the other hand, each frequency which is in the string will be present on the body. Then also the energy transfer from the body to the string via k_1 is only present with frequencies already in the string. The string has taken over the body completely.

As with stringed instrument the condition above always holds, we have a first, rough but very intuitive and simple reasoning for strings forcing the body and not vice versa.

Let us make the discussion more elaborate by changing to the spatial domain and first assume the body to be a plate, furthermore, if it is endless, without boundaries, and we have plucked the string, then the string is 'knocking' on the body resulting in a travelling wave going into all directions around the driving point. This is a free-field condition, the plate has a continues resonance spectrum including all possible frequencies. Then we can write

$$\dot{A}_s = \alpha A_s$$
$$\dot{A}_b = k_2 A_s \ .$$

As there are no reflections of the outgoing wave on the plate back to the driving point, the second term of the string equation vanishes. The equation system is most simple with a string vibrating on its own and the endless plate going with the string frequencies.

Now if we add boundaries to the plate we get reflecting walls which send energy back to the plate. From here on we need to include the phase relations of the waves going out from the driving point and coming back like we would do when using the d'Alambert solution for the string, where two travelling waves going into opposite directions form a standing wave. The equation system then is

$$\dot{A^*}_s = \sum e^{-1/\alpha t_i} A_i^* + k_1 A_b^*$$
$$\dot{A^*}_b = \sum_{i=1}^{N} e^{-1/\beta t_i} A_i^* + k_2 A_s^* \ .$$

Here we sum over all reflecting waves with complex amplitudes A_i^* and different times t_i determined by the different lengths the waves have to travel. The exponential term decays the wave according to the damping of the plate and the time the wave travelled. The same holds for the string with the restriction that we know that only two reflection length can occur. Still each new wave acting on the string will travel on it as long as it is decayed and so we need to sum all over them, too. As both equations also show a dependency upon the other via the coupling parameters k_1 and k_2, we need to determine how those interact with the reflected waves on the geometries.

We start with examining the first term of the body equation and find that for arbitrary complex phases of A_i^*

$$\lim_{N\to\infty}\sum_{n=i}^{N} e^{-1/\beta t_i} A_i^* = 0 \ . \qquad (6)$$

The reason is that for each amplitude A_i^* with a certain phase an amplitude A_j^* with an opposite phase exists. For such a case the coupled equation system would read

$$\dot{A^*}_s = \sum e^{-1/\alpha t_i} A_i^* + k_1 A_b^*$$
$$\dot{A^*}_b = k_2 A_s^* \ ,$$

and we would arrive at the same situation as with the free-field case. The string would take over the body completely.

If we restrict the amount of reflections on the body to a finite number N we find that the cancellation is not complete and eigenfrequencies with associated eigenmodes exist on the body as is the case with real instrument bodies.

Before examining this further we first have a look at the other case of the string with only two reflections from both ends of the string. So instead of the arbitrary sum case we can write

$$\lim_{N\to\infty}\sum_{i=1}^{N} e^{-1/\alpha t_i} A(f)_s^* e^{\iota\Delta t i} = \begin{cases} = 0 \ \text{for} f \neq r f_0 \\ = A_{max} \ \text{for} f = r f_0 \end{cases} \qquad (7)$$

$$\text{with r = 1,2,3...} \qquad (8)$$

Here f_0 is the fundamental frequency of the string and Δt is the time interval once around the string. At first, both results seem to be the same, the first term of each equation vanishes. Still there are two differences.

First, the reason for the vanishing of the body term is because of multiple reflections spatially within a short time span, while the vanishing of the string term is because of multiple reflections over a long time span as only two reflection points are present.

Secondly, the damping of the body is much more than that of the string and so each new impulse onto the body is much stronger than the waves already travelling there. In the string case with lower damping, the new waves meet others still strong.

From this discussion, in the string equation we can include the coupling term into the limit discussion like

$$\lim_{N\to\infty}\sum_{i=1}^{N} e^{-1/\alpha t_i} A(f)_s^* e^{\iota\Delta t i} + k_1 A_b^* = \begin{cases} = 0 \ \text{for} f \neq r f_0 \\ = A_{max} \ \text{for} f = r f_0 \end{cases} \qquad (9)$$

$$\text{with r = 1,2,3... .} \qquad (10)$$

We cannot do this with the body equation that easily as each new and strong impact of the string onto the body does not find a strong wave on the body cancelling it out. The cancellation of waves in the body is much more short-term because of both, the large damping and the complex geometry with many reflection points.

Now we can take up the discussion of the body again and find a similar behaviour for the eigenvalues of the body like

$$\lim_{N\to\infty}\sum_{i=1}^{N} e^{-1/\alpha t_i} A(f)_b^* e^{\iota\Delta t_{body} i} = \begin{cases} = 0 \ \text{for} f \neq f_{body} \\ = A_{max} \ \text{for} f = f_{body} \end{cases} \qquad (11)$$

$$\text{with r = 1,2,3... .} \qquad (12)$$

So the eigenvalues f_{body} and their periodicities $\Delta t_{body} = 1/f_{body}$ may be present in the body spectrum but much more short living as they are damped much stronger. Note that we do not include the string coupling to the limit sum as this is too strong to cancel out with the body reflections.

So in the end we have the equation system two times, one for $N \to \infty$, the steady-state of a tone

$$\dot{A^*}_s = \alpha A_s^*(r f_0)$$
$$\dot{A^*}_b = \sum_{i=1}^{N} e^{-1/\alpha t_i} A(f)_b^* e^{\iota\Delta t_{body} i} + k_2 A_s^*(r f_0) \ ,$$

and one for the case of small N, the initial transient phase

$$\dot{A^*}_s = \sum_{i=1}^{N} e^{-1/\alpha t_i} A_s^*(f) e^{\iota\Delta t i} + k_1 A_b^*$$
$$\dot{A^*}_b = \sum_{i=1}^{N} e^{-1/\beta t_i} A_i^* + \sum_{i=1}^{N} e^{-1/\alpha t_i} A(f)_b^* e^{\iota\Delta t_{body} i} + k_2 A_s^* \ .$$

The first equation system states, that the string determines the vibration of the body, while not be effected vice versa. The eigenmodes of the body are still present in the steady-state equation for the body motion although they are no longer supported by the string and have died out fast. Still they are present in the case where

$$f_{body} \sim r f_0 \ , \qquad (13)$$

where a string frequency is next to a body mode. In this case the body is acting like in resonance adding energy to the body vibration. Note that this resonance phaenomenon is not necessary to cause the body to vibrate with the string frequency! The second term, the string impact onto the body, is the direct input of the string to the body, the 'knocking' of the string onto the body. This periodical knocking which formes the harmonic overtone structure is the first wave distributed onto the body or soundboard. This first wave causes the body to radiate even if no body resonance is there at this frequency. This is often found with guitars in the low register between the Helmholtz resonance around 100 Hz and the first body mode at around 200 Hz. Many string fundamentals lie between these two body resonances and would not be radiated strongly when resonance would be the only reason for radiation. Only the second term in the body equation, the first impact which is much stronger than the distributed waves already present on the body leads to strong radiations of these string frequencies. So for the first wave of the string onto the body at each string cycle, the body is close to the free-field condition.

The body equation also include the body motion for waves which are no eigenfrequencies of the body. This is necessary, as for very small N all frequencies of the very first impulse of the string onto the body are present in the sound. It is indeed the case with stringed instruments that the very first attack phase shows a broad spectrum.

Also, the strings frequencies r f_0 are present within the initial transient case, too, as the sum does not cancel them out. Indeed, string frequencies start right with the first attack in stringed instruments sounds.

The equation for the initial transient also determines the temporal development of the eigenfrequency amplitudes of the resonating body during the attack phase. Those can easily be calculated by stepping through different N = 1, 2, 3,...

Also for very low eigenfrequencies of air modes in the body the situation may be slightly like that of the string, e.g. with the Helmholtz motion of air inside a guitar. It can be assumed as quite strong, because the air shows only one phase and is much less damped than the body. Indeed, in some spectra the Helmholtz frequency is present throughout the tone, still it is not as strong then as body resonances within the initial transient phase. These cases are covered by the 'resonance' term in the body equation of the steady-state system (see figures below).

So we can conclude that there are two reasons for the string forcing the body or soundboard of stringed instruments to go with its frequencies not not vice versa:

1. The vibrating system with less damping takes over the system with more damping.
2. The vibrating system with low dimensionality and therefore a small amount of wave reflection points takes over more complex geometries with more reflection points.

Simple wins against complex, low damping against strong.

We have seen, that the coupled string - body system is not only one of resonance but also one of self-organization. String frequencies not present in the body spectrum are still radiated strongly because of this feature. The nonlinearities normally present in the coupling point is now present in the complex geometry of the 'resonance' body causing many weak reflections which are no longer able to work against the strong impact of the string caused by its simple one-dimensional geometry. The difference in damping is the second parameter necessary.

In Figure 1 and Figure 2 examples of body resonances as appear from the theory driven by a string with 300 Hz are shown. The plots show, how a body resonance would react with its amplitude, if it was really present in the body at a frequency ra-

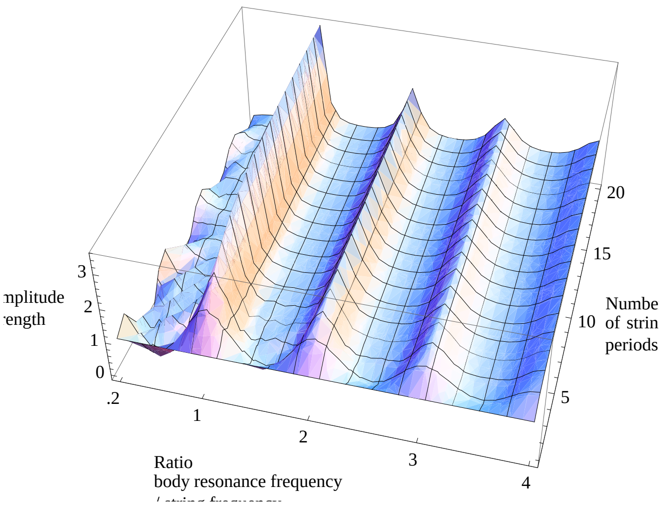

Figure 1: Body resonances driven by a string of 300 Hz periodicity as ratio of string to body frequency. Other than the string frequencies which have a sharp attack (not shown here), the body resonances come in smooth.

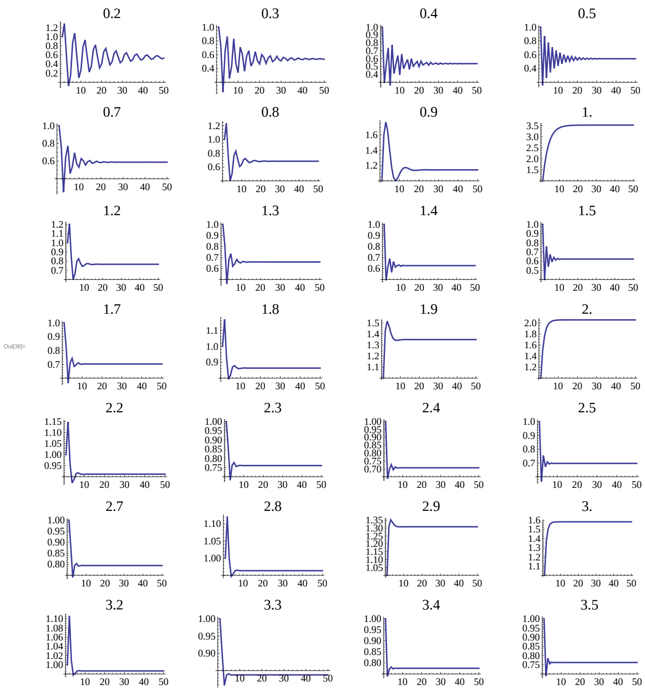

Figure 2: Body resonances driven by a string of 300 Hz periodicity as ratio of string to body frequency as in figure 1, here single plots, scaling to maximum of each resonance amplitude. Periodicities appear in a way known from real instrument behaviour.

tio to the string frequency as indicated in the plots. Of course, if the body frequency meets the string, it is resonating most strongly. Note, that here only the body resonance part of the theory is shown. The energy of the string acting upon the body directly is then superposing. This part comes in immediately with a sharp attack. The body resonances only come in smooth. In Figure 2 the single resonances can be seen scaled to their maximum amplitude. Here, oscillations appear in the amplitudes as well known from real instrument sounds. The cause of these oscillations are the phases cancelling or enhancing each other with each new period of the string. Also note, that even if the amplitude of the resonances are not strong, they are mostly not zero. Still they are not strong enough to radiate a string frequency here strongly. This is only appearing with the forced oscillation part of the equation. The solutions can easily be used to simulate the initial transient of e.g. a guitar tone in the presence of a harmonic overtone spectrum if the body eigenfrequencies are known.

BOW - STRING INTERACTION

With bowed instruments, the interaction between the bow and the string is a nonlinear one leading to a sawtooth motion with normal playing pressure as discussed above. Within this regime the string length determines the pitch of the played note. This allows us to play tones with e.g. a violin by shortening or lengthening of the string, by fingerings. When the bowing pressure is beyond a threshold this pressure takes over and then determines the pitch of the sound. This works very well with cellos. When the pressure is very low a bifurcation scenario appears ending in a noise regime.

Establishing one of the regimes during the initial transient of the sound may go through different stages. This holds for most of the hard attacks of the violin sound. Here a strong pressure is used right from the start to get a scratchy sound. On the other hand, certain soft attacks may establish the sawtooth- or Helmholtz-regime at once.

When again using the amplitudes of frequencies as dependent variables in a coupled equation system we can write two equations, one for the string and one for the bow-string point like

$$\dot{A}_s = \alpha A_s + k_1 A_b$$
$$\dot{A}_b = \beta A_b + k_2 A_s \; .$$

Here A_s is the amplitude of a certain frequency on the string and A_b is the amplitude of that frequency at the bowing point. The string is acting upon itself with a damping α caused by the strings internal damping and the energy loss to the radiating body. The situation is different with the second equation for the bow-string point. Here a Heavyside Θ function is used to represent the stick-slip behaviour of the bow. During sticking phase the bow is acting upon this point and so $\Theta(\gamma A_b(p,v)) = 1$. γ includes the string parameters like rosin, bow width ect. In this case the point is sticking to the string until a certain amplitude or displacement is reached. So then $\beta = A_b(p,v)$ with $A_b(p,v)$ being the amplitude of the string at which the bow is tearing-off the string again. This depends upon the playing pressure p and the playing velocity v. With bowed instruments the volume of the tone is determined by the bow velocity, while the tone color is changed by the pressure.

Additionally, the bow-string point is of course determined by the string itself with a certain coupling constant k_2 which includes the strength of the gluing rosin of the bow, the width of the bow, and all parameters determining if the string is strong enough to tear-off the bow from the string again. So k_2 also depends on the playing pressure as a stronger pressure weakens the ability of the string to tear-off the bow again. If the model is used with changing playing pressure, k_2 is not a constant anymore. As again this tear-off is either there or not we need a Heavyside function to determine if a certain threshold is reached.

As a last parameter, the coupling of the bow-string point upon the string itself via k_1 represents the ability of the bow point to strengthen or weaken a frequency on the string. So we can rewrite this coupled system as

$$\dot{A}_s = \alpha A_s + k_1 A_b$$
$$\dot{A}_b = A_b(p,v)\; \Theta[\gamma A_b(p,v)] + A_s \Theta[k_2 A_s] \; .$$

We can distinguish three phases of the system.

Sawtooth motion

Here the amplitude of the bow-string point A_b is never getting large enough that a tear-off is happening because of a very large displacement of the string. Therefore $\Theta[\gamma A_b(p,v)] = 0$ all the time. In this case the equation system simplifies to

$$\dot{A}_s = \alpha A_s + k_1 A_b$$
$$\dot{A}_b = A_s \Theta[k_2 A_s] \; .$$

Clearly, the bow-string point amplitude A_b is only affected by the string amplitude A_s and therefore the string itself determines the system. In this case the string length sets the pitch of the tone and we can play notes using fingerings.

High-pressure regime

When the pressure is so high that the string is no longer able to tear-off the bow from it two terms can be neglected. First, the second Θ function is always zero. Secondly, A_b is very large. Also, the damping of violin strings is normally quite strong and so α is small. As the periodicity is large here the damping is applied for a long time span. Last, the coupling k_1 of the bow-string point upon the string is strong. So we have

$$\alpha A_s \ll k_1 A_b \; . \qquad (14)$$

Therefore the system reduces to

$$\dot{A}_s = k_1 A_b$$
$$\dot{A}_b = A_b(p,v) \Theta[\gamma A_b(p,v)] \; .$$

The string is determined by the bow alone. It is forced to go with the bow. The bow-string point is only determined by itself, no string influence is happening anymore. So the situation is completely different compared to the sawtooth motion case, where the string determined the pitch. Now the bow is giving the system its frequency via the bowing pressure.

Low-pressure regime

If the bowing pressure is very low, both Θ functions appear in the second equation. The pressure threshold for the bow term is reached because with low pressure values the string easily tears-off the bow. The string term is there as it is also easy for the string to tear-off the bow from it. As the time span between two tear-offs is short, the low value of α does not play a crucial role. The time for damping is not long enough to damp the amplitudes very much. Also the influence of the bow-string point on the string via k_1 is present as it is in the range of the influence of the string upon itself. So we can not neglect any of the terms and have the complete system

$$\dot{A}_s = \alpha A_s + k_1 A_b$$
$$\dot{A}_b = A_b(p,v)\Theta[\gamma A_b(p,v)] + A_s\Theta[k_2 A_s] \ .$$

This leads to very complicated patterns resulting in a noisy sound within this regime. In normal instrument playing this regime is not desirable and one would want to lower the pressure threshold for this regime as much as possible to allow normal sawtooth motion with low playing pressures. This is normally done by using rosin. Rosin glues the string to the bow enlarging the static friction of it. The tear-off caused by the string via k_2 and the tear-off caused by the bow via γ are physically different. The string tears off because the amplitude peak travelling around the string is reaching the bow point acting like a small impulse on the glued string. This causes the tear-off much easier than the force caused by displacement counteracting the strong static friction of the rosin glued bow. So using rosin causes

$$A_b(p,v)\Theta[\gamma A_b(p,v)] \ll A_s\Theta[k_2 A_s] \ , \qquad (15)$$

and therefore enlarging the normal pressure regime to low playing pressures values.

REED INSTRUMENTS

Aschoff was the first to discuss the question why a clarinet or saxophone plays the pitch of the air column and not the one of the reed (Aschoff 1936). He argues that the damping of the reed is much stronger than that of the air column and so the tube length determines the pitch. When writing this in terms of the coupled equation system using A_r and A_a for the reed and air amplitude respectively we have

$$\dot{A}_r = \alpha A_r + k_1 A_a$$
$$\dot{A}_a = \beta A_a + k_2 A_r \ .$$

Here α is the strength of the reed acting upon itself and β the one of the air column. So again a high damping results in low values for these parameters. k_1 and k_2 are the coupling constants again. Now, if the reed is much more damped than the air column we have

$$\alpha \ll \beta \qquad (16)$$

and so by neglecting the α terms the system reduces to

$$\dot{A}_r = k_1 A_a$$
$$\dot{A}_a = \beta A_a + k_2 A_r \ .$$

The reed depends only on the air column and so the air column determines the pitch of the instrument.

Now with reed instruments the situation can also be described in a more complex way taking the nonlinearities of the system into consideration. As the reed is a closing valve being driven by the air column but also by the blowing pressure we can rewrite the equation above as

$$\dot{A}_r = \Gamma[\alpha A_r + k_1 A_a + p_b]$$
$$\dot{A}_a = \beta A_a + k_2 A_r \ .$$

In the first equation the blowing pressure p_b is included. The Γ is the function of the closing value. If the maximum possible amplitude of the reed is A_{max} then

$$\Gamma = \begin{cases} \alpha A_r + k_1 A_a + p_b, & \alpha A_r + k_1 A_a + p_b < A_{max}; \\ A_{max}, & \alpha A_r + k_1 A_a + p_b > A_{max}; \end{cases} \qquad (17)$$

So we can distinguish three cases:

Normal playing condition

Here the reed closes and opens again. So within a period we have both cases of Γ. This means that the action of the air column on the reed and the blowing pressure are strong and therefore

$$\beta \gg \alpha A_r \ll k_1 A_a + p_b \ . \qquad (18)$$

This leads to

$$\dot{A}_r = \Gamma[k_1 A_a + p_b]$$
$$\dot{A}_a = \beta A_a + k_2 A_r \ ,$$

the reed is again dominated by the air column and the blowing pressure.

Saxophones show a characteristic formant region in their spectrum around 3 kHz. This is caused by the fundamental frequency of the reed which is still vibrating a bit of its own and so enhancing this spectral region independently from the played pitch. It is interesting to see, that although the fundamental frequency of a normal saxophone reed is around 1.5 kHz when investigated without the normal playing, this frequency doubles when the reed is in the air flow and interaction with the saxophone embouchure (Bader 2008). When we neglect the α term here we do this to show the basic slavery principle with this instrument. Indeed we neglect it because it is very small compared to the second term of the air column acting on it. If we still take it into consideration we would include this formant structure of the sound. So the formulation used here shows the basic principle of the system, still when including all terms it is capable of describing some of the fine structures of the instruments. This was also the case with the plucked string instruments when initial transients show the amplitude development of the eigenmodes of the body within the sound attack.

Low blowing pressure

When the blowing pressure is low no tone is established and we hear a noise sound only. This noise is still shaped by the eigenfrequencies of the tube but still it is a broadband signal. In this case we are not allowed to neglect α. Also the Γ function is only present with its first case, the reed never closes completely. Therefore we have

$$\dot{A}_r = \alpha A_r + k_1 A_a + p_b$$
$$\dot{A}_a = \beta A_a + k_2 A_r \ .$$

Both systems influence each other and so no clear pitch is established, the sound is noisy. Still the resonance spectrum of the tube and the one of the reed will be present.

Reed instruments show a sudden change from noisy sounds to pitches (with additional background noise) with a clear pressure threshold. This is found in the synergetic formulation within the presence or absence of the second condition of the Γ function. If the maximum amplitude is reach - if only for a short time during a period - we have a completely new condition, the one of normal pressure playing with a clear pitch. If this maximum amplitude is not reached the system does not act in a self-organized way and shows a quite random behaviour, a noisy sound.

High blowing pressure

Theoretically, the saxophone may be played with such a high pressure that no tone can establish. This is an unwanted case, still it would mean that

$$p_b \gg \alpha A_r + k_1 A_a \tag{19}$$

and so

$$\dot{A}_r = \Gamma[p_b] = A_{max}$$
$$\dot{A}_a = \beta A_a + k_2 A_r \ .$$

The reed would only be dominated from a non-changing terms, the blowing pressure. As the reed closes completely here, $\Gamma[p_b] = A_{max}$ and so no sound is produced either noisy or with a pitch.

Multiphonic sounds

Multiphonic sounds are multi-pitch notes played by instruments which can normally only produce one pitch at a time. Here sophisticated players can produce tones with up to five or six notes. With reed instruments three basic playing styles allow such multiphonics

- Blowing pressure at the noise / pitch threshold
- Very high blowing pressure
- Complex fingerings

In the case of a blowing pressure at the pitch threshold the reed undergoes two periodicities instead of one. It tries to establish a normal pitch which is counteracted by the pressure being not strong enough to maintain it. Still right after giving up the pitch, enough energy is again in the system to reestablish it. The time span for this reestablishment is determined by the whole system and so does not depend on the tube length alone and so very likely in an inharmonic relation to the pitch which would establish along with a slightly stronger playing pressure. In this case the Γ function is changing constantly between its single and its double case and therefore two periodicities are present at the same time. Indeed, multiphonics produced by playing pressures at the noise / pitch threshold mostly have only two pitches.

In the case of a very high blowing pressure the case is similar to the one discussed above. The difference is only that then the system changes between the two cases of a normal playing condition and a high blowing pressure constantly. The blowing pressure is not strong enough to maintain a perfectly constant closing although it maintains such a closing for some time. Again, this leads to a most likely inharmonic relation between two time spans, the one of constant closing and the normal pitch. Again, two inharmonic periodicities are produced and two pitches are heard.

The case of complex fingerings is most complicated leading to much more pitches at once than the other two. This is caused by different tube lengths caused by fingerings, where some holes are open while holes deeper down the tube are closed again. This leads to a multiple of reflection points within the tube and so we would need to enlarge our coupled equation system like

$$\dot{A}_r = \Gamma[k_1^1 A_a^1 + k_1^1 A_a^2 + ... + p_b]$$
$$\dot{A}_a^1 = \beta^1 A_a^1 + k_2 A_r$$
$$\dot{A}_a^2 = \beta^2 A_a^2 + k_2 A_r$$
$$...$$

By choosing fingerings which have suitable β and k_1 a multiple of pitches may be produced.

Of course multiphonics are also possible with combinations of blowing pressure thresholds and fingerings, the equation system will be a combination of these cases then, too.

CONCLUSION

Although many phaenomena discussed in this paper have also been investigated many times before, the basic framework derived from the Synergetic formulation of two coupled oscillators works very well with musical instruments. Especially when determining which of the oscillators takes over the other it is well suitable. Also, initial transients can be modelled quite easily. The strength of partials which can not easily be explained as a normal resonance can be understood as the take over of one system by the other. Also nonlinear effects can be included to understand the basic regimes of musical systems like the bow - string or the reed - air column cases.

Synergetics normally formulates an order parameter and a control parameter. The former is the parameter which determines the emergent behaviour, the latter controls the cases. This terminology was omitted here not to confuse musical cases with additional definitions. Still future work will discuss this, too.

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