14th Intermational Congress on

# FREE VIBRATIONS OF THE SYSTEMS OF RIGID BODIES COUPLED BY ELASTIC BARS 

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#### Abstract

It is well known that by using the plückerian coordinates the force at one end of the bar is given by the product between the rigidity matrix and the difference between the displacements at the ends of the bar. With the aid of this relation are obtained the differential equations of the vibrations of a rigid bodies system coupled by elastic bars, the expressions of the rigidity matrix and of the displacements of the rigid bodies being written in the general reference frame. In the final part of the paper is presented an application for the vibrations of a system consisted of two plane shells connected each one with a bar to the fixed system and coupled one to the other by a bar.


## 1. INTRODUCTION

In the classical matrix calculus for the bars systems [1] it is proved that for a linear elastic bar there exist two matrices of rigidity, and the column matrix of the force at one end is expressed with the aid of the sum of the products between those matrices and the column matrices of the bar ends normal sections' displacements. In the case when the normal sections displacements at the ends of the bar are assimilated to some screws expressed in plückerian coordinates [4] or screw coordinates [6], then there exists only one rigidity matrix [1], [2], [4], and the matrix of the plückerian coordinates of the end of bar force is given by the product between the rigidity matrix and the difference between the column matrices of the plückerian coordinates of the normal sections' displacements at the ends of the bar. Starting with this relation, in the paper are established the differential equations of the free vibrations of the systems of rigid bodies coupled by linear elastic bars and then is presented an application for two plane shells coupled by bars and is given a physical interpretation for the vibration modes.

## 2. NOTATIONS

Let's consider the homogenous bar $A B$ by constant section in figure 1 . We denote: $l$ - length of the bar; $A$ - aria of the normal section; $O^{*}$ - middle point of the bar; $O^{*} x$ - longitudinal
axis; $O^{*} y, O^{*} z$ - principal inertial axes of the normal section which passes through $O^{*}$; OXYZ - general reference frame; $I_{y}, I_{z}$ - principal inertial moments of the normal section in $O^{*} ; I_{x}$ - inertial moment for the conventional torsion stress; $k_{1}, k_{2}, \ldots, k_{6}$ - rigidities defined by relations


Figure 1. General notations.

$$
\begin{equation*}
k_{1}=\frac{E A}{l} ; k_{2}=\frac{12 E I_{z}}{l^{3}} ; k_{3}=\frac{12 E I_{y}}{l^{3}} ; k_{4}=\frac{G I_{z}}{l} ; k_{5}=\frac{E I_{y}}{l} ; k_{6}=\frac{E I_{z}}{l} ; \tag{1}
\end{equation*}
$$

$\mathbf{K}_{A B}^{*}$ - bar's rigidity matrix with respect to frame $O^{*} x y z$

$$
\mathbf{K}_{A B}^{*}=\left[\begin{array}{cccccc}
0 & 0 & 0 & k_{1} & 0 & 0  \tag{2}\\
0 & 0 & 0 & 0 & k_{2} & 0 \\
0 & 0 & 0 & 0 & 0 & k_{3} \\
k_{4} & 0 & 0 & 0 & 0 & 0 \\
0 & k_{5} & 0 & 0 & 0 & 0 \\
0 & 0 & k_{6} & 0 & 0 & 0
\end{array}\right]
$$

$\mathbf{D}_{A}^{*}, \mathbf{D}_{B}^{*}$ - column matrices of the plückerian coordinates in the system $O^{*} x y z$ of the screw displacements of the sections in $A$ and $B$

$$
\mathbf{D}_{A}^{*}=\left(\begin{array}{llllll}
\theta_{A x} & \theta_{A y} & \theta_{A z} & \delta_{A x} & \delta_{A y} & \delta_{A z}
\end{array}\right)^{T} ; \mathbf{D}_{B}^{*}=\left(\begin{array}{lllll}
\theta_{B x} & \theta_{B y} & \theta_{B z} & \delta_{B x} & \delta_{B y}  \tag{3}\\
\delta_{B z}
\end{array}\right)^{T} ;
$$

$\mathbf{F}_{A}^{*}, \mathbf{F}_{B}^{*}$ - column matrices of the plückerian coordinates in the system $O^{*} x y z$ of the screw forces in points $A$ and $B ; \mathbf{D}_{A}, \mathbf{D}_{B}$ - column matrices of the plückerian coordinates in the system $O^{*} x y z$ of the screw forces in points $A$ and $B ; \mathbf{F}_{A}, \mathbf{F}_{B}$ - column matrices of the plückerian coordinates in the system $O X Y Z$ of the screw forces in $A$ and $B ; \alpha_{i}, \beta_{i}, \gamma_{i}$, $i=1,2,3$ - director cosines o the axes $O^{*} x, O^{*} y, O^{*} z$ with respect to the system $O X Y Z$; $X, Y, Z$ - coordinates of the point $O^{*}$ with respect to the system $O X Y Z ; \mathbf{R}, \mathbf{G}$ - matrices of rotation, respectively translation

$$
\mathbf{R}=\left[\begin{array}{ccc}
\alpha_{1} & \alpha_{2} & \alpha_{3}  \tag{4}\\
\beta_{1} & \beta_{2} & \beta_{3} \\
\gamma_{1} & \gamma_{2} & \gamma_{3}
\end{array}\right] ; \mathbf{G}=\left[\begin{array}{ccc}
0 & -Z & Y \\
Z & 0 & -X \\
-Y & X & 0
\end{array}\right]
$$

$\mathbf{I}, \mathbf{0}$ - unity, respective null matrix; $\mathbf{T}, \mathbf{T}^{-1}$ - positional matrices of the system $O^{*} x y z$ relative to the system $O X Y Z$;

$$
\mathbf{T}=\left[\begin{array}{cc}
\mathbf{R} & \mathbf{0}  \tag{5}\\
\mathbf{G R} & \mathbf{R}
\end{array}\right] ; \mathbf{T}^{-1}=\left[\begin{array}{cc}
\mathbf{R}^{T} & \mathbf{0} \\
\mathbf{R}^{T} \mathbf{G}^{T} & \mathbf{R}^{T}
\end{array}\right]
$$

$\boldsymbol{\eta}$ - matrix to symmetry;

$$
\boldsymbol{\eta}=\left[\begin{array}{cc}
0 & \mathbf{I}  \tag{6}\\
\mathbf{I} & 0
\end{array}\right]
$$

$\mathbf{K}_{A B}$ - rigidity matrix of the bar in the general frame $O X Y Z$.

## 3. CALCULUS RELATIONS

There exist [1], [2], [4] the following matrix relations:

$$
\begin{gather*}
\mathbf{F}_{A}^{*}=\mathbf{K}_{A B}^{*}\left(\mathbf{D}_{A}^{*}-\mathbf{D}_{B}^{*}\right) ;  \tag{7}\\
\mathbf{D}_{A}=\mathbf{T D}_{A}^{*} ; \mathbf{D}_{A}^{*}=\mathbf{T}^{-1} \mathbf{D}_{A} ; \mathbf{D}_{B}=\mathbf{T D}_{B}^{*} ; \mathbf{D}_{B}^{*}=\mathbf{T}^{-1} \mathbf{D}_{B} ;  \tag{8}\\
\mathbf{F}_{A}=\mathbf{T F}_{A}^{*} ; \mathbf{F}_{A}^{*}=\mathbf{T}^{-1} \mathbf{F}_{A} ; \mathbf{F}_{B}=\mathbf{T F}_{B}^{*} ; \mathbf{F}_{B}^{*}=\mathbf{T}^{-1} \mathbf{F}_{B} ;  \tag{9}\\
\mathbf{K}_{A B}=\mathbf{T K}_{A B}^{*} \mathbf{T}^{-1} ;  \tag{10}\\
\mathbf{F}_{A}=\mathbf{K}_{A B}\left(\mathbf{D}_{A}-\mathbf{D}_{B}\right) . \tag{11}
\end{gather*}
$$

## 4. DIFFERENTIAL EQUATIONS OF THE PROPER VIBRATIONS OF THE SYSTEMS OF RIGID BODIES COUPLED BY ELASTICAL BARS

We consider the system drawn in figure 2 composed by two rigid bodies (denoted by 1,2 ) coupled between them by bars $D E$ and coupled to the fix body $O$ by bars $A A_{0}, B B_{0}$. We use the following notations: $C_{i}, i=1,2$ - weight centers of the bodies; $C_{i} x_{i} y_{i} z_{i}, i=1,2$ systems of the principal inertial axes; $m_{i}, i=\overline{1,2}$ - masses of the bodies; $J_{x_{i}}, J_{y_{i}}, J_{z_{i}}$, $i=\overline{1,2}$ - principal inertial moments; $\mathbf{M}_{i}^{*}, i=1,2$ - inertial matrices with respect to the reference systems $C_{i} x_{i} y_{i} z_{i}$,


Figure 2. System of two rigid bodies coupled by elastic bars.

$$
\mathbf{M}_{i}^{*}=\left[\begin{array}{cccccc}
0 & 0 & 0 & m_{i} & 0 & 0  \tag{12}\\
0 & 0 & 0 & 0 & m_{i} & 0 \\
0 & 0 & 0 & 0 & 0 & m_{i} \\
J_{x_{i}} & 0 & 0 & 0 & 0 & 0 \\
0 & J_{y_{i}} & 0 & 0 & 0 & 0 \\
0 & 0 & J_{z_{i}} & 0 & 0 & 0
\end{array}\right] ;
$$

$O X Y Z$ - general reference system; $\mathbf{T}_{i 0}, i=1,2$ - positional matrices of the systems $C_{i} x_{i} y_{i} z_{i}$ relative to the system $O X Y Z ; \mathbf{M}_{i}, i=1,2$ - inertial matrices relative to the system $O X Y Z$,

$$
\begin{equation*}
\mathbf{M}_{i}=\mathbf{T}_{i 0} \mathbf{M}_{i}^{*} \mathbf{T}_{i 0}^{-1} \tag{13}
\end{equation*}
$$

$\mathbf{D}_{i}^{*}, i=1,2$ - displacements matrices of the rigid bodies in the systems $C_{i} x_{i} y_{i} z_{i} ; \mathbf{D}_{i}, i=1,2$ - displacements matrices of the rigid bodies in the system $O X Y Z$, matrices that verify the following relations:

$$
\begin{equation*}
\mathbf{D}_{i}=\mathbf{T}_{i 0} \mathbf{D}_{i}^{*} ; \mathbf{D}_{1}=\mathbf{D}_{A} ; \mathbf{D}_{2}=\mathbf{D}_{B} . \tag{14}
\end{equation*}
$$

The screw forces that act onto the first rigid body being

$$
\begin{equation*}
-\sum \mathbf{F}_{A}=-\sum \mathbf{K}_{A A_{0}} \mathbf{D}_{1}-\sum \mathbf{K}_{D E}\left(\mathbf{D}_{1}-\mathbf{D}_{2}\right), \tag{15}
\end{equation*}
$$

from the momentum and the moment of the momentum theorems, one obtains:

$$
\begin{align*}
& \mathbf{M}_{1} \ddot{\mathbf{D}}_{1}+\left(\sum \mathbf{K}_{A A_{0}}+\sum \mathbf{K}_{D E}\right) \mathbf{D}_{1}-\left(\sum \mathbf{K}_{D E}\right) \mathbf{D}_{2}=\mathbf{0} ; \\
& \mathbf{M}_{2} \ddot{\mathbf{D}}_{2}-\left(\sum \mathbf{K}_{D E}\right) \mathbf{D}_{1}+\left(\sum \mathbf{K}_{B B_{0}}+\sum \mathbf{K}_{D E}\right) \mathbf{D}_{2}=\mathbf{0}, \tag{16}
\end{align*}
$$

or, if we denote

$$
\begin{equation*}
\mathbf{K}_{11}=\sum \mathbf{K}_{A A_{0}}+\sum \mathbf{K}_{D E} ; \mathbf{K}_{12}=\sum \mathbf{K}_{D E} ; \mathbf{K}_{22}=\sum \mathbf{K}_{B B_{0}}+\sum \mathbf{K}_{D E}, \tag{17}
\end{equation*}
$$

we deduce the differential equations:

$$
\begin{equation*}
\mathbf{M}_{1} \ddot{\mathbf{D}}_{1}+\mathbf{K}_{11} \mathbf{D}_{1}-\mathbf{K}_{12} \mathbf{D}_{2}=\mathbf{0} ; \mathbf{M}_{2} \ddot{\mathbf{D}}_{2}-\mathbf{K}_{12} \mathbf{D}_{1}+\mathbf{K}_{22} \mathbf{D}_{2}=\mathbf{0} . \tag{18}
\end{equation*}
$$

For a system composed by $n$ rigid bodies, we analogously deduce the system of matrix differential equations

$$
\begin{equation*}
\mathbf{M}_{i} \ddot{\mathbf{D}}_{i}+\mathbf{K}_{i i} \mathbf{D}_{i}-\sum_{\substack{j=1 \\ j \neq i}}^{n} \mathbf{K}_{i j} \mathbf{D}_{j}=0 ; i=1,2, \cdots, n, \tag{19}
\end{equation*}
$$

where $\mathbf{K}_{i j}=\mathbf{0}$ if the rigid $i$ is not coupled by bar with the rigid $j$.
We mention that the matrices $\mathbf{M}_{i}, \mathbf{K}_{i j}$ are not symmetric, but we keep this form to use the same positional matrix both to the displacements' transformation and to the forces' transformation. These matrices can be made symmetric using the relations:

$$
\begin{equation*}
\widetilde{\mathbf{M}}_{i}=\mathbf{M}_{i} \boldsymbol{\eta} ; \widetilde{\mathbf{K}}_{i j}=\mathbf{K}_{i j} \boldsymbol{\eta} . \tag{20}
\end{equation*}
$$

## 5. EXAMPLE



Figure 3. Example.
Let's study the free vibrations of the system drawn in figure 3 composed by two identical rectangular rigid shells by mass $m$ and dimensions $2 a, 2 b$, the shells being coupled by three identical bars of length $2 l$ and by circular section of diameter $d$.

The bars being identical they have relative to the local reference systems $O_{i}^{*} x_{i}^{*} y_{i}^{*} z_{i}^{*}$, $i=1,2,3$ the same rigidity matrix $\mathbf{K}^{*}$ given by relation (2), where $k_{1}=\frac{E A}{2 l}$, $k_{2}=k_{3}=\frac{4 E I}{3 l^{3}}, k_{4}=\frac{G I}{l}, k_{5}=k_{6}=\frac{E I}{2 l}, I$ being the inertial moment, $I=\frac{\pi d^{4}}{64}$. Writing
the first equation (16) in the system $C_{1} x_{1} y_{1} z_{1}$ and the second equation in the system $C_{2} x_{2} y_{2} z_{2}$ (fig. 3), it results:

$$
\begin{align*}
& \mathbf{M}_{1}^{(1)} \ddot{\mathbf{D}}_{1}^{*}+\left[\mathbf{K}_{A A_{0}}^{(1)}+\mathbf{K}_{D E}^{(1)}\right] \mathbf{D}_{1}^{*}-\mathbf{K}_{D E}^{(1)} \mathbf{T}_{21} \mathbf{D}_{2}^{*}=\mathbf{0} ; \\
& \mathbf{M}_{2}^{(1)} \ddot{\mathbf{D}}_{2}^{*}-\mathbf{K}_{D E}^{(2)} \mathbf{T}_{21}^{-1} \mathbf{D}_{1}^{*}+\left[\mathbf{K}_{B B_{0}}^{(2)}+\mathbf{K}_{D E}^{(2)}\right] \mathbf{D}_{2}^{*}=\mathbf{0} . \tag{21}
\end{align*}
$$

The principal inertial moments of the shells being $J_{x}=\frac{m b^{2}}{3}, \quad J_{y}=\frac{m a^{2}}{3}$, $J_{z}=\frac{m\left(a^{2}+b^{2}\right)}{3}$, one obtains

$$
\mathbf{M}_{1}^{(1)}=\mathbf{M}_{2}^{(2)}=\left[\begin{array}{cccccc}
0 & 0 & 0 & m & 0 & 0  \tag{22}\\
0 & 0 & 0 & 0 & m & 0 \\
0 & 0 & 0 & 0 & 0 & m \\
J_{x} & 0 & 0 & 0 & 0 & 0 \\
0 & J_{y} & 0 & 0 & 0 & 0 \\
0 & 0 & J_{z} & 0 & 0 & 0
\end{array}\right] .
$$

For the rigidity matrices we successively obtain $\mathbf{K}_{A A_{0}}^{(1)}=\mathbf{K}_{B B_{0}}^{(2)}=\mathbf{T}_{3^{*} 1} \mathbf{K}^{*} \mathbf{T}_{3^{1} 1}^{-1}$, $\mathbf{K}_{D E}^{(1)}=\mathbf{T}_{3^{*} 1} \mathbf{K}^{*} \mathbf{T}_{3^{* 1}}^{-1}$.

Further on, keeping into account the expression of the positional matrix,

$$
\mathbf{T}_{21}=\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0  \tag{23}\\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 2(l+b) & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
-2(l+b) & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

and the relations: $A_{1}=k_{1}+k_{2}, \quad A_{2}=-k_{2}(l+b), \quad A_{3}=-k_{1} a-k_{2}(l+a), \quad A_{4}=2 k_{3}$, $A_{5}=k_{2}(l+2 a), A_{6}=k_{4}+k_{5}+k_{2}(l+b)^{2}, A_{7} k_{2} a(l+b), A_{8}=k_{4}+k_{5}+k_{2} a^{2}+k_{2}(l+a)^{2}$, $A_{9}=2 k_{5}+k_{2}(l+a)^{2}+k_{2}(l+b)^{2}+k_{1} a^{2}, \quad B_{1}=k_{2} ; \quad B_{2}=k_{2}(l+b), \quad B_{3}=k_{1}, \quad B_{4}=-k_{1} a$, $B_{6}=k_{3} a, \quad B_{7}=k_{5}-k_{2}(l+b)^{2}, \quad B_{8}=k_{3} a(l+b), \quad B_{9}=k_{4}+k_{5} a^{2}$, $B_{10}=k_{5}+k_{1} a^{2}-k_{2}(l+b)^{2}$, one obtains the rigidity matrices

$$
\mathbf{K}_{A_{0}}^{(1)}+\mathbf{K}_{D E}^{(1)}=\left[\begin{array}{cccccc}
0 & 0 & A_{2} & A_{1} & 0 & 0  \tag{24}\\
0 & 0 & A_{3} & 0 & A_{1} & 0 \\
-A_{2} & A_{5} & 0 & 0 & 0 & A_{4} \\
A_{6} & A_{7} & 0 & 0 & 0 & -A_{2} \\
A_{7} & A_{8} & 0 & 0 & 0 & A_{5} \\
0 & 0 & A_{9} & A_{2} & A_{3} & 0
\end{array}\right] ; \quad \mathbf{K}_{D E}^{(1)} \mathbf{T}_{21}=\left[\begin{array}{cccccc}
0 & 0 & B_{2} & B_{1} & 0 & 0 \\
0 & 0 & B_{4} & 0 & B_{3} & 0 \\
-B_{2} & B_{6} & 0 & 0 & 0 & B_{1} \\
B_{7} & B_{8} & 0 & 0 & 0 & B_{2} \\
-B_{8} & B_{9} & 0 & 0 & 0 & B_{6} \\
0 & 0 & B_{10} & -B_{2} & B_{4} & 0
\end{array}\right],
$$

$$
\mathbf{K}_{B B_{0}}^{(2)}+\mathbf{K}_{D E}^{(2)}=\left[\begin{array}{cccccc}
0 & 0 & -A_{2} & A_{1} & 0 & 0  \tag{25}\\
0 & 0 & A_{3} & 0 & A_{1} & 0 \\
A_{2} & A_{5} & 0 & 0 & 0 & A_{4} \\
A_{6} & -A_{7} & 0 & 0 & 0 & A_{2} \\
-A_{7} & A_{8} & 0 & 0 & 0 & A_{5} \\
0 & 0 & A_{9} & -A_{2} & A_{3} & 0
\end{array}\right] ; \mathbf{K}_{D E}^{(2)} \mathbf{T}_{21}^{-1}=\left[\begin{array}{cccccc}
0 & 0 & -B_{2} & B_{1} & 0 & 0 \\
0 & 0 & B_{4} & 0 & B_{2} & 0 \\
B_{2} & B_{6} & 0 & 0 & 0 & B_{1} \\
B_{2} & -B_{8} & 0 & 0 & 0 & -B_{2} \\
B_{8} & B_{9} & 0 & 0 & 0 & B_{6} \\
0 & 0 & B_{10} & B_{2} & B_{4} & 0
\end{array}\right]
$$

and if we keep into account the local displacements $\mathbf{D}_{1}^{*}, \mathbf{D}_{2}^{*}, \mathbf{D}_{1}^{*}=\left(\begin{array}{lllll}\theta_{1 x} & \theta_{1 y} & \theta_{1 z} & \delta_{1 x} & \delta_{1 y} \\ \delta_{1 z}\end{array}\right)^{T}$, $\mathbf{D}_{2}^{*}=\left(\theta_{2 x} \theta_{2 y} \theta_{2 z} \delta_{2 x} \delta_{2 y} \delta_{2 z}\right)^{T}$, then the matrix equations (23) lead to the independent systems

$$
\begin{gather*}
m \ddot{\delta}_{1 x}+A_{1} \delta_{1 x}+A_{2} \theta_{1 z}+B_{1} \delta_{2 x}+B_{2} \theta_{2 z}=0 ; m \ddot{\delta}_{1 x}+A_{1} \delta_{1 x}+A_{2} \theta_{1 z}+B_{1} \delta_{2 x}+B_{2} \theta_{2 z}=0 ; \\
J_{z} \ddot{\theta}_{1 z}+A_{2} \delta_{1 x}+A_{3} \delta_{1 y}+A_{9} \theta_{1 z}-B_{2} \delta_{2 x}+B_{4} \delta_{2 y}+B_{10} \theta_{2 z}=0 ; \\
m \ddot{\delta}_{2 x}+B_{1} \delta_{1 x}-B_{2} \theta_{1 z}+A_{1} \delta_{2 x}-A_{2} \theta_{2 z}=0 ; m \ddot{\delta}_{2 y}+B_{3} \delta_{1 y}+B_{4} \theta_{1 z}+A_{1} \delta_{2 y}+A_{3} \theta_{2 z}=0 ;  \tag{26}\\
J_{z} \ddot{\theta}_{2 z}+B_{2} \delta_{1 x}+B_{4} \delta_{1 y}+B_{10} \theta_{1 z}-A_{2} \delta_{2 x}+A_{3} \delta_{2 y}+A_{9} \theta_{2 z}=0 ; \\
m \ddot{\delta}_{1 z}+A_{4} \delta_{1 z}-A_{2} \theta_{1 x}+A_{5} \theta_{1 y}+B_{5} \delta_{2 z}-B_{2} \theta_{2 x}+B_{6} \theta_{2 y}=0 ; \\
J_{x} \ddot{\theta}_{1 x}-A_{2} \delta_{1 z}+A_{6} \theta_{1 x}+A_{7} \theta_{1 y}+B_{2} \delta_{2 z}+B_{7} \theta_{2 x}+B_{8} \theta_{2 y}=0 ; \\
J_{y} \ddot{\theta}_{1 y}+A_{5} \delta_{1 z}+A_{7} \theta_{1 x}+A_{8} \theta_{1 y}+B_{6} \delta_{2 z}-B_{8} \theta_{2 x}+A_{5} \theta_{2 y}=0 ; \\
m \ddot{\delta}_{2 z}+B_{5} \delta_{1 z}+B_{2} \theta_{1 x}+B_{6} \theta_{1 y}+A_{4} \delta_{2 z}+A_{2} \theta_{2 x}+A_{5} \theta_{2 y}=0 ;  \tag{27}\\
J_{x} \ddot{\theta}_{2 x}-B_{2} \delta_{1 z}+B_{7} \theta_{1 x}-B_{8} \theta_{1 y}+A_{2} \delta_{2 z}+A_{6} \theta_{2 x}-A_{7} \theta_{2 y}=0 ; \\
J_{y} \ddot{\theta}_{2 y}+B_{6} \delta_{1 z}+B_{8} \theta_{1 x}+B_{9} \theta_{1 y}+A_{1} \delta_{2 z}-A_{7} \theta_{2 x}+A_{8} \theta_{2 y}=0 .
\end{gather*}
$$

For the system (26) looking for solutions as

$$
\begin{equation*}
\delta_{i x}=\delta_{i x}^{0} \cos (p t-\varphi) ; \delta_{i y}=\delta_{i y}^{0} \cos (p t-\varphi) ; \theta_{i z}=\theta_{i z}^{0} \cos (p t-\varphi), \tag{28}
\end{equation*}
$$

by linear combinations one obtains the independent systems

$$
\begin{gather*}
\left(A_{1}+B_{1}-m p^{2}\right)\left(\delta_{1 x}^{0}+\delta_{2 x}^{0}\right)+\left(A_{2}-B_{2}\right)\left(\theta_{1 z}^{0}-\theta_{2 z}^{0}\right)=0 ; \\
\left(A_{1}-B_{3}-m p^{2}\right)\left(\delta_{1 y}^{0}-\delta_{2 y}^{0}\right)+\left(A_{3}-B_{4}\right)\left(\theta_{1 z}^{0}-\theta_{2 z}^{0}\right)=0 ;  \tag{29}\\
\left(A_{2}-B_{2}\right)\left(\delta_{1 x}^{0}+\delta_{2 x}^{0}\right)+\left(A_{3}-B_{4}\right)\left(\delta_{1 y}^{0}-\delta_{2 y}^{0}\right)+\left(A_{9}+B_{10}-J_{z} p^{2}\right)\left(\theta_{1 y}^{0}-\theta_{2 y}^{0}\right)=0, \\
\left(A_{1}-B_{1}-m p^{2}\right)\left(\delta_{1 x}^{0}-\delta_{2 x}^{0}\right)=0 ; \\
\left(A_{1}+B_{3}-m p^{2}\right)\left(\delta_{1 y}^{0}+\delta_{2 y}^{0}\right)+\left(A_{3}+B_{4}\right)\left(\theta_{1 z}^{0}+\theta_{2 z}^{0}\right)=0 ;  \tag{30}\\
\left(A_{3}+B_{4}\right)\left(\delta_{1 y}^{0}+\delta_{2 y}^{0}\right)+\left(A_{9}+B_{10}-J_{z} p^{2}\right)\left(\theta_{1 z}^{0}+\theta_{2 z}^{0}\right)=0 .
\end{gather*}
$$

Let's denote by $d_{1}\left(p^{2}\right)$ the determinant of the coefficients of the unknowns $\delta_{1 x}^{0}+\delta_{2 x}^{0}$, $\delta_{1 y}^{0}-\delta_{2 y}^{0}, \theta_{1 z}^{0}-\theta_{2 z}^{0}$ in the system (28) and let's denote by $d_{2}\left(p^{2}\right)$ the determinant of the coefficients of the unknowns $\delta_{1 x}^{0}-\delta_{2 x}^{0}, \delta_{1 y}^{0}+\delta_{2 y}^{0}, \theta_{1 z}^{0}+\theta_{2 z}^{0}$ in system (30). The first six eigenpulsations result by solving the equations $d_{1}\left(p^{2}\right)=0 ; d_{2}\left(p^{2}\right)=0$. At the eigenpulsation
$p_{1}=\sqrt{\frac{A_{1}-B_{1}}{m}}$ corresponds the vibration mode where the shells have translation vibration in their own plane, one anti-phase to another on the axes $C_{1} x_{1}, C_{2} x_{2}$. For the other two solutions of the equation $d_{2}\left(p^{2}\right)=0$ one obtains that the vibration modes that correspond to these pulsation are represented by oscillatory rotational motions in phase around the points $P_{1}, P_{2}$ situated onto the axes $C_{i} x_{i}$ at the distance $-\lambda$ by the points $C_{i}$ (fig. 4).


Figure 4. Vibration modes.
For those three pulsation that are obtained from equation $d_{1}\left(p^{2}\right)=0$, vibration modes consist of oscillatory rotational motions in anti-phase, the first shell having the oscillation center at the point $P_{1}(-\eta, \xi)$, and the second at the point $P_{2}(-\eta, \xi)$. Analogously, for the system (27) the six eigenpulsations that are obtained correspond vibration modes that consist of oscillatory rotational motions in phase and in anti-phase around some straight lines situated in the planes $C_{1} x_{1} y_{1}$, respectively $C_{2} x_{2} y_{2}$ the straight lines being symmetric (fig. 3) relative to the axis $O_{3}^{*} y_{3}^{*}$.

## 6. CONCLUSIONS

In our paper we obtained the matrix equation of the free vibrations of a system of rigid bodies coupled by elastic bars, using the plückerian coordinates and the screw expressions of the forces and displacements. The equations are simple and easy to handle. Finally, an example is also presented.

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