ICSV14
Cairns • Australia
9-12 July, 2007
14th International Congress on Sound & Vibration

NON LINEAR RESPONSE OF BEAMS USING FINITE ELEMENT METHOD WITH SYMPLECTIC INTEGRATION SCHEME

R.R.Dash, T. Rout, B.B.Maharathi

Department of Mechanical Engineering,
Indira Gandhi Institute of Technology, Sarang, Orissa, India
rati_igit@yahoo.co.in

Abstract

The beam finite element matrices are formed with the assumption that it is subjected to large deflection with small strain and rotation. The finite element equations are represented in Hamiltonian form and are solved by the appropriate symplectic integration scheme. The induced axial forces are not averaged and the first second and third order stiffness matrices are formed to incorporate the effect of axial force. All conservative laws are observed during the numerical integration. The response of the beam is studied for free and forced vibration with un-damping cases.

1. INTRODUCTION

The geometrically non-linear or large amplitude vibrations of beam are a challenging field. Approximate analytical methods and numerical methods such as finite difference methods, finite element methods are used for solving these problems. The large deflections of beam make their response non-linear. In this article large deflection with small rotation of beams are considered for response analysis. The beam finite element matrices in Lagrangian form are developed for the purpose. Then the beam finite element equations in lagrangian form are expressed in Hamiltonian form to be solved by a suitable symplectic integration scheme. The equations of motion are linearised to obtain the finite element formulation. The errors caused by the finite element discretization are reduced to a large extent by increased number of iterations. The geometrical stiffness matrix is reformed in every incremental step to cope with the non-linear response of the beam. The direct integration method like Jacobian method is used to solve these problems but is non-symplectic. The result obtained from the solution using non-symplectic schemes may be different from the actual response of the system. Jacobian of the time transformation differs slightly from unity, thereby showing the system to damp artificially. In this article a finite element formulation is built directly by integrating the Lagrangian in a non-linear sense without averaging the axial forces. The second and third order stiffness matrices are derived which are independent of nodal displacement. The Hamiltonian and the corresponding Hamilton's equation are formulated for free vibration problems. For undamped forced vibrations the Hamiltonian and the corresponding Hamilton's equation are extended.

2. LAGRANGIAN

The axial strain ε and the curvature Γ for an initially straight Euler-Bernoulli beam undergoing large displacement with small strain and rotation, are given by

$$\varepsilon = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \ \Gamma = \left(\frac{\partial^2 w}{\partial x^2}\right)$$

where u(x, t) and w(x, t) are the axial and transverse displacements, respectively. The axial resultant force S and the bending moment M may be written as

S = $EA\varepsilon$, $M = EI\Gamma$ respectively

where *E A and I are* Young's modulus, cross-sectional area and moment of inertia of the cross-section respectively.

The corresponding un-damped Lagrangian is

$$L = \frac{1}{2}\int_0^l \rho A\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]dx - \frac{1}{2}\int_0^l \left(EA\varepsilon^2 + EI\Gamma^2\right)dx + \int_0^l \left(F_u u + F_w w\right)dx \tag{1}$$

where ρ is density of material t is time $F_u(x,t)$ and $F_w(x,t)$ are the axial and transverse loading, respectively. This formulation is different from the others as the induced axial forces S are not averaged, instead are function of displacements.

3. FINITE ELEMENT DISCRETIZATION

Let us consider an initially straight beam of uniform cross-section. The displacement functions u(x, t) and w(x, t) are interpolated by their nodal values u(t), *w(t)*, respectively, so that u(x, t) =[n(x)]{u(t)},

w(x, t) = [N(x)]{w(t)} where $[n(x)] = [1-\xi, \ \xi]$ and

$[N(x)] = [1-3\xi^2 + 2\xi^3, \xi\left(\xi^2 - 2\xi + 1\right)l, 3\xi^2 - 2\xi^3, \left(\xi^3 - \xi^2\right)l]$ are shape functions where $\xi = x/l$

After integration, the Lagrangian of equation (1) becomes

$$\begin{aligned} L = {} & \frac{1}{2}\{\dot{u}\}^T[M_u]\{\dot{u}\} + \frac{1}{2}\{\dot{w}\}^T[M_w]\{\dot{w}\} - \frac{1}{2}\{u\}^T[K_u]\{u\} - \frac{1}{2}\{w\}^T[K_w]\{w\} \\ & - \frac{1}{2}\{u\}^T[K_{uq}]\{w_q\} - \frac{1}{8}\{w_q\}^T[K_q]\{w_q\} + \{F_u\}^T\{u\} + \{F_w\}^T\{w\} \end{aligned} \tag{2}$$

where for element e with node i and j,

$$\{w\}^e = [w_i, \theta_i, w_j, \theta_j]^T, \ \{u\}^e = [u_i, u_j]^T$$

$$[M_u]^e = \int_0^l \rho A[n]^T[n]dx, \quad [M_w]^e = \int_0^l \rho A[N]^T[N]dx$$

$$[K_u]^e = \int_0^l EA[n']^T[n']dx, \quad [K_w]^e = \int_0^l EI[N'']^T[N'']dx$$

$$\{F_u\}^e = \int_0^l [n]^T F_u dx, \quad \{F_w\}^e = \int_0^l [N]^T F_w dx$$

$$\{w_q\}^e = [w_i^2, w_i\theta_i, w_i w_j, w_i\theta_j, \theta_i^2, \theta_i w_j, \theta_i\theta_j, w_j^2, w_j\theta_j, \theta_j^2]^T$$

$$[K_{uq}]^e = \int_0^l EA[n']^T[Q]dx$$

$$\left[K_q\right]^e = \int_0^l EA[Q]^T[Q]dx$$

Where [Q] is given by

$$[Q] = [\frac{36}{l^2}\left(\xi^2 - 2\xi^3 + \xi^4\right), \frac{12}{l}\left(-\xi + 5\xi^2 - 7\xi^3 + 3\xi^4\right), \frac{72}{l^2}\left(-\xi^2 + 2\xi^3 - \xi^4\right),$$
$$\frac{12}{l}\left(2\xi^2 - 5\xi^3 + 3\xi^4\right), 1 - 8\xi + 22\xi^2 - 24\xi^3 + 9\xi^4, \frac{12}{l}\left(\xi - 5\xi^2 + 7\xi^3 - 3\xi^4\right),$$
$$-4\xi + 22\xi^2 - 36\xi^3 + 18\xi^4, \frac{36}{l^2}\left(\xi^2 - 2\xi^3 + \xi^4\right), \frac{12}{l}\left(-2\xi^2 + 5\xi^3 - 3\xi^4\right), 4\xi^2 - 12\xi^3 + 9\xi^4]$$

[Q] has been obtained as given below

$$\left(\frac{\partial w}{\partial x}\right)^2 = [Q]\left\{w_q\right\}^e$$

It is clear from the above expressions that $\left[K_u\right]^e$ and $\left[K_w\right]^e$ are first order, $\left[K_{uq}\right]^e$ is second order and $\left[K_q\right]^e$ is the third order stiffness matrix. Similarly $\left\{w_q\right\}^e$ which is a quadratic function and $\{w\}^e$ is the second order displacement. The stiffness matrices are independent of the nodal displacements.

4. FINITE ELEMENT EQUATIONS

The finite element equations are obtained by applying the Hamilton principle to the Lagrangian given by equation (2)

$$\left[M_u\right]\{\ddot{u}\} + \left[K_u\right]\{u\} + \frac{1}{2}\left[K_{uq}\right]\left\{w_q\right\} = \left\{F_u\right\}$$
$$\left[M_w\right]\{\ddot{w}\} + \left[K_w\right]\{w\} + \frac{1}{2}\left[w_{q,w}\right]^T\left[K_{uq}\right]^T\{u\} + \frac{1}{4}\left[w_{q,w}\right]^T\left[K_q\right]\left\{w_q\right\} = \left\{F_w\right\} \quad (3)$$

First, let us consider the autonomous Hamiltonian system, where $w_{q,w}$ is differentiation of w_q with respect to w. For damped forced vibration problems, the finite element equations are obtained by adding the damping terms as follows.

5. SYMPLECTIC INTEGRATION SCHEMES

The Lagrangian formulation is more popular than the Hamiltonian due to it's direct application for solution of problems. But the Hamiltonian methods are advantageous if the resulting equations are symplectic in structure which will be very convenient in theoretical and numerical studies. Ruth [1] have developed some symplectic numerical methods for the Hamiltonian system.

The given system may be transformed to Hamiltonian system as written below.

$$\frac{\partial p}{\partial t} = -H_q(p,q,t) \quad , \quad \frac{\partial q}{\partial t} = H\ (p,q,t) \quad (4)$$

where $H(p_1, p_2, \ldots p_n, q_1, q_2, \ldots q_n, t)$ is Hamiltonian energy for inertial frame of reference.

For autonomous Hamiltonian system we can have the following.

$$\dot{z} = J\frac{\partial H}{\partial z}, \ z \in R^{2n}, \quad (5)$$

where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (6)

and $z_i = p_i$, $z_{i+1} = q_i$; $i \le n$

it's phase flow is denoted as $g^t(z) = g(z,t) = g_H(z,t)$, being a one parameter group of canonical maps. i.e.

g^0=identity, $g^{t_1+t_2} = g^{t_1 t_2}$

and if z_0 is taken as an initial condition, then $z(t) = g^t(z_0)$ is the solution of (5) with the initial value z_0.Different symplectic integration schemes to the system (5) have been constructed [1-4]. All the symplectic schemes for Hamiltonian systems preserve all the linear conservative quantities. Moreover, the time-centred symplectic scheme, which is an implicit scheme, preserves all the linear and quadratic conservative quantities. Let us consider the first- and second-order canonical difference schemes to eqns (5)

$$\begin{aligned} p_i^{k+1} &= p_i^k - hH_{qi}\left(p^{k+1}, q^k\right) \\ q_i^{k+1} &= q_i^k + hH_{pi}\left(p^{k+1}, q^k\right) \end{aligned} \tag{7}$$

$$\begin{aligned} p_i^{k+1} &= p_i^k - hH_{qi}\left(p^{k+1}, q^k\right) - \frac{h^2}{2}\left(\sum_{j=1}^{n} H_{qj} H_{pj}\right)_{qi} \left(p^{k+1}, q^k\right) \\ q_i^{k+1} &= q_i^k + hH_{pi}\left(p^{k+1}, q^k\right) + \frac{h^2}{2}\left(\sum_{j=1}^{n} H_{qj} H_{pj}\right)_{qi} \left(p^{k+1}, q^k\right) \end{aligned} \tag{8}$$

where h is the time step. These schemes are explicit with respect to the variable q. In general, the Hamiltonian for non-linear vibration problems of structures is separable, i.e. ***H* =** U(p) + V(q), where U(p) is quadratic in p, representing the kinetic energy. Thus, in this case, eqns (7) and (8) are a set of linear equation in p. The time-centered Euler scheme [1] for eqns (5) is

$$z^{k+1} = z^k + hJH_z\left(\frac{z^{k+1} + z^k}{2}\right)$$

The scheme may be written as

$$\begin{aligned} p_i^{k+1} &= p_i^k - hH_{qi}\left(\frac{p^{k+1} + p^k}{2}, \frac{q^{k+1} + q^k}{2}\right) \\ q_i^{k+1} &= q_i^k + hH_{pi}\left(\frac{p^{k+1} + p^k}{2}, \frac{q^{k+1} + q^k}{2}\right) \end{aligned} \tag{9}$$

which is an implicit second-order scheme. Thus the scheme (9) requires iterations.

For non-autonomous Hamiltonian systems we regard the time t as an additional dependent variable. That is, letting q_{n+1} = t can choose a parameter τ as a new independent variable. It is well known that

$$p_{n+1} = -H$$

which has a unit of energy, is the generalized momentum conjugate to the time t. For this special choice, the function $K(w) = p_{n+1} + H(z)$ with

$w = (q_1, q_2, \ldots. q_n, t, p_1, p_2, \ldots p_n, -H)^T$ will take the place of the Hamiltonian function H**.**

6. SEPARABLE HAMILTONIAN

If the Hamiltonian can be expressed as H=U(p)+V(q,t), then it is a separable Hamiltonian. The third-order three-step scheme for the above Hamiltonian is given below

$$p^{k+1} - p^{k} = c_1 h V_{,q}\left(q^{k}, t_0\right)$$
$$q^{k+1} - q^{k} = d_1 h U_{,p}\left(p^{k+1}\right)$$
$$p^{k+2} - p^{k+1} = c_2 h V_{,q}\left(q^{k+1}, t_0 + d_1 h\right)$$
$$q^{k+2} - q^{k+1} = d_2 h U_{,p}\left(p^{k+2}\right)$$
$$p^{k+3} - p^{k+2} = c_3 h V_{,q}\left(q^{k+2}, t_0 + \left(d_1 + d_2\right)h\right)$$
$$q^{k+3} - q^{k+2} = d_3 h U_{,p}\left(p^{k+3}\right)$$

where cs and ds must satisfy the following equations

$$c_1 + c_2 + c_3 = 1,\ d_1 + d_2 + d_3 = 1,$$
$$c_2 d_1 + c_3\left(d_1 + d_2\right) = 1/2$$
$$c_2 d_1^2 + c_3\left(d_1 + d_2\right)^2 = 1/3$$
$$d_3 + d_2\left(c_1 + c_2\right)^2 + d_1 c_1^2 = 1/3$$

Table 1. Convergence of linear and non-linear frequency parameters
For hinged-hinged fixed-hinged fixed-fixed beams at $A_{max}/\zeta = 1$

	Hinged-hinged			Fixed-hinged			Fixed-fixed		
No of elements	2	4	8	2	4	8	2	4	8
λ_L	98.18	97.46	97.41	242.14	238.03	237.74	516.93	501.89	500.65
λ_{NL}	124.20	117.19	116.01	277.92	265.93	262.50	549.16	532.52	525.14

λ_L linear frequency parameter= $\dfrac{\rho A \omega_0^2 l^4}{EI}$ λ_{NL} non-linear frequency parameter= $\dfrac{\rho A \omega^2 l^4}{EI}$

There are five equations for six unknowns; thus, there are many solutions. One particularly simple solution is obtained by setting $d_3 = 1$

$$c_1 = 7/24,\quad c_2 = 3/4,\quad c_3 = -1/24$$
$$d_1 = 2/3,\quad d_2 = -2/3,\quad d_3 = 1$$

The symplectic integration schemes mentioned above are used to solve Hamiltonian equations. For the non-linear problems, it is better to use explicit symplectic integration schemes than implicit schemes. One reason is the simplicity of computation and the other reason is the convenience to study its mapping $z^{n+1} = f\left(z^{n}\right)$. Explicit schemes, which do not require iteration procedures, suffer only the rounding error. Thus, the explicit symplectic integration schemes will preserve the symplectic structure better than the implicit symplectic integration schemes. Of course, from the viewpoint of computational stability, implicit schemes are better. By increasing the number of iterations, implicit symplectic integration schemes can also preserve the symplectic structure accurately. In practical implementation, variable time steps may be used for guaranteeing the stability of computation. On the other hand, from the numerical examples below, we know that the time-centered Euler scheme is better than the other schemes mentioned above.

Hamiltonian Equations

The Hamiltonian of the Lagrangian can be given as follows

$$H = \frac{1}{2}\{p_u\}^T[M_u]^{-1}\{p_u\} + \frac{1}{2}\{p_w\}^T[M_w]^{-1}\{p_w\} + \frac{1}{2}\{u\}^T[K_u]\{u\} + \frac{1}{2}\{w\}^T[K_w]\{w\}$$

$$\frac{1}{2}\{u\}^T[K_{uq}]\{w\} + \frac{1}{8}\{w_q\}^T[K_q]\{w_q\} - \{F_u\}^t\{u\} - \{F_w\}^T\{w\}$$

where $\{p_u\} = [M_u]\{\dot{u}\}$, $\{p_w\} = [M_w]\{\dot{w}\}$ are the momenta.

The resulting Hamilton's equation can be given as

$$\frac{\partial\{u\}}{\partial t} = [M_u]^{-1}\{p_u\}$$

$$\frac{\partial\{w\}}{\partial t} = [M_w]^{-1}\{p_w\}$$

$$\frac{\partial\{p_u\}}{\partial t} = -[K_u]\{u\} - \frac{1}{2}[K_{uq}]\{w_q\} + \{F_u\}$$

$$\frac{\partial\{p_w\}}{\partial t} = -[K_w]\{w\} - \frac{1}{2}[w_{q,w}]^T[K_{uq}]^T\{u\} - \frac{1}{4}[w_{q,w}]^T[K_q]\{w_q\} + \{F_w\}$$

7. NUMERICAL EXAMPLES

The finite element equations are solved by different integration schemes for the case of free vibrations, and un-damped forced vibrations. A uniform straight beam with immovable edges is considered for the purpose with hinged-hinged, fixed-hinged, and fixed-fixed boundary conditions.

7.1 Free vibrations

The free vibration of the beam for different end conditions is investigated first. The non-linear natural frequencies are dependent on the amplitude of vibration. The frequency ratios (non-linear/linear) are tabulated in Tables 1 and 2, where A_{max} is the amplitude of the midpoint of the beam, ζ is the radius of gyration. Table 1 shows the convergence of the linear and non-linear frequencies for $A_{max}/\zeta = 1$with different numbers of finite elements. Table 2 presents the frequency ratios at various amplitudes for hinged-hinged, fixed-hinged and fixed-fixed boundary conditions with four elements. The present results are obtained using the first-order and the third-order three-step symplectic integration schemes with equal time steps, respectively. The results of ref [1] are also presented in Table 2.

Table 2. Frequency ratios at various amplitude ratios for a uniform beam for hinged-hinged, fixed-hinged, fixed-fixed end conditions.

	Hinged-hinged		Fixed-hinged		Fixed-fixed	
A_{max}/ζ	Present	Ref[1]	present	Ref[1]	present	Ref[1]
0.2	1.0038	1.0040	1.0021	1.0023	1.0014	1.0014
0.4	1.0162	1.0165	1.0100	1.0100	1.0051	1.0051
0.6	1.0035	1.0036	1.0215	1.0210	1.0108	1.0108
0.8	1.063	1.063	1.0369	l .0367	1.0201	1.020
1.0	1.069	1.069	1.0570	1.0570	1.0301	1.0301
2.0	1.3432	1.3431	1.2077	1.2077	1.1175	1.1175
3.0	1.673	1.673	1.4156	1.4156	1.3219	1.3218
4.0	2.0392	2.0392	1.6580	1.6580	1.3911	1.3911

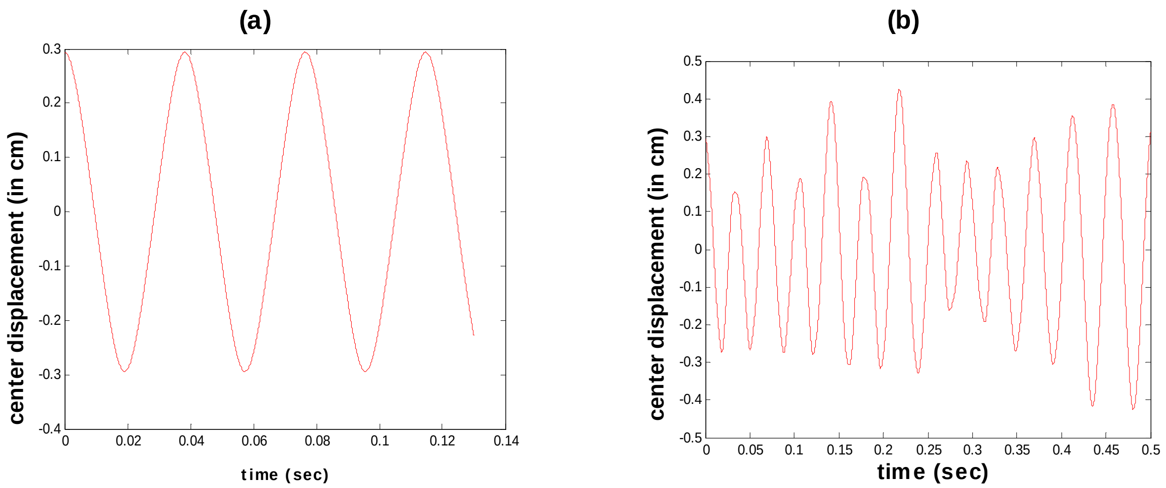

Figure 1 [The free vibration of an undamped beam[left fig. Response using one mode, right fig. response using two modes]

7.2 Undamped forced vibrations

A clamped-clamped beam under a static concentrated load of 284.3919 N acting at the mid-span at time t=0 was considered. The modulus of elasticity is $2.07 \times 10^{8}\,\mathrm{kN/m}$, the mass density is $2.71 \times 10^{-3}\,\mathrm{kg/cm^{3}}$ the length is l = 50.8 cm and the cross-section is 2.54 x 0.3175 cm. Because of symmetry of loading only one-half of the beam was modeled by six finite elements. The time history of the mid-span deflection computed by various schemes: first order, second order, third order and time-centred schemes, are shown in Figs 2(a-d), respectively. For the comparison of different schemes, we use a time step, $\Delta t = 3.0\mu s$. From Fig. 2 it is clear that the results obtained using different schemes are almost identical. This problem has been studied by many investigators. Mondkar and Powell [5] used five eight-node plane stress elements to model one-half of the beam. Yang and Saigal [6] used six beam elements with $\Delta t = 5{,}10\,\mu s$. McNamara [7] used five beam bending elements based on a central-difference operator with $\Delta t = 5.0\mu s$ The maximum displacement and the period of the first cycle were 0.02286 m and $2884\mu s$ in [7], 0.019558m and $2300\mu s$ in [5,6] and 0.019456m and $2151\mu s$ in this study. In addition, the response of the beam was studied only for first $5000\mu s$ in the existing references.

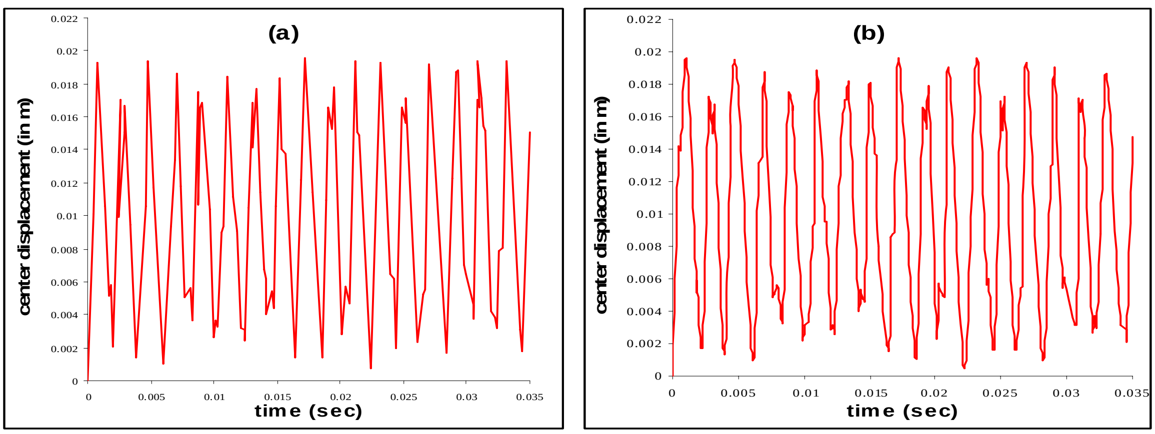

Figure.2 Mid-span displacement of example 2 from 0 to 35,000 μs using different schemes. (a) The first-order scheme with $\Delta t = 3\mu s$. (b) The second-order scheme with $\Delta t = 3\mu s$.

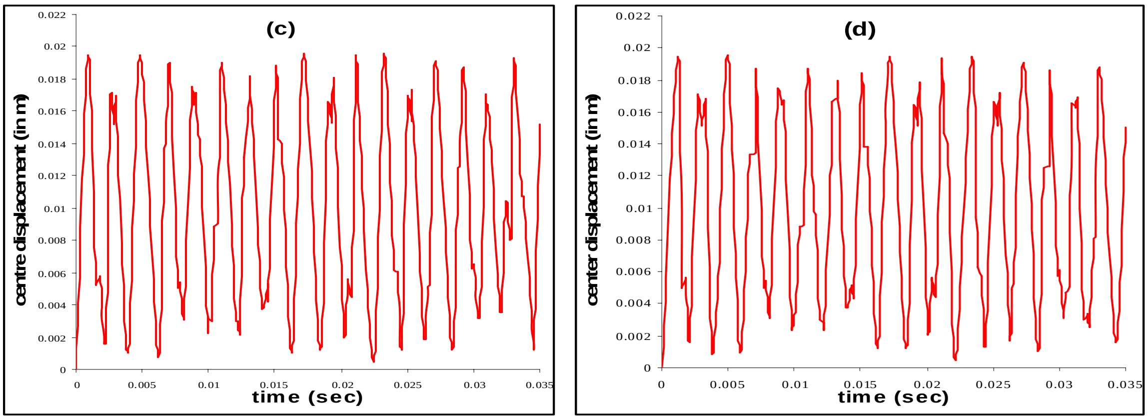

Figure 2. Mid-span displacement for example 2 from 0 to 35,000 μs using different schemes.
(c) The third-order three-step scheme with $\Delta t = 3\mu s$ (d) The time-centred Euler scheme with $\Delta t = 3\mu s$

8. CONCLUSIONS

The second-order and the third-order stiffness matrices are introduced for the geometrical non-linear finite element analysis of beams. An accurate formulation of non-linear finite elements is derived. The accuracy of the equations provides a good foundation for studying non-linear vibrations, chaos and dynamic bifurcation of structures. Symplectic integration methods have been successfully used to solve the Hamilton's equations. The results obtained for the three examples show that the present method is efficient for the dynamic analysis of beam problems with large deflections. The ideas may be extended to the problems of plates and shells. Thus, the procedure provides a good prospect to non-linear finite element static and dynamic analysis of solids and structures.

REFERENCES

[1] R. D. Ruth, A canonical mtegration technique. IEEE Trans. Nuclear Sci. NS-30, 2669-2671 (1983).

[2] K. Feng and M. Z. Qin, Hamiltonian algorithms for Hamiltonian systems and a comparative numerical study. Comput. Phys. Commun. 65, 1733187 (1991).

[3] K. Feng, H. M. Wu, M. Z. Qin and D. L. Wang, Construction of canonical difference schemes for Hamiltonian formalism via generating functions. J. Comuut. Math. 11. 71-96 (1989).

[4] M. Robert and A. Pau, The accuracy of symplectic integrators. Nonlinearity 5, 541-565 (1992).

[5] D. P. Mondkar and G. H. Powell, Finite element analysis of nonlinear static and dynamic response Zint. *J. Numer. Meth. Enang* 11, 499-520 (1977)

[6] T. Y. Yang and S. Sargal,.A simple element for static and dynamic response of beams with material and geometric nonlinearities. Int. J. Numer. Meth. Engng 20, 851-867 (1984).

[7] J. E. McNamara, Solutions schemes for problems of nonlinear structural dynamics, J. Pressure Vessel Technol. ASME 96, 966102 (1974).

[8] A. V. Srinivasan, Large amplitude free oscillations of beams and plates. Am. Inst. Aeronaut. J. 3, 1951-1953 (1965).

[9] Klaus-Jurgen Bathe, Finite Element Procedures, Prentice Hall of India, 1996.

[10] J. Mathews, K.D. Fink, Numerical Methods Using Matlab. Prentice Hall, 2004