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# VIBRATION OF ELASTIC RINGS EXCITED BY PERIODICALLY-SPACED MOVING SPRINGS

Sripathi Vangipuram Canchi[1] and Robert G. Parker[2]

[1]Department of Mechanical Engineering, University of California Berkeley
5146 Etcheverry Hall, Berkeley, CA 94720
[2]Department of Mechanical Engineering, Ohio State University
201 W. 19th Avenue, Columbus, OH 43210
parker.242@osu.edu

## Abstract

This work investigates parametric instabilities of in-plane bending vibrations of a thin elastic ring subject to forces from discrete rotating springs of arbitrary number, spacing, and orientation. Several configurations are examined, including systems with symmetric and asymmetric circumferential spring spacing. The method of multiple scales is applied to analytically identify instability boundaries as closed-form expressions, and two different numerical approaches are used to verify these results. The effects of different system parameters on the instability boundaries are studied analytically: the bending stiffness of the ring, the number of springs, and their stiffness, location, orientation and rotation speed. For several cases, well-defined properties for the occurrence or suppression of instabilities are obtained as simple relations in the system parameters.

## 1 INTRODUCTION

The present work addresses parametric excitation of bending vibrations in a stationary, thin ring subjected to forces from rotating springs. The motivation is from planetary gears, which are commonly used in automotive transmissions, helicopters, aircrafts and wind turbines. Planetary gear dynamics have historically been analyzed using lumped-parameter models that take the ring, planets, carrier and the sun as rigid bodies and the gear tooth meshes as springs. Recent studies, however, indicate that the deformable nature of the gear bodies, especially a thin ring gear, must be incorporated to accurately model the mechanics. Bending vibrations of the ring are parametrically excited by forces from the moving springs (ring-planet meshes), causing instability under certain conditions.

The bending vibrations of rings have been extensively studied analytically and through experiments [1-9]. However, little work on the vibration of rings subject to moving loads is found. Huang and Soedel [10, 11] presented closed-form solutions for the forced vibration of rotating rings subjected to harmonic and periodic point forces and spatially distributed forces. They compared those results with the inverted problem of a stationary ring with a moving point force. Metrikine and Tochilin [12] studied the vibrations of an elastic ring with a time-varying, moving point force to model train wheels. There seems to be no prior work on the vibration of rings subjected to forces from moving springs. A ring subjected to moving springs manifests as a parametrically excited system because the stiffness operator of the governing equation changes as the spring locations change. In this work, assuming that the moving spring stiffnesses are small compared to the bending stiffness of the ring, perturbation methods are employed to analytically identify parametric instability boundaries as closed-form expressions.

## 2 PROBLEM FORMULATION

Figure 1a shows a stationary, thin ring of uniform cross-section with mean radius $r$ subject to forces at its centroidal surface from $M$ multiple spring-sets, $j = 1, 2, ..., M$. Each spring-set consists of two springs of constant stiffness $k_{1j}$ and $k_{2j}$ oriented in mutually perpendicular directions. The orientation angle $\beta_j$ ($0 \le \beta_j < \pi/2$) is the angle between the spring $k_{2j}$ and the radial direction. The spring-sets are arbitrarily spaced so that $\phi_j$ ($0 \le \phi_j < 2\pi$) is the angular coordinate of the $j^{th}$ spring-set measured from fixed $\mathbf{E_1}$ at initial time $t = 0$. The above system describes the most general case of discrete spring forces on a stationary ring. The spring-sets rotate around the ring with a constant angular speed $\Omega_{sp}$. As they rotate, the orientation angles $\beta_j$ and the relative angular spacing $(\phi_j - \phi_{j-1})$ between any two adjacent spring-sets do not change. Referring to Figure 1b, $\theta$ is the angular coordinate of any point on the ring in the inertial reference frame $OE_1E_2$, and $\varphi$ is the angular coordinate of the same point in the rotating spring-fixed reference frame $Oe_1e_2$. The angular coordinates of a material point on the ring are related by $\theta = \varphi + \Omega_{sp} t$.

Only in-plane bending vibrations of the ring are considered in this work. Using inextensibility of the ring centroidal axis and applying Hamilton's principle, the equation for the tangential displacement $\hat{u}$ is obtained in nondimensional form as [13]

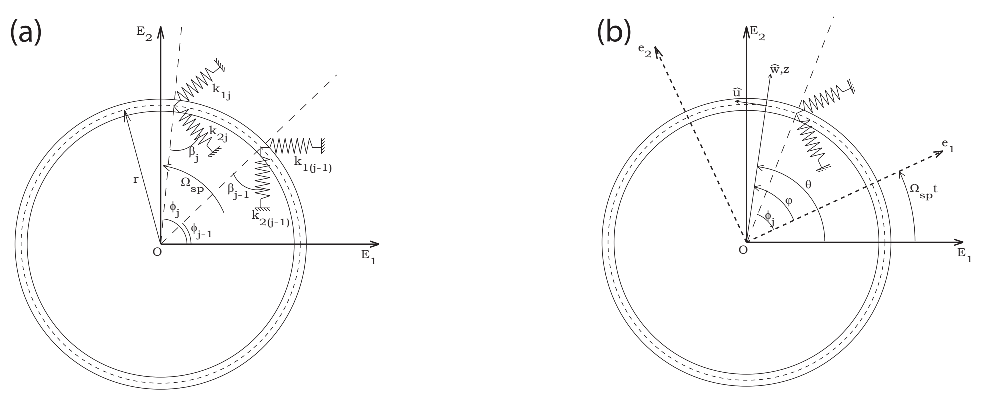


Figure 1 (a) Rotating ring on multiple rotating spring-sets. (b) Definition of reference frames.

$$\frac{\partial^2}{\partial\tau^2}\left(u-u''\right)-\left(u^{VI}+2u^{IV}+u''\right)+2\pi\varepsilon\left[\sum_{j=1}^{M}\left\{(c_{1j}u+c_{2j}u')\delta(\theta-v_{sp}\tau-\phi_j)\right\}\right.$$
$$\left.-\frac{\partial}{\partial\theta}\sum_{j=1}^{M}\left\{(c_{3j}u+c_{4j}u')\delta(\theta-v_{sp}\tau-\phi_j)\right\}\right]=0 \tag{1}$$

$$u=\frac{\hat{u}}{r},\quad \tau=\omega t,\quad \omega^2=\frac{EI}{\rho Ar^4},\quad v_{sp}=\frac{\Omega_{sp}}{\omega},\quad \varepsilon=\frac{k}{2\pi k_b},\quad k_b=EI/r^3,$$
$$k=\max(k_{1j},k_{2j}),\quad j=1,2,...,M;\quad c_{1j}=\frac{1}{k}(k_{1j}\cos^2\beta_j+k_{2j}\sin^2\beta_j), \tag{2}$$
$$c_{2j}=c_{3j}=\frac{1}{k}(k_{1j}-k_{2j})\cos\beta_j\sin\beta_j,\quad c_{4j}=\frac{1}{k}(k_{1j}\sin^2\beta_j+k_{2j}\cos^2\beta_j)$$

Here, $E$ is the Young's modulus, $A$ is the cross-sectional area and $\rho$ is the density of the ring. The important nondimensional parameters are $\varepsilon$, which represents the ratio of the stiffness of the spring-sets to the bending stiffness $k_b$ of the ring, and the nondimensional spring rotation speed $v_{sp}$. The time-varying spring forces parametrically excite the system as the angular locations of the spring-sets changes periodically. Parametric instabilities occur for particular values of the magnitude ($\varepsilon$) and frequency ($v_{sp}$) of the time-varying excitation. Under the assumption that $\varepsilon$ is a small quantity (stiffness of all the springs are of the same order and small compared to the bending stiffness of the ring), perturbation methods are used to obtain closed-form approximations for the regions of parametric instability in the $v_{sp}$-$\varepsilon$ plane.

## 3 PARAMETRIC INSTABILITY ANALYSIS

To capture principal and combination instabilities, a two-term Galerkin discretization is applied to (1) using the expansion $u(\theta,\tau)=\psi_n(\tau)e^{in\theta}+\psi_m(\tau)e^{im\theta}+cc$, where $cc$ represents the complex conjugate of all preceding terms with $m,n\geq 2$ to eliminate rigid body motion. Substituting into (1), and forming the inner product of the resulting equation with each of the basis functions yields the coupled equations

$$\frac{d^2\psi_n}{d\tau^2}+p_n^2\psi_n+\varepsilon q_n\{G_{nn}\psi_n+H_{nn}e^{-i2nv_{sp}\tau}\overline{\psi}_n+G_{nm}e^{i(m-n)v_{sp}\tau}\psi_m+H_{nm}e^{-i(m+n)v_{sp}\tau}\overline{\psi}_m\}=0$$
$$\frac{d^2\psi_m}{d\tau^2}+p_m^2\psi_m+\varepsilon q_m\{G_{mm}\psi_m+H_{mm}e^{-i2mv_{sp}\tau}\overline{\psi}_m+G_{mn}e^{-i(m-n)v_{sp}\tau}\psi_n+H_{mn}e^{-i(m+n)v_{sp}\tau}\overline{\psi}_n\}=0 \tag{3}$$

$$G_{nm}=\sum_{j=1}^{M}\left[\varsigma_{nm}^j+i\xi_{nm}^j\right]e^{i(m-n)\phi_j}=\sum_{j=1}^{M}\left[(c_{1j}+nmc_{4j})+i(mc_{2j}-nc_{3j})\right]e^{i(m-n)\phi_j}$$
$$H_{nm}=\sum_{j=1}^{M}\left[\gamma_{nm}^j+i\lambda_{nm}^j\right]e^{-i(m+n)\phi_j}=\sum_{j=1}^{M}\left[(c_{1j}-nmc_{4j})+i(-mc_{2j}-nc_{3j})\right]e^{-i(m+n)\phi_j} \tag{4}$$
$$p_n^2=n^2(n^2-1)^2/(1+n^2),\quad q_n=1/(1+n^2)$$

where $G_{nm}$ and $H_{nm}$ depend on the spring-set parameters, and $p_n$ represents the nondimensional natural frequency for the bending vibrations of a free ring in the $n$ nodal diameter mode ($e^{\pm in\theta}$). Although the only excitation frequency in (1) is $v_{sp}$, Galerkin discretization to the modal coordinates in (3) results in the two excitation frequencies $(m+n)v_{sp}$ and $(m-n)v_{sp}$. The $O(\varepsilon)$ terms in (3) arise from the projection onto the $m$ nodal diameter mode of the force the rotating springs exert when the ring deflects in the $n$ nodal diameter mode. Such a projection results in forces varying with the modulated frequencies $(m+n)v_{sp}$ and $(m-n)v_{sp}$. Therefore, the system (3) may be viewed as having the two different excitation frequencies $(m+n)v_{sp}$ and $(m-n)v_{sp}$.

Parametric instabilities arise when the nondimensional spring rotation speed $v_{sp}$ is close to particular combinations of the free ring natural frequencies $p_m$ and $p_n$. Application of the method of multiple scales [13] shows that terms leading to resonant response (secular terms) may arise when $(m \pm n)v_{sp} \approx p_m + p_n$, which are conditions for summation type combination instabilities of the first (plus sign: $m+n$) and second (minus sign: $m-n$) kind, or when $(m \pm n)v_{sp} \approx p_m - p_n$, which are difference type combination instabilities of the first and second kind. Principal instability corresponding to the $n^{th}$ mode is obtained with $m = n$, giving $nv_{sp} \approx p_n$. (In arriving at these expressions, $m \geq n$ and $v_{sp} \geq 0$ are assumed without loss of generality.)

Considering the parametric instability when $(m+n)v_{sp} \approx p_m + p_n$, let $(m+n)v_{sp} = p_m + p_n + \varepsilon\hat{\sigma}$ where $\hat{\sigma}$ is the detuning parameter. Elimination of terms leading to unbounded, aperiodic response in (3) yields the expression for summation combination instability boundaries of the first kind as

$$v_{sp} = \frac{(p_m + p_n) + \varepsilon\hat{\sigma}}{(m+n)}, \qquad \hat{\sigma} = \frac{q_n}{2p_n}\sum_{j=1}^{M}\varsigma_{nn}^{j} + \frac{q_m}{2p_m}\sum_{j=1}^{M}\varsigma_{mm}^{j} \pm \sqrt{\frac{q_n q_m \hat{\Delta}}{p_n p_m}}$$
$$\hat{\Delta} = \hat{I}^2 + \hat{R}^2, \quad \hat{I} = \sum_{j=1}^{M}(\lambda^j \cos\hat{\mu}_j - \gamma^j \sin\hat{\mu}_j) \quad \hat{R} = \sum_{j=1}^{M}(\gamma^j \cos\hat{\mu}_j + \lambda^j \sin\hat{\mu}_j) \quad \hat{\mu}_j = (m+n)\phi_j \tag{5}$$

and the $n^{th}$ mode principal instability boundaries are given by $v_{sp} = (2p_n + \varepsilon\hat{\sigma})/2n$. Considering the parametric instability when $(m-n)v_{sp} \approx p_m + p_n$, let $(m-n)v_{sp} = p_m + p_n + \varepsilon\tilde{\sigma}$, and the summation combination instability boundaries of the second kind are similarly obtained as

$$v_{sp} = \frac{(p_m + p_n) + \varepsilon\tilde{\sigma}}{(m-n)}, \qquad \tilde{\sigma} = \frac{q_n}{2p_n}\sum_{j=1}^{M}\varsigma_{nn}^{j} + \frac{q_m}{2p_m}\sum_{j=1}^{M}\varsigma_{mm}^{j} \pm \sqrt{\frac{q_n q_m \tilde{\Delta}}{p_n p_m}}$$
$$\tilde{\Delta} = \tilde{I}^2 + \tilde{R}^2, \quad \tilde{I} = \sum_{j=1}^{M}(\varsigma^j \cos\tilde{\mu}_j + \xi^j \sin\tilde{\mu}_j) \quad \tilde{R} = \sum_{j=1}^{M}(\varsigma^j \cos\tilde{\mu}_j - \xi^j \sin\tilde{\mu}_j) \quad \tilde{\mu}_j = (m-n)\phi_j \tag{6}$$

It can be shown mathematically that difference type instabilities when $(m \pm n)v_{sp} \approx p_m - p_n$ cannot occur because such a condition yields complex solutions for the detuning parameter.

Numerical verification of the analytical solution is performed using two approaches. In the first method, Galerkin discretization of (1) with basis functions as $e^{\pm in\theta}$ yields a time-varying state matrix form $\dot{\mathbf{x}}(\tau) = \mathbf{P}(\tau)\mathbf{x}(\tau)$ with period $T = 2\pi / v_{sp}$. Floquet's theorem is applied to obtain the regions of instability in the $v_{sp}$-$\varepsilon$ parameter plane. This method is computationally intensive because of the numerical time integration, especially for small $v_{sp}$ and so long period $T$. Alternatively, the system may be analyzed in a spring-fixed reference frame which allows a computationally efficient evaluation of system stability. In this reference frame, the governing equation attains a time-invariant state matrix form $\dot{\mathbf{x}}(\tau) = \mathbf{Q}\mathbf{x}(\tau)$ and the system stability is dictated by the real part of the eigenvalues of $\mathbf{Q}$. By computing $\mathbf{Q}$ and its eigenvalues for ranges of $\varepsilon$ and $v_{sp}$ (or other parameters), the regions of instability are obtained [13].

## 4 RESULTS AND DISCUSSION

The analytical and numerical results for the stability boundaries are plotted in the $v_{sp}$-$\varepsilon$ plane. Figure 2a shows these results for the case of one radial rotating spring. The analytical stability boundaries are obtained using (5) and (6) considering the first three bending modes, namely, modes with 2, 3 and 4 nodal diameters. The numerical instability regions are also computed taking the first three modes (starting with $n = 2$) to discretize the tangential displacement. The agreement between the analytical and numerical results is evident, even for relatively large values of the spring stiffness to bending stiffness ratio $\varepsilon$. Both numerical methods yield the same instability regions.

### 4.1 Effect of number of spring-sets, symmetry and asymmetry

The width of the parametric instability regions are governed by $\hat{\Delta}$ and $\tilde{\Delta}$ defined in (5) and (6), and instabilities appear only if $\hat{\Delta}$ or $\tilde{\Delta}$ are non-zero. When all the spring-sets are identical with the same individual spring stiffnesses and orientation angles, and when the spring-sets are equally spaced, then $\hat{\Delta}$ is non-zero only when the nodal diameters $m$ and $n$ are related to the total number of spring-sets $M$ by $m + n = sM$, and $\tilde{\Delta}$ is non-zero only when $m - n = sM$ where $s = 1, 2, 3, ...$ [13]. Hence, For the case of identical and equally spaced spring-sets (referred to as the symmetric case), symmetry of the system suppresses many principal and combination instabilities. An example with three radial springs is presented considering the 2, 3 and 4 nodal diameter modes. The only instabilities that appear in the symmetric case (Figure 2b) are principal instability due to the 3 nodal diameter mode, and combination instability of the first kind from interaction of the 2 and 4 nodal diameter modes. In the asymmetric cases of identical, unequally spaced spring-sets (Figure 2c), or non-identical, equally spaced spring-sets (Figure 2d), all possible instabilities occur.

### 4.2 Parametric study

The analytical solution shows how parametric instability regions change due to a variation in the system parameters. As examples, the effects of orientation angle ($\beta$), stiffness angle ($\alpha$) and modulation angle ( $\Upsilon$ ) are considered. For the case of one rotating spring, the orientation angle is varied from $\beta = 0^o$ (radial) to $\beta = 90^o$ (tangential), and the result is shown in Figure 3a. Larger instability regions appear when the same spring is oriented in the radial direction versus the tangential direction, and this result is easily verified analytically [13].

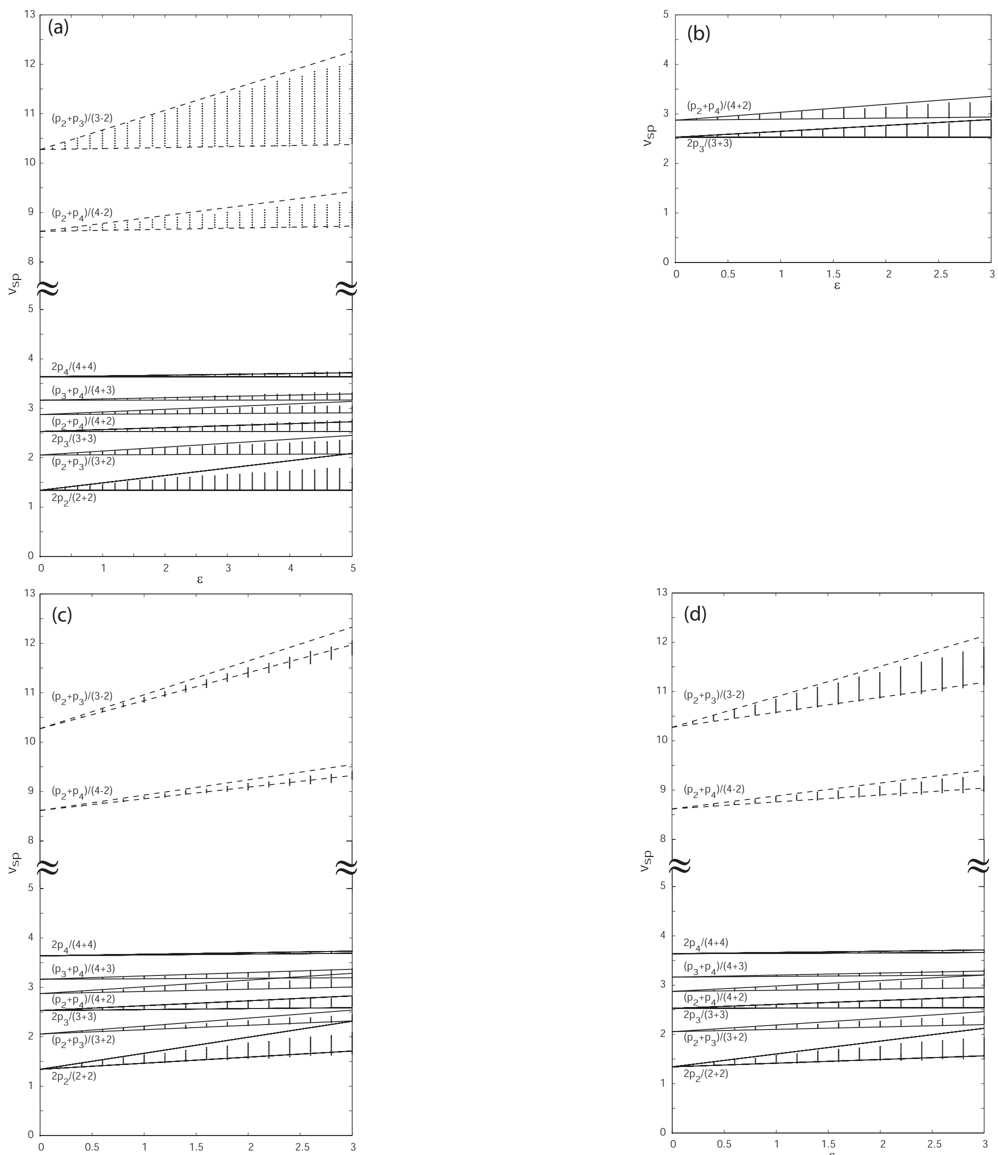


Figure 2 Parametric instability regions (a) One radial rotating spring with $k_{11} = 0$ , $k_{21} = k$ , $\beta_1 = 0^o$ (b) Symmetric case of three identical and equally spaced rotating radial springs with $k_{1j} = 0$ , $k_{2j} = k$ , $\beta_j = 0^o$ . Asymmetric cases: (c) Three identical but unequally spaced rotating radial springs with $k_{1j} = 0$ , $k_{2j} = k$ , $\beta_j = 0^o$ , $\phi_1 = 0^o$ , $\phi_2 = 110^o$ , $\phi_3 = 250^o$ . (d) Three non-identical but equally spaced rotating radial springs with $k_{21} = k_{23} = k_{12} = k$ , $\beta_j = 0^o$ . –, principal and combination instabilities of first kind; --, combination instability of second kind; ***, numerical solution.

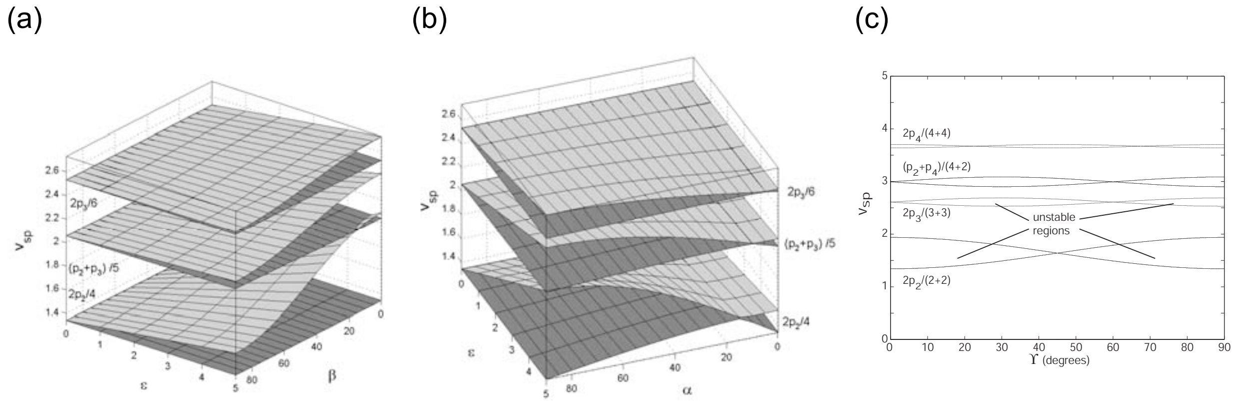


Figure 3 Parametric instability regios (a) Effect of spring orientation angle: one rotating spring with $k_{11}=0$, $k_{21}=k$. (b) Effect of stiffness angle: one rotating spring-set with $k_1=k$, $\beta=0^o$. (c) Effect of modulation angle: two pairs of diametrically opposed radial springs with $\varepsilon=k/2\pi k_b=1$, $k_{1j}=0$, $k_{2j}=k$, $\beta_j=0^o$, $\phi_1=0^o$, $\phi_2=90^o+\Upsilon^o$, $\phi_j=180^o$, $\phi_j=270^o+\Upsilon^o$.

The stiffness angle is defined as $\alpha_j=\tan^{-1}\left(k_{2j}/k_{1j}\right)$ ($0^o\le\alpha_j\le 90^o$), so that $k_{1j}=k_j\cos\alpha_j$, $k_{2j}=k_j\sin\alpha_j$ where $k_j=\sqrt{k_{1j}^2+k_{2j}^2}$. Considering a single rotating spring-set defined by $k,\alpha$, Figure 3b shows the instability zones for different values of $\alpha$ with $\beta=0^o$. Interestingly, for $\beta=0^o$, the different combination instabilities of the first kind, including principal instabilities, vanish for particular values of $\alpha$, as indicated in Figure 3b by closing of the instability regions in the $v_{sp}$-$\varepsilon$ plane. Analytical investigation shows that combination instabilities of the first kind vanish for the $n$ and $m$ nodal diameter modes if $\tan\alpha=1/nm$ [13]. Consequently, principal instabilities corresponding to 2 and 3 nodal diameter modes vanish when $\alpha=14.03^o$ and $\alpha=6.34^o$, respectively, and the combination instability due to their interaction vanishes when $\alpha=9.46^o$ (Figure 3b). These results hold for arbitrary spacing of spring-sets with different $k_j$, so long as $\beta_j=0^o$, and the stiffness angles for all the spring-sets are the same. Combination instabilities of the second kind do not exhibit similar behavior because there is no value of $\alpha$ ($0^o\le\alpha\le 90^o$) for which $\tilde{\Delta}=0$.

Diametrically opposed spring-set pair configuration is of practical importance in planetary gear systems where equal planet spacing is not possible due to assembly requirements. The effect of angular spacing between the diameters on the parametric instabilities is shown in Figure 3c for two pairs of diametrically opposed radial springs ($\varepsilon=1$) located at $\phi_1=0^o$, $\phi_2=90^o+\Upsilon^o$, $\phi_3=180^o$ and $\phi_4=270^o+\Upsilon^o$, where $\Upsilon$ is the modulation angle. The width of the instability regions vary with the modulation angle and are plotted for the range $\Upsilon=0^o$ to $\Upsilon=90^o$. It may be easily shown that for identical and diametrically opposed spring-sets, parametric instabilities cannot occur if $m\pm n$ is odd [13]. This is confirmed from Figure 3c. If $m\pm n$ is even, however, $\hat{\Delta}$ or $\tilde{\Delta}$ may become zero depending on the values of $m,n,\phi_j$ and $M$. For example, in Figure 3c, the principal instability from $n=2$ vanishes when $\Upsilon=45^o$.

## 5 CONCLUSIONS

In-plane bending vibrations of a stationary ring are parametrically excited when subject to multiple, rotating spring-sets of arbitrary stiffness and orientation. Instability boundaries are obtained analytically as closed-form expressions using a first order perturbation method, and these analytical results compare well with numerical results. Although there is essentially one independent excitation frequency (spring-set rotation speed $v_{sp}$), it is coupled to the nodal diameters $m, n$ by projections of the spring force onto the vibration modes in Galerkin discretization. As a result, the modal coordinate equations have the parametric excitation frequencies $(m+n)v_{sp}$ and $(m-n)v_{sp}$. Summation combination instabilities of two kinds occur corresponding to two different values of $v_{sp}$: one at lower frequency $(p_m + p_n)/(m+n)$ and another at higher frequency $(p_m + p_n)/(m-n)$. Difference type instabilities do not exist for this problem. The stiffness, orientation, and relative spacing between spring-sets govern the occurrence and width of the instability regions. Equally spaced, identical spring-sets and diametrically opposed, identical spring-sets are shown to suppress several of the instabilities. Simple rules relating the nodal diameters of the suppressed instabilities and the number of spring-sets are obtained and demonstrate the advantages symmetry can play in physical systems. The effect of fixed spring-sets together with moving spring-sets, and the effects of a rotating ring and time-varying stiffness spring-sets have also been investigated in related studies [13, 14].

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