ICSV14
Cairns • Australia
9-12 July, 2007

14th International Congress on Sound & Vibration

MODIFIED GENERAL TRANSFER MATRIX METHOD FOR LINEAR ROTOR BEARING SYSTEMS

B B Maharathi[1], R R Dash[1] and T Rout[1]

[1]Department of Mechanical Engineering, IGIT, Sarang
769 146, Dhenkanal, Orissa, India
bbmaharathi@yahoo.com

Abstract

A modified general transfer matrix method is developed for the steady state response analysis of linear flexible rotor-bearing systems in the frequency domain with fixed matrix size. In this paper, the modifications of the transfer matrix method based on Timoshenko Beam Theory are derived from the concept of continuous systems instead of the conventional lumped system concept and the paper tries to extend the transfer matrix method to fit a synchronous elliptical orbit and a non-synchronous multi-lobed whirling orbit. To demonstrate the applicability of this method, a three-disk rotor-bearing system is used as a physical model in the numerical analysis.

1. INTRODUCTION

Dynamic characteristics of rotor-bearing systems are obtained by various methods such as; transfer matrix method (Lumped system and continuous system), finite element technique and dynamic stiffness method considering different influencing parameters related to rotor, disk and bearings [1-5]. In this work, an attempt has been made to formulate the general transfer matrix method based on continuous system model and superimposed of vibrations of the shaft in both the plains for dynamic response and critical speed of rotor systems.

2. TRANSFER MATRIX OF SHAFT

The elastic relations of the shaft element based on Timoshenko beam theory as shown in Figs. 1 & 2 are given as follows :

In the X-Z plane;

$$\frac{\partial^4 X}{\partial Z^4}-\left(\frac{\rho}{K_sG}+\frac{\rho}{EI}\right)\frac{\partial^4 X}{\partial Z^2\partial t^2}+\frac{\rho^2}{K_sGE}\frac{\partial^4 X}{\partial t^4}+\frac{\rho A}{EI}\frac{\partial^2 X}{\partial t^2}-\frac{2\rho\omega}{E}\left(\frac{\partial^3 Y}{\partial Z^2\partial t}-\frac{\rho}{K_sG}\frac{\partial^3 Y}{\partial t^3}\right)-\frac{P}{EI}\frac{\partial^2 X}{\partial Z^2}$$
$$-\frac{T\rho}{EIK_sG}\frac{\partial^3 Y}{\partial Z\partial t^2}+\frac{T}{EI}\frac{\partial^3 Y}{\partial Z^3}=0 \tag{1a}$$

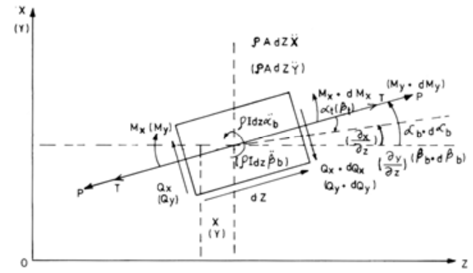


Figure 1. A rotating shaft element.

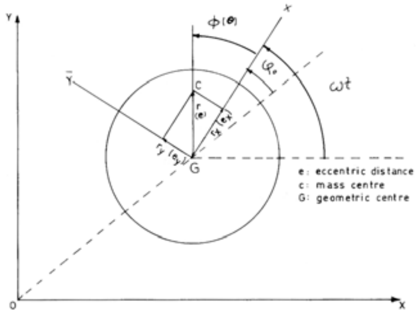


Figure 2. Geometries of shaft and disk unbalance.

In the X-Y plane;

$$\frac{\partial^4 Y}{\partial Z^4}-\left(\frac{\rho}{K_s G}+\frac{\rho}{EI}\right)\frac{\partial^4 Y}{\partial Z^2 \partial t^2}+\frac{\rho^2}{K_s GE}\frac{\partial^4 Y}{\partial t^4}+\frac{\rho A}{EI}\frac{\partial^2 Y}{\partial t^2}+\frac{2\rho\omega}{E}\left(\frac{\partial^3 X}{\partial Z^2 \partial t}-\frac{\rho}{K_s G}\frac{\partial^3 X}{\partial t^3}\right)-\frac{P}{EI}\frac{\partial^2 Y}{\partial Z^2}$$

$$+\frac{T\rho}{EIK_s G}\frac{\partial^3 X}{\partial Z \partial t^2}-T\frac{\partial^3 X}{\partial Z^3}=0 \tag{1b}$$

Since the whirling orbit is elliptical and synchronous $(\Omega = w)$ as shown in Fig.3, the steady state solutions can be written as;

$$X(z,t)=X_C(t)\cos\Omega t+X_S(t)\sin\Omega t\,,\qquad Y(z,t)=Y_C(t)\cos\Omega t+Y_S(t)\sin\Omega t \tag{2}$$

where, X_C, X_S, Y_C, Y_S are mode functions. Introducing $\overline{X}=X_C+jX_S$ and $\overline{Y}=Y_C+jY_S$, the general solutions can be written as;

$$\overline{X}(z)=U_C e^{\lambda z}+jU_S e^{\lambda z} \quad \text{and} \quad \overline{Y}(z)=V_C e^{\lambda z}+jV_S e^{\lambda z}.$$

U_C, U_S, V_C, V_S are real constants and λ is the characteristic value with respect to a specific natural mode. Separating real and imaginary parts, it yields as;

$$\begin{aligned}
&(\lambda^4+f_1\lambda^2+g)U_C+[-(h\lambda^2+k)+j(c\lambda^3+d\lambda)]\,V_S=0\\
&j(\lambda^4+f_1\lambda^2+g)U_S+[j(h\lambda^2+k)+(c\lambda^3+d\lambda)]\,V_C=0\\
&[(h\lambda^2+k)-j(c\lambda^3+d\lambda)]\,U_S+(\lambda^4+f_1\lambda^2+g)\,V_C=0\\
&[-j(h\lambda^2+k)-(c\lambda^3+d\lambda)]\,U_C+j(\lambda^4+f_1\lambda^2+g)\,V_S=0
\end{aligned} \tag{3}$$

Since U_C, U_S, V_C, V_S being non-trival, the characteristic equation can be obtained by setting the determinant of Eq. (3) to zero. It yields as;

$$\{(\lambda^4+f_1\lambda^2+g)+[j(c\lambda^3+d\lambda)-(h\lambda^2+k)]\}^2\cdot\{(\lambda^4+f_1\lambda^2+g)-[j(c\lambda^3+d\lambda)-(h\lambda^2+k)]\}^2=0 \tag{4}$$

where, $f_1=\left(\frac{\rho\omega^2}{E}\right)+\left(\frac{\rho\omega^2}{K_S G}\right)-\frac{P}{EI}$, $\qquad g=\left(\frac{\rho^2\omega^4}{K_S GE}\right)-\left(\frac{\rho A\omega^2}{EI}\right)$

$$h=\frac{2\rho\omega^2}{E},\; k=\frac{2\rho^2\omega^4}{K_S GE}, \qquad c=\frac{T}{EI},\; d=\frac{T\rho\omega^2}{K_S GEI}$$

By separating Eq. (4) into two parts as follows :

$$\left\{\left(\lambda^4+f_1\lambda^2+g\right)+\left[\,j\left(c\lambda^3+d\lambda\right)-\left(h\lambda^2+k\right)\right]\right\}^2=0 \tag{5a}$$

$$\left\{\left(\lambda^4+f_1\lambda^2+g\right)-\left[\,j\left(c\lambda^3+d\lambda\right)-\left(h\lambda^2+k\right)\right]\right\}^2=0 \tag{5b}$$

The four complex roots of Eq. (5a) are $\lambda_i = a_i + jb_i$ for i = 1 ~ 4 and other four complex roots of Eq. 5(b) are $\lambda_i = a_i + jb_i$ for i = 5 ~ 8.

For $\lambda_i = a_i + jb_i$, i = 1 ~ 4, $U_C = V_S$, $U_S = -V_C$

For $\lambda_i = a_i + jb_i$, i = 5 ~ 8, $U_C = -V_S$, $U_S = V_C$

Thus, the four homogeneous solutions are as follows :

$$
\begin{aligned}
X_C(Z) &= \sum_{i=1}^{4} A_i e^{a_i Z}\cos b_i Z + \sum_{i=5}^{8} A_i e^{a_i Z}\cos b_i Z - \sum_{i=1}^{4} B_i e^{a_i Z}\sin b_i Z + \sum_{i=5}^{8} B_i e^{a_i Z}\sin b_i Z \\
X_S(Z) &= \sum_{i=1}^{4} A_i e^{a_i Z}\sin b_i Z + \sum_{i=5}^{8} A_i e^{a_i Z}\sin b_i Z + \sum_{i=1}^{4} B_i e^{a_i Z}\cos b_i Z - \sum_{i=5}^{8} B_i e^{a_i Z}\cos b_i Z \\
Y_C(Z) &= -\sum_{i=1}^{4} A_i e^{a_i Z}\sin b_i Z + \sum_{i=5}^{8} A_i e^{a_i Z}\sin b_i Z - \sum_{i=1}^{4} B_i e^{a_i Z}\cos b_i Z - \sum_{i=5}^{8} B_i e^{a_i Z}\cos b_i Z \\
Y_S(Z) &= \sum_{i=1}^{4} A_i e^{a_i Z}\cos b_i Z - \sum_{i=5}^{8} A_i e^{a_i Z}\cos b_i Z - \sum_{i=1}^{4} B_i e^{a_i Z}\sin b_i z - \sum_{i=5}^{8} B_i e^{a_i Z}\sin b_i Z
\end{aligned}
\tag{6}
$$

where,

A_i and B_i are real constants (i = 1 ~ 8).

Differentiating Eq. (6) with respect to z and substituting Z=0, it becomes

$$
[W(Z=0)] = [M]_{17\times17}\begin{bmatrix} A \\ B \\ 1 \end{bmatrix}_{17\times1} \tag{7}
$$

where,

$$[W(Z=0)] = [X_C(0), X_S(0), Y_C(0), Y_S(0), 1]^t$$

$$X_C = [X_C, X_C^{'}, X_C^{''}, X_C^{'''}]^t, \quad X_S = [X_S, X_S^{'}, X_S^{''}, X_S^{'''}]^t$$

$$Y_C = [Y_C, Y_C^{'}, Y_C^{''}, Y_C^{'''}]^t, \quad Y_S = [Y_S, Y_S^{'}, Y_S^{''}, Y_S^{'''}]^t$$

$A = [A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8]^t$, $B = [B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8]^t$

t = transpose of the array.

$$
\begin{bmatrix} A \\ B \\ 1 \end{bmatrix} = [M]_{17\times17}^{-1}[W(z=0)] \tag{8}
$$

At Z = L,

$$
[W(Z=L)] = [H]_{17\times17}\begin{bmatrix} A \\ B \\ 1 \end{bmatrix} \tag{9}
$$

where, $[W(Z=L]=[X_C(L), X_S(L), Y_C(L), Y_S(L)]^t$

Now,

$$
[W(Z=L)] = [H]_{17\times17}[M]_{17\times17}^{-1}[W(Z=0)] = [N]_{17\times17}[W(Z=0)] \tag{10}
$$

The state variables can be written as;

$$\overline{X} = X_C + jX_S, \quad \overline{X}' = \overline{\alpha} - \frac{\overline{Q}_X}{K_S GE}, \quad \overline{X}'' = \left(1 - \frac{P}{K_S GE}\right)\frac{\overline{M}_X}{EI} - \frac{\rho\omega^2}{K_S G}\overline{X}$$

$$\overline{X}''' = \left(-C_f\rho I\omega^2 - \frac{\rho\omega^2}{K_S G}\right)\overline{\alpha} - j2C_f\rho I\omega^2\overline{\beta} + \left[C_f + \frac{\rho\omega^2}{K_S^2 G^2 A}\right]\overline{Q}_X - \frac{C_f T}{EI}\overline{M}_X \tag{11a}$$

and, $\overline{Y} = Y_C + jY_S, \quad \overline{Y}' = \beta - \frac{\overline{Q}_y}{K_S GE}, \quad \overline{Y}'' = \left(1 - \frac{P}{K_S GE}\right)\frac{\overline{M}_y}{EI} - \frac{\rho\omega^2}{K_S G}\overline{Y}$

$$\overline{Y}''' = \left(-C_f\rho I\omega^2 - \frac{\rho\omega^2}{K_S G}\right)\overline{\beta} - j2C_f\rho I\omega^2\overline{\alpha} + \left[C_f + \frac{\rho\omega^2}{K_S^2 G^2 A}\right]\overline{Q}_y + \frac{C_f T}{EI}\overline{M}_y \tag{11b}$$

where, $C_f = \frac{(K_S GA - P)}{EIK_S GA}$

Further, the state vectors can be written as;

$$\overline{X} = X_C + jX_S, \quad \overline{Y} = Y_C + jY_S, \quad \overline{\alpha} = \alpha_c + j\alpha_S, \quad \overline{\beta} = \beta_c + j\beta_S$$
$$\overline{M}_X = M_{XC} + jM_{XS}, \quad \overline{M}_Y = M_{YC} + jM_{YS}$$
$$\overline{Q}_X = M_{XC} + jQ_{XS}, \quad \overline{Q}_Y = Q_{YC} + jQ_{YS} \tag{12}$$

Combining Eqs. (11) and (12) and separating real and imaginary terms, it yields as;

$$[W(Z = L)]_{17\times1} = [A]_{17\times17}[S_l]_{17\times1} \tag{13}$$

where,

$$[W(Z = L)] = \{X_C, X_C', X_C'', X_C''', X_S, X_S', X_S'', X_S''', Y_C, Y_C', Y_C'', Y_C''', Y_S, Y_S', Y_S'', Y_S''', 1\}^t$$

$$[S] = \{X_C, X_S, Y_C, Y_S, \alpha_C, \alpha_S, \beta_C, \beta_S, M_{XC}, M_{XS}, M_{YC}, M_{YS}, Q_{XC}, Q_{XS}, Q_{YC}, Q_{YS}, 1\}^t$$

At Z = 0, $[W(Z = 0)] = [A]\{S_0\}_{17\times1}$ (14)

Combining Eqs. (13) and (14), it yields

$$\{S_i\}_{17\times1} = [A]^{-1}[N][A]\{S_0\}_{17\times1} = [T]_{17\times17}[S_0]_{17\times1} \tag{15}$$

where, $[T]_{17\times17}$ is the transfer matrix of the rotor segment (Z = L).

3. TRANSFER MATRIX OF THE DISK

The equilibrium condition, the relations of the state variables between the right and left side of an unbalanced disk is expressed as [1-4];

$$\begin{Bmatrix} S^R \\ 1 \end{Bmatrix}_{17\times1} = [T_d]_{17\times17}\begin{Bmatrix} S^L \\ 1 \end{Bmatrix}_{17\times1} \tag{16}$$

4. TRANSFER MATRIX OF THE BEARING

From the force equilibrium, the relationship of the state variables between the left and right sides can be written as [1-5];

$$\begin{Bmatrix} S^R \\ 1 \end{Bmatrix}_{17\times1} = [T_b]_{17\times17}\begin{Bmatrix} S^L \\ 1 \end{Bmatrix}_{17\times1} \tag{17}$$

5. OVERALL TRANSFER MATRIX

The overall transfer matrix of the rotor from one end to another can be written as;

$$\{S_n\}_{17\times1} = [U]_{17\times17}\{S_0\}_{17\times1} \tag{18}$$

where, $[U] = [T_n][T_{bj}][T_{n-1}][T_{dj}]........[T_{db}][T_2][T_{bi}][T_1]\{S_0\}$ (19)

6. SOLUTION ANALYSIS

Since the shear force and bending moments are zero at both the free ends of the shaft, Eq. (18) becomes;

$$\begin{Bmatrix} S_n^{'} \\ 0 \\ 1 \end{Bmatrix} = \begin{bmatrix} [U_{11}]_{8\times8} & [U_{12}]_{8\times8} & [U_1]_{8\times1} \\ [U_{21}]_{8\times8} & [U_{22}]_{8\times8} & [U_2]_{8\times1} \\ \{0\}^t_{8\times1} & \{0\}^t_{8\times1} & 1 \end{bmatrix}_{17\times17} \begin{Bmatrix} S_0^{'} \\ 0 \\ 1 \end{Bmatrix} \tag{20}$$

where,

$$S' = \{X_C, X_S, Y_C, Y_S, \alpha_C, \alpha_S, \beta_C, \beta_S\}^t, \quad O = \{O, O, O, O, O, O, O, O\}^t$$

By deleting all elements, which are related to moments and shear forces, it gives;

$$\{S_n^{'}\}_{8\times1} = [U_{11}]_{8\times8}\{S_0^{'}\} + [U_1]_{8\times1} \tag{21a}$$

and $[U_{21}]_{8\times8}.\{S_0^{'}\}_{8\times1} + [U_2]_{8\times1} = \{O\}_{8\times1}$ (21b)

Subscripts ‘O’ and ‘n’ are labelled for stages.

Eight simultaneous equations in Eq. (21b) are solved to obtain the eight state vectors $\{S_0^{'}\}_{8\times1}$.

$$\{S_0^{'}\}_{8\times1} = \{X_{CO}, X_{SO}, Y_{CO}, Y_{SO}, \alpha_{CO}, \alpha_{SO}, \beta_{CO}, \beta_{SO}\}^t$$

Then, the state vectors at any desired stage (p^{th} stage) can be obtained by using Eq. (18) through matrix operation for the determination of dynamic response.

$$\{S_p\}_{17\times1} = [U_p]_{17\times17}\{S_o\}_{17\times1} \tag{22}$$

where, $\{S_o\}_{17\times1} = \{\,\{S_O^{'}\}^t_{8\times1} \quad \{O\}^t_{8\times1} \quad 1\}^t$

7. FINITE ELEMENT ANALYSIS

A typical flexible rotor-bearing system consists of a rotor composed of discrete disks and rotor segments, and discrete elastic bearings as shown in Fig.4. Each rotor element is modelled as an eight degree of freedom element with two rotations and two translations at each end in each plane. The co-ordinates $(q_i^e,\ i = 1 \sim 8)$ are the time-dependent and point displacements of the finite rotor element.

$$\{q_i^e\}^t = \{q_1^e, q_2^e, q_3^e, q_4^e, q_5^e, q_6^e, q_7^e, q_8^e\}^t = \{v_1, w_1, \theta_1, \phi_1, v_2, w_2, \theta_2, \phi_2\}^t \tag{23}$$

The Lagrangian equation of motion for the finite rotor element at the constant speed can be written as;

$$[M^e]\{\ddot{q}^e\} - \omega[G^e]\{\dot{q}^e\} + [K^e]\{q^e\} = [F^e] \tag{24}$$

The Lagrangian equation of motion of the unbalanced rigid disk with gyroscopic effect at the constant angular speed can be written as;

$$\{[M_t^d]+[M_r^d]\}\{\ddot{q}^d\}-\omega[G^d]\{\dot{q}^d\}=\{F^d\} \tag{25}$$

where, $\{F^d\}=[F_s^d]\sin\omega t+[F_c^d]\cos\omega t=\begin{Bmatrix}-u_z^d\omega^2\\ u_y^d\omega^2\\ 0\\ 0\end{Bmatrix}\sin\omega t+\begin{Bmatrix}-u_y^d\omega^2\\ u_z^d\omega^2\\ 0\\ 0\end{Bmatrix}\cos\omega t$

$$u_z^d=m_i\cdot e_i\sin\beta,\quad u_y^d=m_i\cdot e_i\cos\beta$$

The governing equation of the bearing can be written as;

$$[O]\{\ddot{q}^b\}+[C^b]\{\dot{q}^b\}+[K^b]\{q^b\}=\{F^b\}$$

or
$$\begin{bmatrix}C_{xx} & C_{xy}\\ C_{yx} & C_{yy}\end{bmatrix}\begin{Bmatrix}\dot{v}^b\\ \dot{w}^b\end{Bmatrix}+\begin{bmatrix}K_{xx} & K_{xy}\\ K_{yx} & K_{yy}\end{bmatrix}\begin{Bmatrix}v^b\\ w^b\end{Bmatrix}=\{F^b\} \tag{26}$$

The assembled undamped system equation is of the form :

$$[M^s]\{\ddot{q}^s\}+\{-\omega[G^s]+[C^s]\}\{\dot{q}^s\}+[K^s]\{q^s\}=[F^s] \tag{27}$$

The steady state solution is

$$\{q^s\}=\{q_c^s\}\cos\omega t+\{q_s^s\}\sin\omega t \tag{28}$$

Differentiating any separating cosine and sine terms, it yields

$$\begin{bmatrix}([K^S]-\omega^2[M^S]) & \omega^2([C^S]-[G^S])\\ -\omega^2([C^S]-[G^S]) & ([K^S]-\omega^2[M^S])\end{bmatrix}\begin{Bmatrix}\{q_c^s\}\\ \{q_s^s\}\end{Bmatrix}=\begin{Bmatrix}\{F_c^s\}\\ \{F_s^s\}\end{Bmatrix} \tag{29}$$

Then

$$\begin{Bmatrix}\{q_c^s\}\\ \{q_s^s\}\end{Bmatrix}=\begin{bmatrix}([K^s]-\omega^2[M^s]) & \omega^2([C^s]-[G^s])\\ -\omega^2([C^s]-[G^s]) & ([K^s]-\omega^2[M^s])\end{bmatrix}^{-1}\begin{Bmatrix}\{F_c^s\}\\ \{F_s^s\}\end{Bmatrix} \tag{30}$$

The solution of Eq.(30) provides $\{q_c^s\}$ and $\{q_s^s\}$ and back substitution in Eq.(28) determines the unbalance response of the rotor system at the required position.

8. NUMERICAL ANALYSIS

In order to illustrate the accuracy of the theoretical analysis, a three disks rotor system mounted on fluid-film bearing is considered as physical model as shown in Fig.5 [disk mass (M^d) = 13.47 kg, polar mass moment of inertia (I_p^d) = 1.02 x 10^{-1} kg m^2, transverse mass moment of inertia (I_T^d) = 5.11 x 10^{-2} kg m^2, direct stiffness coefficients (K_{xx} = K_{yy}) = 1x10^7 N/m, cross-coupled stiffness coefficients (K_{yx} = K_{xy}) = 5 x 10^6 N/m, direct damping and cross damping coefficients are C_{xx} = C_{yy} = 2 x 10^3 N/m/sec, C_{xy} = C_{yx} = 0, respectively].

Two cases are considered in the numerical analysis.

Case-1 : Bearing without damping (K_{xx}, K_{xy}, K_{yy}, K_{yx})

Case-2 : Bearing with stiffness and damping (K_{xx}, K_{xy}, K_{yy}, K_{yx}, C_{xx}, C_{yy})

9. DISCUSSION & CONCLUSION

The amplitudes of response are calculated by the absolute values of the major axis of the elliptical orbits. The calculated dynamic responses at disk-1 are plotted as solid curves in Figs.6 & 7 for the first natural mode region based on modified transfer matrix method. The dashed curves are calculated from the finite element method by six elements. It is shown that two curves for both the cases are close to each other. The cross-stiffness make the critical speed split into two values for its asymmetry, as evaluated in case-2 at 44.83 Hz and 51.82 Hz. The critical speed can be located from the maximum response peak of the frequency response curve. For asymmetrical bearing due to cross-stiffness and damping, the whiling orbit becomes elliptical rather than circular. Synchronous response due to unbalance mass and non-synchronous response due to the journal motion with frequency being three times of the rotating speed is considered $(\Omega = 3\omega)$ and the numerical results are produced in Figs.8 & 9.

The modified general transfer matrix method is a versatile technique for the determination of dynamic characteristics of any rotor-bearing system considering various influencing parameters related to rotor, disk and bearings in the frequency domain.

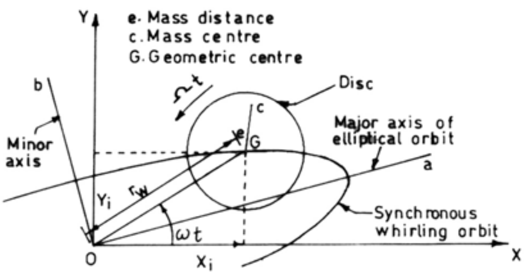


Figure 3. Response of a synchronous whirling orbit.

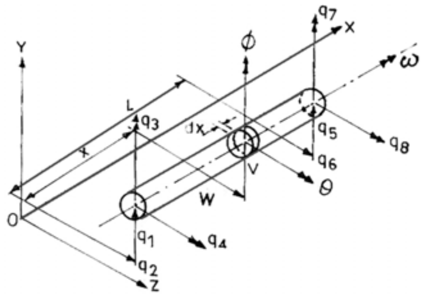


Figure 4. A finite rotor element.

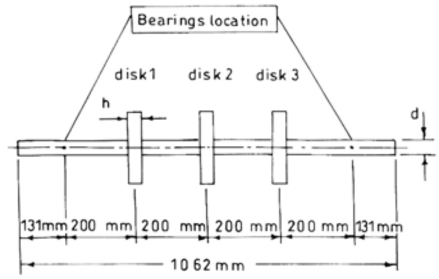


Figure 5. A three-disk rotor bearing system (h=0.04 m, d = 0.04 m).

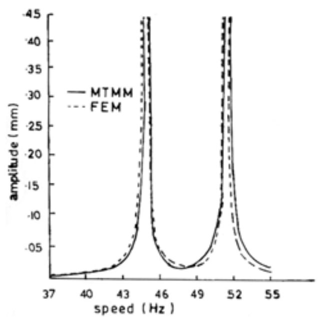
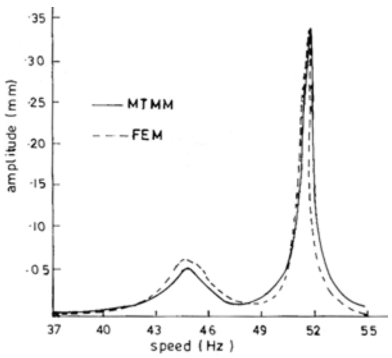


Figure 6. Dynamic responses at disk-1 (case-1).



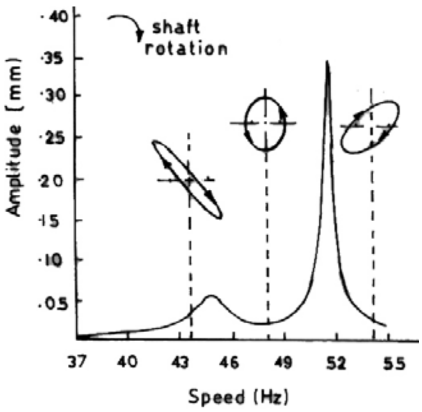


Figure 7. Dynamic responses of disk-1, case-2.

Figure 8. Synchronous whirling orbit of disk-1, case-2.

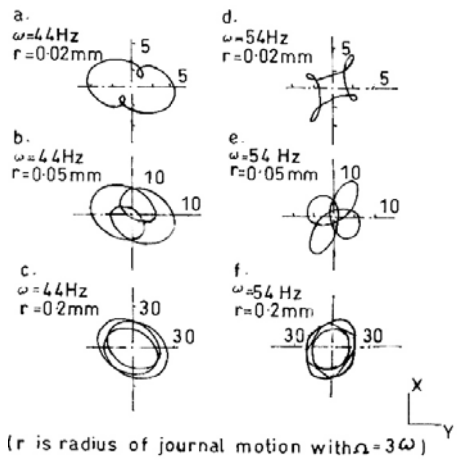


Figure 9. Non-synchronous whirling orbit of disk-1, case-2.

10. REFERENCES

[1] B. B. Maharathi, *"Dynamic behaviour analysis of linear rotor-bearing system using the complex transfer matrix technique"*, Int. Journal of Acoustics and Vibration, **10**, 115-130 (2005).

[2] B. B. Maharathi, et al, *"Dynamic behaviour analysis of a dual-rotor system using the transfer matrix method"*, Int. Journal of Acoustics and Vibration, **10,** 115-128 (2004).

[3] B. B. Maharathi, and A. K. Behera, *"A transfer matrix method for dynamic behaviour of rotor-bearing systems"*, Trans. of Canadian Society of Mechanical Engineering, 99-CSME-11, June 15 (2001).

[4] B. B. Maharathi, and A. K. Behera, *"Transfer matrix analysis of high speed rotors by distributed elements"*, Trans. of Mechanical Engineering, Australia, **Me-25,** 21-30 (2001).

[5] T. Someya, *Data book of Journal-Bearing*, Springer-Verlag, Tokyo, 1988.