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A CENTER MANIFOLD ALGORITHM FOR NONLINEAR DYNAMIC SYSTEMS WITH TIME-DELAY AND PARAMETRIC EXCITATION

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Abstract

A technique for center manifold reduction of nonlinear delay differential equations with time-periodic coefficients is presented. The DDEs considered here have at most cubic nonlinearities multiplied by a perturbation parameter. The periodic terms and matrices are not assumed to have predetermined norm bounds, thus making the method applicable to systems with strong parametric excitation. Perturbation expansion converts the nonlinear response problem into solutions of a series of non-homogenous linear ordinary differential equations with time periodic coefficients. One set of linear non-homogenous ODEs is solved for each power of the perturbation parameter. Each ODE is solved by a Chebyshev spectral collocation method. Thus we compute a finite approximation to the nonlinear infinite-dimensional map for the DDE. The accuracy of the method is demonstrated with a nonlinear delayed Mathieu equation, a milling model, and a single inverted pendulum with a periodic retarded follower force and nonlinear restoring force in which the amplitude of the limit cycle associated with a flip bifurcation is found analytically and compared to that obtained from direct numerical simulation.

1. INTRODUCTION

Dimensional reduction of nonlinear DDEs has been considered by researchers in the past using different approaches. A center manifold algorithm for constant coefficient DDEs near Hopf bifurcation points was first formulated in [1]. Unlike the case for center manifold reduction of ODEs [2], the algorithm for DDEs is necessarily stated in the language of functional analysis and requires the description of the adjoint system. The method was first applied to a practical system (Hopf bifurcation in machine tool vibrations) much later [3,4]. Recently, alternative techniques for dimensional reduction of nonlinear DDEs have been proposed. These include the use of stiff and soft substructures [5], the method of multiple scales [6], and a Galerkin projection technique which reduces a DDE to a small number of ODEs [7]. It should be noted that all of these techniques are designed for autonomous nonlinear DDEs.

The study of nonlinear time-periodic DDEs has included the computation of the response [8] and the use of normal forms and continuation algorithms. Much of the practical

work in the engineering community has focused on the milling problem which, unlike the case of turning, is modeled by DDEs with time-periodic coefficients. It has been observed in numerous studies that, in addition to the secondary Hopf or Neimark-Sacker bifurcation, such a system can also exhibit flip or period-doubling bifurcations. The center manifold algorithms in [1,3,4] cannot be directly applied to the time-periodic case without additional complexities associated with redefining the adjoint equation. Nevertheless a center manifold computation for the case of flip bifurcation in single DOF low immersion milling was carried out in [9] by assuming that the system can be accurately modeled using a 2-dimensional map of the cutting tool's position and velocity from one instantaneous cut to the next. For such a system, the coefficients are essentially periodically-spaced Dirac delta functions. The problem of milling with a finite cutting time, on the other hand, requires a calculation using functional analytic tools to decompose the dynamics on the center manifold [10]. This algorithm is based on previous efforts in the mathematical literature to generalize the adjoint equation and formalize the center manifold theory for the case of periodic DDEs. The motivation for the present paper is in a possible alternative method which does not involve the explicit computation of the adjoint.

The method here for dimensional or center manifold reduction of periodic DDEs makes use of a nonlinear extension of the infinite dimensional monodromy operator defined in [11] for linear periodic DDEs. The nonlinear operator maps the initial function onto each subsequent delay interval. Iteration of the nonlinear operator produces the solution via the 'method of steps'. The linear part of this operator has been previously approximated by a finite dimensional matrix using a variety of numerical techniques, including orthogonal polynomial expansion [12] and collocation [13] for the purpose of stability analysis. In this paper we use the method of Chebyshev spectral collocation which is discussed at length by Trefethen [15] and was applied to study the stability of linear periodic DDEs in [13-14]. It is assumed that the periodically modulated nonlinearities are multiplied by a perturbation parameter. Using a classical perturbation technique, the solution in any interval is expanded using the solution from the previous interval in the in the zeroth order part. The expansion proceeds in a usual way by powers of perturbation parameter. Collecting terms for like powers of the perturbation parameter leads to a series of linear non-homogenous ordinary differential equations which are then solved using Chebyshev spectral collocation. Quadratic and cubic nonlinearities result in similar nonlinearities in the resulting approximate nonlinear DDE solution map. Increased accuracy is obtained by increasing the number of collocation points and by adding more terms to the perturbation expansion. Next, center manifold reduction is applied to the nonlinear map. An algorithm that performs center manifold reduction on large nonlinear maps in MATLAB© is developed for the purpose of making the problem computationally tractable.

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2. DEVELOPMENT OF NONLINEAR MAP

Consider a time periodic nonlinear DDE given by

$$\begin{aligned} \dot{x} &= A(t)x + B(t)x(t-\tau) + \varepsilon f(x(t), x(t-\tau), t) \\ x(s) &= \phi(s) \qquad \forall\ s \in [-\tau, 0] \end{aligned} \tag{1}$$

where $x(t)$ is an n-dimensional state vector and $A(t) = A(t+T)$, $B(t) = B(t+T)$ are $n \times n$ coefficient matrices with principal period T. Here τ is a fixed delay and ε is the perturbation parameter. The nonlinear term $f(\alpha, \beta, t) = f(x(t), x(t-\tau), t)$ is a vector-valued function, with n components, which is periodic with period T in its third input. For the purpose of simplicity, it is assumed that the delay τ is the same as the principal period T.

(Our method, however, generalizes to $\tau \neq T$ case.) Writing equation (1) with the summation convention and denoting $x^{\tau}(t) = x(t-\tau)$ yields

$$\dot{x}_i = A_i^j(t)x_j + B_i^j(t)x_j^{\tau} + \varepsilon f_i(x, x^{\tau}, t) \tag{2-a}$$

We now assume that $f(\alpha, \beta, t)$ is a polynomial of degree 3 in α and β. That is, we may write (2-a) in the form

$$\begin{aligned}\dot{x}_i = A_i^j(t)x_j + B_i^j(t)x_j^{\tau} + \varepsilon[C_i^{jk}(t)x_j x_k + D_i^{jk}(t)x_j^{\tau}x_k^{\tau} + F_i^{jk}(t)x_j^{\tau}x_k] + \\ \varepsilon[G_i^{jkl}(t)x_j x_k x_l + H_i^{jkl}(t)x_j x_k x_l^{\tau} + J_i^{jkl}(t)x_j x_k^{\tau}x_l^{\tau} + K_i^{jkl}(t)x_j^{\tau}x_k^{\tau}x_l^{\tau}]\end{aligned} \tag{2-b}$$

For the linear system obtained by setting $\varepsilon = 0$ in (2), one may analyze stability by considering the infinite dimensional monodromy operator which maps the solution $x^{(p-1)}(t)$ in $[(p-1)\tau, p\tau]$ onto the solution $x^{(p)}(t)$ in $[p\tau, (p+1)\tau]$. The eigenvalues of the monodromy operator, the Floquet multipliers, determine the stability of the linear system. An accurate finite dimensional approximation of the monodromy operator, that is, a "monodromy matrix", can be computed using Chebyshev spectral collocation [13-15] and other techniques.

For the nonlinear DDE (2-b), we will approximate the corresponding nonlinear map $x^{(p-1)}(t) \mapsto x^{(p)}(t)$ by a perturbation series in ε. Note that (1) typically has parameters but these are not explicitly included in the analysis for now. Obtaining the nonlinear map in a parameter dependent ("symbolic") manner would allow for bifurcation analysis of equation (2-b) at the critical points. Later we re-introduce the bifurcation parameter via versal deformation theory as in [16] for the ODE case.

Suppose the solution $x(t)$ in the interval $[0, \tau] = [0, T]$ for (2-b) is written in the form

$$x(t) = \phi(t) + \xi_0(t) + \varepsilon\xi_1(t) + \varepsilon^2\xi_2(t) + \varepsilon^3\xi_3(t) + ... \tag{3}$$

Here $\phi(t)$ is a vector of known functions as in (1) defined over $[0, \tau]$ by translation from $[-\tau, 0]$. The functions $\xi_i(t), i = 0,1,2,..$ are unknown n-dimensional vectors for each power of ε. The initial conditions for these unknown vectors are

$$\xi_0(0) = \phi(\tau) - \phi(0), \quad \xi_1(0) = 0, \quad \xi_2(0) = 0, \quad \xi_3(0) = 0,... \tag{4}$$

so that $x(0) = \phi(\tau)$. Substituting (3) in (2-b) and retaining terms up to third order, the following sequence of linear time-periodic ODEs is obtained after collecting like powers of ε:

$$\varepsilon^0: \quad \dot{\xi}_{0i} = A_i^j\xi_{0j} + (A_i^j + B_i^j)\phi_j - \dot{\phi}_i \tag{5-a}$$

$$\begin{aligned}\varepsilon^1: \quad \dot{\xi}_{1i} = A_i^j\xi_{1j} + C_i^{jk}(\xi_{0j} + \phi_j)(\xi_{0k} + \phi_k) + F_i^{jk}(\xi_{0j} + \phi_j)\phi_k + D_i^{jk}\phi_l\phi_n \\ + G_i^{jkl}(\xi_{0j} + \phi_j)(\xi_{0k} + \phi_k)(\xi_{0l} + \phi_l) + H_i^{jkl}(\xi_{0j} + \phi_j)(\xi_{0k} + \phi_k)\phi_l + \\ J_i^{jkl}(\xi_{0j} + \phi_j)\phi_k\phi_l + K_i^{jkl}\phi_j\phi_k\phi_l\end{aligned} \tag{5-b}$$

$$\begin{aligned}\varepsilon^2: \quad \dot{\xi}_{2i} &= A_i^j \xi_{2j} + C_i^{jk}[(\xi_{0j}+\phi_j)\xi_{1k} + \xi_{1j}(\xi_{0k}+\phi_k)] + F_i^{jk}\xi_{1j}\phi_k \\ &\quad G_i^{jkl}[(\xi_{0j}+\phi_j)\xi_{0k}\xi_{1l} + \xi_{1j}(\xi_{0k}+\phi_k)\xi_{0l} + \xi_{0j}\xi_{1k}(\xi_{0l}+\phi_l)] + \\ &\quad H_i^{jkl}[(\xi_{0j}+\phi_j)\xi_{0k}\phi_l + \xi_{1j}(\xi_{0k}+\phi_k)\phi_l] + J_i^{jkl}\xi_{1j}\phi_k\phi_l\end{aligned} \tag{6-a}$$

$$\begin{aligned}\varepsilon^3: \quad \dot{\xi}_{3i} &= A_i^j \xi_{3j} + C_i^{jk}[\xi_{2j}(\xi_{0k}+\phi_k) + \xi_{1j}\xi_{1k} + (\xi_{0j}+\phi_j)\xi_{2k}] + \\ &\quad G_i^{jkl}[\xi_{2j}(\xi_{0k}+\phi_k)(\xi_{0l}+\phi_l) + (\xi_{0j}+\phi_j)\xi_{2k}(\xi_{0l}+\phi_l) + \\ &\quad (\xi_{0j}+\phi_j)(\xi_{0k}+\phi_k)\xi_{2l} + \xi_{1j}\xi_{1k}(\xi_{0l}+\phi_l) + \xi_{1j}(\xi_{0k}+\phi_k)\xi_{1l} + \\ &\quad (\xi_{0j}+\phi_j)\xi_{1k}\xi_{1l}] + H_i^{jkl}[(\xi_{0j}+\phi_j)(\xi_{0k})\phi_l + \xi_{2j}(\xi_{0k}+\phi_k)\phi_l] + \\ &\quad J_i^{jkl}\xi_{2j}\phi_k\phi_l\end{aligned} \tag{6-b}$$

We now consider a sequence of infinite-dimensional maps corresponding to equations (5,6) of the form

$$\begin{aligned}\varepsilon^0: \quad & m_{\xi 0} = V m_\phi \\ \varepsilon^1: \quad & m_{\xi 1} = P_2^1(m_\phi, V m_\phi) + P_3^1(m_\phi, V m_\phi) = P^1(m_\phi) \\ \varepsilon^2: \quad & m_{\xi 2} = P_2^2(m_\phi, V m_\phi, P^1(m_\phi)) + P_3^2(m_\phi, V m_\phi, P^1(m_\phi)) = P^2(m_\phi) \\ \varepsilon^3: \quad & m_{\xi 3} = P_2^3(m_\phi, V m_\phi, P^1(m_\phi), P^2(m_\phi)) + P_3^3(m_\phi, V m_\phi, P^1(m_\phi), P^2(m_\phi)) \\ & \quad = P^3(m_\phi)\end{aligned} \tag{7}$$

Here P_i^j denotes a polynomial of homogeneous degree i in its $j+1$ coefficients. Equation (7) is exactly equivalent to (5,6) until one chooses a particular *finite representation* for the infinite-dimensional vector m_w corresponding to an arbitrary function $w(t)$. Using equations (3,7), we express the solution $x(t) = x^{(p)}(t)$ in an interval $[p\tau, (p+1)\tau]$ as a nonlinear transformation of the solution in the previous interval $\phi(t) = x^{(p-1)}(t)$ in $[(p-1)\tau, p\tau]$ as

$$m_x(p) = (I+V)m_\phi + \varepsilon P^1(m_\phi) + \varepsilon^2 P^2(m_\phi) + \varepsilon^3 P^3(m_\phi) \tag{8}$$

which is exactly equivalent to equation (1) with the perturbation expansion of (3). In the this paper we do not discuss the numerical approximation of this nonlinear map in detail using the method of Chebyshev collocation. For more information we refer the reader to [13-15].

3. CENTER MANIFOLD REDUCTION

The nonlinear map in equation (8) can be written for a particular value of ε as

$$m_x(k+1) = U m_x(k) + \overline{C}_Q[m_x(k)]^2 + \overline{C}_C[m_x(k)]^3 \tag{9}$$

where $m_x(k+1)$ is the collocation vector in a particular interval, $m_x(k)$ is the collocation vector in the preceding interval, $[m_x(k)]^i$ is the vector of all the possible independent nonlinear terms of the order $i = 2,3$ of the collocation vector $m_x(k)$ and, $\overline{C}_Q$ and $\overline{C}_C$ are the

quadratic and cubic coefficient matrices. The order of nonlinearities in $[m_x(k)]^i$ is a lexicographic order. The coefficient matrix $\overline{C}_Q$ will have appropriate entries for each of the nonlinearities appearing in the map.

In equation (9), only up to cubic powers of nonlinearities are retained, even though the powers in equation (8) are of much higher order. If *n* is the dimension of the state space in equation (1) and *N* is the number of Chebyshev points used in collocation, then there are $\sum_{i=1}^{nN} i = o(nN^2)$ independent quadratic and $\sum_{i=1}^{nN} i(i+1)/2 = o(nN^3)$ independent cubic terms.

Equation (9) can be written in the modal coordinates by means of a modal transformation of the state

$$m_x(k) = M_U m_z(k) \tag{10}$$

$$m_z(k+1) = J_U m_z(k) + M_U^{-1}\overline{C}_Q\overline{M}^2[m_z(k)]^2 + M_U^{-1}\overline{C}_C\overline{M}^3[m_z(k)]^3 \tag{11}$$

or

$$m_z(k+1) = J_U m_z(k) + \overline{C}_Q^M[m_z(k)]^2 + \overline{C}_C^M[m_z(k)]^3 \tag{12}$$

Here, M_U is the modal decoupling matrix, J_U is the Jordan canonical form of U and, $\overline{M}^2$ and $\overline{M}^3$ are the matrices resulting from the quadratic and cubic terms in equation (11) because of the state transformation (10). Now the states can be partitioned according to the eigenvalues of J_U into center and stable states as

$$\begin{bmatrix} m_{zc}(k+1) \\ m_{zs}(k+1) \end{bmatrix} = \begin{bmatrix} \mu^c & 0 \\ 0 & \mu^s \end{bmatrix}\begin{bmatrix} m_{zc}(k) \\ m_{zs}(k) \end{bmatrix} + \begin{bmatrix} [\overline{C}_Q^M]^c \\ [\overline{C}_Q^M]^s \end{bmatrix}[m_{zc}, m_{zs}]^2 + \begin{bmatrix} [\overline{C}_C^M]^c \\ [\overline{C}_C^M]^s \end{bmatrix}[m_{zc}, m_{zs}]^3 \tag{13}$$

and a modified method of center manifold reduction for nonlinear maps [30] is applied to equation (13). A nonlinear transformation is defined of the form

$$m_{zs}(k) = U_2[m_{zc}(k)]^2 + U_3[m_{zc}(k)]^3 \tag{14}$$

where U_2 and U_3 are undetermined coefficient matrices of quadratic and cubic terms. Substituting equation (14) into equation (13), a nonlinear algebraic equation of the form

$$\begin{aligned} &U_2(\mu^c m_{zc} + [\overline{C}_Q^M]^c[m_{zc}, m_{zs}]^2 + [\overline{C}_C^M]^c[m_{zc}, m_{zs}]^3)^2 + \\ &U_3(\mu^s m_{zs} + [\overline{C}_Q^M]^c[m_{zc}, m_{zs}]^2 + [\overline{C}_C^M]^c[m_{zc}, m_{zs}]^3)^3 \\ &\quad = \mu^s m_{zs} + [\overline{C}_Q^M]^s[m_{zc}, m_{zs}]^2 + [\overline{C}_C^M]^s[m_{zc}, m_{zs}]^3 \end{aligned} \tag{15-a}$$

is obtained with the independent variable *k* suppressed for brevity. Only quadratic and cubic terms are retained in equation (14) and then coefficients of the like powers are collected for determining U_2 and U_3. The resulting coefficient equations for quadratic and cubic terms are

$$[m_{zc}]^2: \quad U_2\overline{\mu}^c = \mu^s U_2 + s([\overline{C}_Q^M]^s) \tag{15-b}$$

with

1. $s([\overline{C}_Q^M]^s) =$ surviving coefficients from matrix $[\overline{C}_Q^M]^s$ due to order 2 truncation

2. $\overline{\mu}^c =$ a coefficient matrix resulting from squaring each element of vector $\mu^c m_{zc}$

and

$$[m_{zc}]^3 : \quad U_3\hat{\mu}^s + U_2\, cs(\mu^c m_{zc} \times [\overline{C}_Q^M]^c [m_{zc}, m_{zs}]^2) = \mu^s U_3 + s([\overline{C}_C^M]^s) + cs([\overline{C}_Q^M]^c [m_{zc}, m_{zs}]^2) \tag{15-c}$$

with

3. $s([\overline{C}_Q^M]^s) =$ surviving coefficients of matrix $[\overline{C}_Q^M]^s$ due to order 3 truncation

4. $cs(\mu^c m_{zc} \times [\overline{C}_Q^M]^c [m_{zc}, m_{zs}]^2) =$ surviving coefficients of the coefficient matrix resulting from the product in the parentheses, due to order 3 truncation

5. $cs([\overline{C}_Q^M]^c [m_{zc}, m_{zs}]^2) =$ surviving coefficients of the coefficient matrix resulting from the term in the parentheses due to order 3 truncation

6. $\hat{\mu}^s =$ a coefficient matrix resulting from cubing each element of vector $\mu^s m_{zc}$

Equations (15) are linear generalized Lyapunov equations which are solved for unknown coefficient matrices U_2 and U_3 by using Kronecker products. Solving (15-b) first for U_2 and (15-c) later for U_3 using U_2, the original map (13) yields a one- (for fold and flip bifurcations) or two- (for secondary Hopf bifurcations) dimensional map on the center subspace given by

$$m_{zc}(k+1) = \mu^c m_{zc}(k) + a_2[m_{zc}(k)]^2 + a_3[m_{zc}(k)]^3 \tag{16}$$

4. EXAMPLE

Consider the single inverted pendulum in the horizontal plane with a linear and a quadratic torsional spring at the base, a linear torsional damper at the base, and acted upon by a *T*-periodic follower force proportional to the delayed (with delay period *T*) angular displacement. The equation of motion is given as

$$ml^2\ddot{q} + c\dot{q} + k_l q + k_n q^2 + [P_1 + P_2\cos(\omega t)]l\sin(q(t) - \eta q(t-T)) = 0 \tag{17}$$

where $T = 2\pi/\omega$ and m is kilograms, l is in meters, c is torsional damping in N.m.s/rad, k_l is a linear torsional stiffness and k_n is nonlinear torsional stiffness in N.m/rad. Expanding in a Taylor series about the zero equilibrium position and retaining up to cubic terms, equation (17) can be written in the state space form with $\overline{P} = (P_1 + P_2\cos(\omega t))/ml$ and $x(t) = q(t)$. Also, $(k_n/ml^2) = (\eta^3/6ml)$ can be designated as ε, which conforms to the form of equation (1). At a particular parameter set given by

$$k_l / ml^2 = 1.75,\ c / ml^2 = 0.0482979,\ P_1 / ml = 0.4025,$$
$$P_2 / ml = 0.734,\ k_n / ml^2 = \eta^3 / 6ml = 1/6,\ \eta = 1,\ T = 2.4 \tag{18}$$

the system undergoes a flip bifurcation which is evidenced by a -1 eigenvalue of the monodromy matrix $(I+V)$ in equation (8). We choose c/ml^2 as the bifurcation parameter.

The numerical center manifold algorithm is programmed in MATLAB and applied to system (17) with parameter set (18) with *N*=32 (collocation points) and *n*=2 (states) which translates into a map having a 64-dimensional collocation vector, 2080 quadratic terms and 45760 cubic terms. The size of the coefficient matrices are 64 x 2080 and 64 x 45760, for quadratic and cubic terms, respectively. The reduced scalar map on the center manifold takes the form of equation (16) with $\mu^c = -1+\gamma$ where γ is a versal deformation parameter and $a_2 = -0.3310,\ a_3 = -0.0511$. The post-bifurcation limit cycle amplitude is derived using the second iterate of equation (16) as

$$m_{zc}(k+2) = (1-2\gamma)m_{zc}(k) - 2\delta(m_{zc}(k))^3 \tag{19}$$

where $\delta = a_2{}^2 + a_3$ and terms higher than cubic have been dropped. The fixed points of equation (19) (from which the amplitude of the periodic orbit can be recovered) are obtained by setting $m_{zc}(k+2) = m_{zc}(k)$ as

$$m_{zc}(k) = \pm\sqrt{-\frac{\gamma}{a_2{}^2 + a_3}} \tag{20}$$

where $\gamma = \mu^c + 1$. Now the bifurcation parameter c/ml^2 (which was temporarily omitted to allow for the numerical computations) is re-introduced (compare with [33] for the ODE case) by computing the gradient g_c of γ with respect to c/ml^2. This is found to be $g_c = 0.5195$ and hence $\delta = 0.0583$ which implies that the bifurcation is super-critical. Using equation (20), with $\gamma = g_c \Delta(c/ml^2)$, the amplitude of the limit cycle is computed and compared with the one obtained from the direct numerical simulation. Figure 1 plots the displacement and velocity variables for equation (17) for half the doubled period after steady state is reached computed via the MATLAB DDE23 routine and the proposed Center Manifold Algorithm. The center manifold calculations confirm the amplitude and the doubled period.

5. CONCLUSIONS

A method to compute center manifold reductions of delay differential equations with periodic coefficients has been proposed and illustrated. The method uses classical perturbation analysis assuming that the nonlinearities are multiplied by the perturbation parameter. An approximate nonlinear map is constructed using Chebyshev spectral collocation which takes the solution in each interval of length equal to the principal period to the solution in the next interval. The method can be extended to the analysis of Hopf bifurcation and to order reduction problems associated with DDEs with periodic coefficients. The scope of the systems considered can also be widened by considering DDEs with a rational relationship between the delay and the principal periods. The use of versal deformation theory allows a bifurcation analysis in terms

of a parameter as shown in the example, although the nonlinear map is numerically constructed (using MATLAB) to allow for computational tractability.

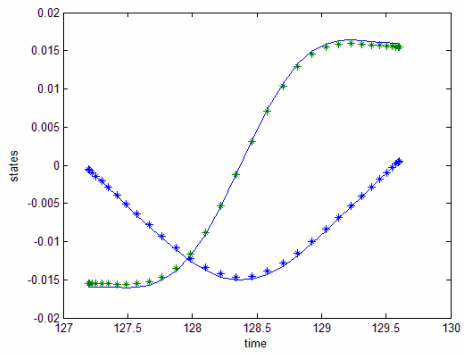

Figure 1. Displacement and velocity for equation (17) for half of the doubled period with DDE23 (+) and Center Manifold Algorithm (continuous)

REFERENCES

[1] Hassard, B. and Wan, Y. H.:Bifurcation formulae derived from center manifold, *Journal of Mathematical Analysis and Applications*, 63, 297-312 (1978).
[2] Carr, J.: *Applications of Centre Manifold Theory,* Springer, New York (1981).
[3] Kalmar-Nagy, T., Stepan, G., and Moon F. C.: Subcritical Hopf bifucration in the delay equation model for machine tool vibrations, *Nonlinear Dynamics*, 26, 121-142 (2001).
[4] Gilsinn, D. E.: Estimating critical Hopf bifurcation parameters for a second order delay differential equation with application to machine tool chatter, *Nonlinear Dynamics*, 30, 103-154 (2002).
[5] Wang, Z. H. and Hu, H. Y.: Dimensional reduction for nonlinear time-delayed systems composed of stiff and soft substructures, *Nonlinear Dynamics*, 25, 317-331 (2001).
[6] Das, S. L. and Chaterjee, A.: Multiple scales without center manifold reductions for delay differential equations near Hopf bifurcations, *Nonlinear Dynamics*, 30, 323-335 (2002).
[7] Wahi, P. and Chaterjee, A.: Galerkin projections for delay differential equations, Proceedings of ASME DETC'03/VIB-48570, Chicago, IL, Sept. 2-6, 2003.
[8] Ma, H., Butcher E. A. and Bueler, E.: Chebyshev expansion of linear and piecewise linear dynamic systems with time delay and periodic coefficients under control excitations, ASME *Journal of Dynamic Systems Measurement and Control*, 125, 236-243 (2003).
[9] Szalai, R., Stepan, G., and Hogan, S. J.: Global dynamics of low-immersion high-speed milling, *Chaos* 14, 1-9 (2004).
[10] Szalai, R., and Stepan, G.: Period doubling bifurcation and center manifold reduction in a time-periodic and time-delayed model of machining, *J. Nonlinear Science*, in press.
[11] Halanay, S., *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, New York (1966).
[12] Butcher, E. A., Ma, H., Bueler, E., Averina, V., and Szabo, Z.: Stability of linear time-periodic delay-differential equations via Chebyshev polynomials, *International Journal of Numerical Methods in Engineering* 59, 895-922 (2004).
[13] Bueler, E.: A Chebyshev collocation for linear, periodic ordinary and delay differential equations: a posteriori estimates, SIAM *J. Numerical Analysis*, in press.
[14] Butcher, E. A., Nindujarla, P., and Bueler, E.: Stability of up- and down-milling using Chebyshev collocation method, Proc. ASME DETC 2005-84880, Sept. 24-28, 2005, Long Beach, CA (2005).
[15] Trefethen, L. N.: *Spectral Methods in MATLAB*, Software, Environments, and Tools, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (2000).
[16] David, A. and Sinha, S. C., Versal Deformation and Local Bifurcation Analysis of Time-Periodic Nonlinear Systems, *Nonlinear Dynamics* 21, 317-336 (2000).