# A MODEL OF THE WEAR DUE TO THE ROLLING CONTACT OF A WHEEL OVER A GENERAL RAIL PROFILE 

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#### Abstract

Recent research has suggested that the contact mechanics between the wheel and rail in railway systems may act as a geometric filter for wear-type rail corrugation. This filter is hypothesized to work alongside the dynamics to create the wear pattern, attenuating some wavelengths (and associated frequencies) while promoting others, and hence influencing the spectra of noise produced. It has been proposed that this effect may give rise to the field observation of the speed insensitivity of the dominant wavelength of short pitch corrugation. This type of corrugation is responsible for an annoying tonal noise in the range of 500-800 Hz.

In this paper a simplified model of the wear due to the rolling contact of a wheel over a general rail profile is presented. This model is simple to implement and allows for an investigation into the non-linear behaviour of the dynamic wear, not discussed in previous research. The relevance of these results to the transportation noise phenomenon of wear-type rail corrugation is discussed.


## 1. INTRODUCTION

Rail corrugation is a significant problem in railway engineering, which manifests as an unwanted periodic wear pattern that develops on the surface of the wheel and rail with use. This variation from a flat profile induces unwanted vibrations, noise and other associated problems. Currently the only reliable method to eliminate corrugations is removal by grinding at significant expense to the railway operator. For a comprehensive review on recent corrugation studies refer to Sato et al. [1].

One difficult to explain feature of wear-type rail corrugation is the apparent insensitivity of corrugation wavelength to vehicle speed, when the dominant wavelength is short. This implies that the vibration dynamics alone may not be sufficient to describe the evolution of corrugation. For a discussion of this trend refer to Grassie and Kalousek [2]. To explain this behaviour it has been hypothesized in some studies that a geometric filter due to the contact mechanics of the wheel and the rail may be present. It is well understood that the dynamics of the wheel/rail system are insensitive to wavelengths shorter than the contact patch, the socalled contact filtering effect. One of the most renowned contact filters is that proposed by Remington [3]. Remington derived an semi-analytic model for how the roughness spectrum of the rail profile influenced the wheel/rail dynamics, primarily concerned with modelling
observations of noise spectra. This model predicts that very short wavelengths will have little effect on the wheel/rail dynamics. In this paper however, we are concerned with how the deformation of the wheel and rail due to the initial rail profile influence the subsequent wear that occurs, hence the results of [3] are not directly applicable. It has been proposed in Muller [4], Nielsen [6] and Wu and Thompson [6] that this contact mechanics induced wear provides a band in which a range of corrugation wavelengths are amplified.

In [5] an interesting model of the contact mechanics between a smooth and a corrugated cylinder has been developed by considering an infinite series of sinusoids to describe the corrugation and using this to solve the contact integral equations. This provides qualitative evidence of an amplification band due purely to the contact mechanics, but does not quantify how significant this effect may be in terms of corrugation growth rate with realistic parameter values. Also the effect of corrugation amplitude on the filter was not investigated. In [6] the contact mechanics are simplified by making use of the Carter solution for the tangential contact problem and a varying relative curvature. This is combined with a varying normal force and the wear behaviour is investigated; however there is no quantification of the filtering behaviour of the contact mechanics and the effect of a time dependant shift of the contact patch is neglected in the analysis.

In this paper the contact mechanics induced wear is modeled by using a truncated Taylor Series of the corrugated profile and using a modification of the Carter Solution for the rolling contact between two smooth cylinders, which takes account of the varying relative curvature of the surfaces and the shifting of the contact patch. From this model the wear due purely to the contact mechanics of traversing a corrugated profile is much more easily calculated, allowing an investigation into the non-linear behaviour of the wear that has not previously been performed. The relevance of these results to the transportation noise phenomenon of wear-type rail corrugation is discussed.

In particular the novel contributions of this paper are

1) Development of a closed form analytical solution for an extended Hertzian model of the normal and tangential contact problems for the rolling contact of a wheel and a general rail profile and the subsequent evaluation of the wear profile history.
2) Quantification of this wear filtering effect in terms of corrugation growth rate.
3) Investigation and discovery of the nonlinear behaviour of the filter in relation to corrugation amplitude.

## 2. HERTZ AND CARTER CONTACT MODELS

It is necessary to first review the Hertz and Carter Solutions for an uncorrugated profile to provide the basis for developing the extended theory where corrugations are present. In this model we are concerned with the quasi-static elastic contact of two 2D bodies as shown in Fig. 1.


Figure 1. Elastic Contact of two non-conformal bodies

It is assumed that the contact is non-conformal and that a one point contact occurs, which can be assured by only looking at cases where there is a sufficient difference in the curvature of the bodies. The normal pressure distribution for the contact between two 2D bodies under the elastic half space approximation can be shown (see Johnson [7]) to be given by,

$$
\begin{equation*}
\int_{a}^{b} \frac{p(\zeta)}{x-\zeta} d \zeta=\frac{\pi E}{4\left(1-v^{2}\right)} \frac{d}{d x}\left(Z_{1}(x)-Z_{2}(x)\right) \tag{1}
\end{equation*}
$$

where $Z_{1}(x)$ and $Z_{2}(x)$ are the shapes of the undeformed bodies, $x$ is the local longitudinal coordinate, $E$ is the Elastic Modulus of the bodies, $v$ is Poisson's Ratio for the two bodies, $\zeta$ is the longitudinal position and $p(x)$ is the normal pressure distribution.

The solution of singular integral equations like Eq.(1) is described in Muskhelishvili [8]. In the general case it is difficult to find a simple expression for the pressure distribution. The Hertz solution of the contact between two cylinders makes a second order Taylor Series approximation to the undeformed profiles (representing the cylinders as quadratics), i.e.

$$
\begin{equation*}
Z_{i}(x) \approx(-1)^{i+1} \frac{x^{2}}{2 R_{i}}, \quad i=1,2 \tag{2}
\end{equation*}
$$

where $\mathrm{R}_{\mathrm{i}}$ is the radius of the $i^{\text {th }}$ undeformed cylinder. Under this approximation it can be shown by using Eq. (2) in Eq. (1) that the solution for the pressure distribution will be given by,

$$
\begin{equation*}
p(x)=\frac{p_{0}}{a_{0}} \sqrt{a_{0}^{2}-x^{2}}, \tag{3}
\end{equation*}
$$

where

$$
\begin{equation*}
p_{0}=\sqrt{\frac{N E}{2\left(1-v^{2}\right) \pi R}}, \quad a_{0}=\sqrt{\frac{8\left(1-v^{2}\right) R N}{\pi E}}, \quad \frac{1}{R}=\frac{1}{R_{1}}+\frac{1}{R_{2}}, \tag{4}
\end{equation*}
$$

and $N$ is the normal force. In Eq. (4) $a_{0}$ is the half contact patch width, i.e. $a=-a_{0}$ and $b=a_{0}$.
The solution to the tangential contact problem, where a torque is applied to one of the cylinders in the above problem, is given by the Carter solution [9]. The solution to the tangential stress distribution $q(x)$ is given by the sum of two ellipses as

$$
\begin{equation*}
q(x)=q_{1}(x)+q_{2}(\hat{x}), \tag{5}
\end{equation*}
$$

where

$$
\begin{gather*}
q_{1}(x)=\left\{\begin{array}{rr}
\frac{\mu p_{0}}{a_{0}} \sqrt{a_{0}^{2}-x^{2}}, & -a_{0}<x<a_{0} \\
0, & \text { otherwise }
\end{array}\right\}, \quad q_{2}(\hat{x})=\left\{\begin{array}{cc}
\frac{\mu p_{0}}{a_{0}} \sqrt{\hat{a}_{0}^{2}-\hat{x}^{2}}, & -\hat{a}_{0}<\hat{x}<\hat{a}_{0} \\
0, & \text { otherwise }
\end{array}\right\},  \tag{6}\\
\hat{a}_{0}=a_{0}+\frac{R}{\mu} \xi_{0} \text { for }-\frac{\mu a_{0}}{R} \leq \xi_{0} \leq 0, \quad \hat{x}=x+a_{0}-\hat{a}_{0}
\end{gather*}
$$

$\mu$ is the friction coefficient and $\xi_{0}$ is the creepage (the relative velocity of the bodies normalised by the mean velocity). It can also be shown that for low values of creepage that

$$
\begin{equation*}
\xi_{0}=\frac{\mu}{R}\left(\hat{a}_{0}-a_{0}\right) \quad \text { and } \quad Q_{0}=\frac{\mu N}{a_{0}^{2}}\left(a_{0}^{2}-\hat{a}_{0}^{2}\right), \tag{8}
\end{equation*}
$$

where $Q_{0}$ is the tangential force. Furthermore the local relative velocity (or slip), $s$, between
the bodies can be shown to be given by,

$$
s(x)=\left\{\begin{array}{cc}
-\frac{\mu}{R}\left(\sqrt{x^{2}-a_{0}^{2}}-\sqrt{\hat{x}^{2}-\hat{a}_{0}^{2}}\right), & x \leq-a_{0}  \tag{9}\\
0, & -a_{0}<x<2 \hat{a}_{0}-a_{0} \\
-\frac{\mu}{R} \sqrt{\hat{x}^{2}-\hat{a}_{0}^{2}}, & 2 \hat{a}_{0}-a_{0} \leq x<a_{0} \\
\frac{\mu}{R}\left(\sqrt{x^{2}-a_{0}^{2}}-\sqrt{\hat{x}^{2}-\hat{a}_{0}^{2}}\right), & a_{0} \leq x
\end{array}\right.
$$

Eq. (9) shows that within the contact region there will be a "stick" zone with no local relative sliding velocity and also a "slip" zone. In the model developed in the next section, only the contact conditions in the slip region will be important because no frictional work, and hence no wear, will occur in the stick region.

## 3. EXTENDED HERTZIAN ROLLING CONTACT OVER A GENERAL RAIL PROFILE

In this section a model for the rolling contact of a cylinder over a general profile will the developed. The purpose of this model is to provide a simple method of investigating and quantifying the contact filtering effect on roughness discussed in [4], [5] and [6].


Figure 2. Wheel-Rail Contact.
To do this Eq. (1) is solved when one undeformed body is given by the superposition of a cylinder and a general profile as shown in Fig. (2), i.e.

$$
\begin{equation*}
Z_{1}(x) \approx \frac{x^{2}}{2 R_{1}} \quad Z_{2}(x) \approx \frac{-x^{2}}{2 R_{2}}+g(x-V t) \tag{10}
\end{equation*}
$$

where $g(x-V t)$ is the general profile, moving under contact with velocity $V$. In [5] Eq. (1) is solved by assuming that the roughness profile, $g(x)$, can be represented as an infinite series of sinusoids. In this paper a truncated McLaurin Series of the moving profile, as given in Eq. (10), will instead be taken to allow closed form solutions for the normal pressure distribution and traction. Using the series expansion approximation for Eq. (10) in Eq (1) gives,

$$
\begin{equation*}
\int_{a}^{b} \frac{p(\zeta)}{x-\zeta} d \zeta=\frac{\pi E}{4\left(1-v^{2}\right)}\left(\frac{1}{R} x-g^{\prime}(-V t)-x g^{\prime \prime}(-V t)\right) \tag{11}
\end{equation*}
$$

First the new variable $\mathrm{R}^{*}$ is introduced, and the coordinates transformed according to

$$
\begin{equation*}
\frac{1}{R^{*}}=\frac{1}{R}-g^{\prime \prime}(-V t), \quad \chi=x-R^{*} g^{\prime}(-V t) \quad \text { and } \quad \eta=s-R^{*} g^{\prime}(-V t), \tag{12}
\end{equation*}
$$

which when substituted into Eq. (11) yields

$$
\begin{equation*}
\int_{\eta=a-R^{*} g^{\prime}\left(-V_{t}\right)}^{\eta=b-R^{*} g^{\prime}\left(-V_{t}\right)} \frac{\hat{p}(\eta)}{\chi-\eta} d \eta=\frac{\pi E}{4\left(1-v^{2}\right)} \frac{\chi}{R} . \tag{13}
\end{equation*}
$$

Eq. (13) is simply the integral equation used to derive the Hertz Solution (Eq. (1)), perturbed to account for the undulating profile. Thus by comparing Eq. (13) to the solutions shown in Eq. (3)-(9), the solution for the normal pressure can be derived as,

$$
\begin{equation*}
p(x, t)=\frac{p_{0}}{a_{0}} \sqrt{a_{0}^{2}-\left(x-V t-R^{*} g^{\prime}(-V t)\right)^{2}}, \tag{14}
\end{equation*}
$$

Where $p_{0}$ and $a_{0}$ are now functions of time defined by,

$$
\begin{equation*}
p_{0}(t)=\sqrt{\frac{N E\left(1-R g^{\prime \prime}(-V t)\right)}{2\left(1-v^{2}\right) \pi R}} \quad \text { and } \quad a_{0}(t)=\sqrt{\frac{8\left(1-v^{2}\right) R N}{\pi E\left(1-R g^{\prime \prime}(-V t)\right)}} . \tag{15}
\end{equation*}
$$

Similarly the modified Carter solution for the tangential stress distribution in the slip region will be given by,

$$
\begin{gather*}
q(x, t)=\frac{\mu p_{0}}{a_{0}} \sqrt{a_{0}^{2}-\left(x-V t-R^{*} g^{\prime}(-V t)\right)^{2}}, \quad s(x, t)=-\frac{\mu}{R} \sqrt{\left(\left(x-V t-R^{*} g^{\prime}(-V t)+a_{0}-\hat{a}_{0}\right)^{2}-\hat{a}_{0}^{2}\right)} \\
\text { for } 2 \hat{a}_{0}-a_{0}+R^{*} g^{\prime}(-V t)<x-V t<a_{0}+R^{*} g^{\prime}(-V t), \tag{16}
\end{gather*}
$$

where

$$
\begin{equation*}
\hat{a}_{0}=a_{0}+\frac{R^{*}}{\mu} \xi_{0} \tag{17}
\end{equation*}
$$

and the creep and traction relations remain the same as Eq. (8), with the new definitions given by Eqs. (12) and (17).

Note that the solution given is general in a sense as it is valid for any initial profile $g(x)$. These solutions for the normal and tangential contact problems will be valid for a wide range of initial profiles provided the amplitudes are small enough and the wavelengths long enough to avoid a multiple point contact, specifically when $A \omega^{2} \ll 1 / R$ where $A$ is the amplitude and $\omega$ is the angular spatial frequency of the corrugation.

Fig. 3 depicts the evolution of the pressure distribution as it passes over an arbitrary surface. Two important properties of the solution can be observed. The first, depicted in Fig. 3 a ) and 3 b ), is the shift of the contact patch centre forwards uphill and backwards downhill, which can be observed in Eq. (14), where the shift is a function of the slope of the profile, $g^{\prime}(-V t)$. The other property of the solution, depicted in Fig. 3c) and 3d), is that the pressure distribution becomes narrow and tall on the peaks and short and fat in the troughs of the corrugation, which can be observed in Eq. (15) where $a_{0}$ and $p_{0}$ depend on the second derivative of the profile, $g^{\prime \prime}(-V t)$. These properties are important when considering the change in frictional power which occurs in the slip region as the wheel traverses the rail profile.


Figure 3. Wheel-Rail Contact Solution Behaviour. Shaded area indicates slip region.

## 4. WEAR DUE TO ROLLING CONTACT

To evaluate how much wear this rolling contact induces the frictional work hypothesis shall be used. The frictional work hypothesis states that the wear rate is proportional to the amount of frictional work done, i.e.

$$
\begin{equation*}
\frac{d m}{d t}=k_{0} P_{\text {fricition }} \tag{18}
\end{equation*}
$$

where $m$ is the mass of material removed, $k_{0}$ is the wear coefficient and $P_{\text {friction }}$ is the frictional power. In this paper the primary concern is the change of profile height at each point along the rail, thus transforming Eq. (18) into this form yields

$$
\begin{equation*}
W(x)=\frac{k_{0}}{\rho} \int_{-\infty}^{\infty} V_{\text {contact }} q(x, t) s(x, t) d t \tag{19}
\end{equation*}
$$

where $W(x)$ is the profile height change at position $x, \rho$ is the density and $V_{\text {contact }}$ is the speed of the contact patch, which is found from

$$
\begin{equation*}
V_{\text {contact }}=V+R^{*} \frac{d}{d t}\left(g^{\prime}(-V t)\right)+R^{* 2} g^{\prime}(-V t) \frac{d}{d t}\left(g^{\prime \prime}(-V t)\right) . \tag{20}
\end{equation*}
$$

## 5. SIMULATION RESULTS

To examine the behaviour of the rolling contact wear model, the integral in Eq. (19) has been evaluated numerically, using the intermediate equations from section 4 and the assumption of constant traction force. This allows the profile change after one pass to be evaluated. In this case the initial profile was chosen to be a sinusoid, so as to approximate the wear on a corrugation and also to allow analytic solutions for the profile derivatives to be used. Evaluating multiple iterations of the same profile would require an estimate of these derivatives to be made after each pass. Since it is unlikely that an analytic solution to Eq. (19) can be developed, numerical estimates of the derivatives of the new profile would be required, which shall be examined at a later date.

The simulation parameters are chosen to represent those that may occur in practice in the contact between a rail and a wheel (see for example [10]) and are summarized in table 1. It should be noted that the solutions obtained are independent of $V$, which is to be expected as a faster velocity may cause more wear per unit time, however this wear will be spread over a larger distance, cancelling any speed dependence of the spatial wear.

Table 1. Simulation Parameters

| $k_{0}[\mathrm{~kg} / \mathrm{Nm}]$ | $5 \times 10^{-9}$ | $b$ ( contact width) $[\mathrm{m}]$ | 0.01 | $\rho\left[\mathrm{~kg} / \mathrm{m}^{3}\right]$ | 7700 |
| :--- | :---: | :--- | :---: | :--- | :---: |
| $P[\mathrm{~N}]$ | 66000 | $E\left[\mathrm{~N} / \mathrm{m}^{2}\right]$ | $2.1 \times 10^{11}$ | $v$ | 0.29 |
| $Q[\mathrm{~N}]$ | 3000 | $R[\mathrm{~m}]$ | 0.46 | $\mu$ | 0.4 |

The first simulations are intended to show the non-linear nature of the wear. The wear in space is shown in Fig. 4 for long ( 10 cm ) and short ( 2 cm ) wavelengths, with an initial amplitude of 10 microns. It can be seen in this plot that for long wavelengths more wear occurs in the troughs than on the peaks, which causes an amplification of the corrugations peak to peak amplitude. This is due to the second property identified in section 3 of the contact patch and slip region becoming larger in the trough due to the more closely conformal contact and dissipating more frictional power. In the shorter wavelength case the maximum wear no longer occurs in the initial profile trough, but slightly offset to it. This may be ascribed to the asymmetrical nature of the distribution of frictional power in the contact patch and also the motion of the contact patch (the first property described in section 3). In the long wavelength case, it can be seen that the wear is approximately sinusoidal, whereas in the short wavelength case it is clearly not. This indicates that the wear is becoming non-linear as the wavelength decreases.


Figure 4. Profile and wear after traversing long ( 10 cm , left) and short ( 2 cm , right) wavelength corrugations.

To show the amplifying behaviour of the contact induced wear, the ratio of the root mean squared (RMS) value of the worn profile over the initial profile has been calculated to obtain an exponential growth rate parameter equivalent to that used in [10], i.e.

$$
\begin{equation*}
G_{r}=\frac{\sqrt{\frac{1}{x_{2}-x_{1}} \int_{x_{1}}^{x_{1}} \text { output }(x)^{2} d x}}{\sqrt{\frac{1}{x_{2}-x_{1}} \int_{x_{1}}^{x} \operatorname{input}(x)^{2} d x}}-1 \tag{21}
\end{equation*}
$$

Thus if this growth rate, $G_{r}$, is greater than zero the profile will be growing and if it is less than zero it will be diminishing. The result of this analysis is shown in Fig. 5 for a range of initial profile amplitudes.



Figure 5. Ratio of profile RMS values. Left plot shows results for initial amplitudes of 1, 5, 10, 50 and 100 microns. Right plot shows close-up for initial amplitudes of $1,2,3,4$ and 5 microns.

It can be seen from Fig. 5 that a wavelength band does seem to be actively promoted, as found in [4], [5] and [6]. However, in contrast to previous findings, it can also be seen that the
amount of amplification tends to be stronger with smaller amplitudes. This implies that the contact mechanics induced amplification and filtering is non-linear and is more significant for the corrugation initiation phase. It is further apparent that the peak of the amplification shifts slightly to longer wavelengths as the amplitude becomes larger. The order of the amplification is also similar to observed growth rates of approximately $5 \times 10^{-6}$ [11] for amplitudes smaller than $10 \mu \mathrm{~m}$ but becomes insignificant for amplitudes larger than $50 \mu \mathrm{~m}$. This indicates that the wear contact filtering effect on corrugation growth is only significant for small amplitude growth.

## 6. CONCLUSIONS

An extended Hertzian model for the wear due to a rolling 2D contact is presented. Numerical integration of the model for realistic railway parameters has been performed in order to quantify the magnitude of contact filtering effects and also to investigate non-linear behaviour. It is shown that a mid-wavelength amplification band exists for this model in accordance with previous research. However, the form of the solution provides full insight into the mechanism that may cause the profile amplification and its non-linear behaviour. The filtering effect is quantized and shown to be comparable to field measurements of growth rates only when corrugation amplitudes are small. It is also shown that the amplification peak decreases with corrugation amplitude and shifts to longer wavelengths.

Extensions of this work that are under consideration are the evaluation of the contact behaviour with parameter variations, the evaluation of the profile evolution over multiple passes and the comparison with numerical solutions to the contact integral equations in 2D and 3D.

## 7. ACKNOWLEDGEMENTS

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