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A METHOD OF MEASURING THE AXIAL ELASTIC MODULUS OF A SANDWICH CYLINDER BASED ON MODAL TESTING

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Abstract

The axial elastic modulus is one of the most important parameters of the sandwich cylinder, which is made of metal and one or two kinds of complex material. But it was difficult to measure the elastic modulus exactly by means of routine material test because the cylinder was not easy to be made into a standard specimen. This paper presents a non-destructive method of measuring the axial elastic modulus of the sandwich cylinder based on modal testing. In this method, the axial elastic modulus can be determined by solving the bending vibration equations of the cylinder, and the modal testing provides the required bending frequencies in the equations. The shear modulus can also be determined simultaneously. The method was proved to be practicable by measuring an aluminium cylinder whose elastic modulus is known, and the errors were evaluated. The results of testing sandwich cylinders are also presented.

1. INTRODUCTION

The cylinder studied in this paper is made of metal and one or two kinds of complex material, therefore it was called sandwich cylinder. The sandwich cylinder is a primary part of a rotor. The highest rotary speed of this rotor is restricted by its critical speed when the rotor is sub-critical, and the critical speed has relations with the axial elastic modulus of the cylinder. So the axial elastic modulus is one of the most important parameters of the cylinder. But the elastic modulus was difficult to be measured by means of conventional material parameters test. There were two reasons, one was that the cylinder was not easy to be made into a standard specimen, the other was that the edges of specimen were usually broken before getting the test results during the conventional material test even if the cylinder was made into the non-standard test specimens. Furthermore the sandwich cylinder was not expected to be destroyed in most of conditions.

In the last decades, the use of dynamic test techniques for the elastic characterization of both isotropic and anisotropic materials has increased due to their non-destructive nature, simple operating procedures and superior precision over the static test methods[1]. And this test technique has also been applied in the identification of elastic properties of sandwich materials[2]. A practicable method of measuring the axial elastic properties of the sandwich cylinder based on modal testing was introduced in this paper. The first step of this method was to get the bending vibration frequency equation relating the nature bending frequencies to the axial elastic properties of the sandwich cylinder. Then the required frequencies were provided by means of the modal testing. So the axial elastic modulus and shear modulus can be determined by solving the frequencies equations.

2. THEORY OF THIS MEASURING METHOD

2.1 Bending Vibration Equation of Cylinder

The key of the dynamic test method is the frequency equation relating the nature frequencies to the elastic properties, mass and dimension. The frequency equation for determining axial elastic properties of the constant section shell cylinder can be obtained by solving the following differential equation of motion [3]

$$EJ\frac{\partial^4 y}{\partial x^4}+\rho S\frac{\partial^2 y}{\partial t^2}-\rho J(1+\frac{E}{KG})\frac{\partial^4 y}{\partial t^2\partial x^2}+\frac{\rho^2 J}{KG}\frac{\partial^4 y}{\partial t^4}=0 \tag{1}$$

Where y is the displacement, E is the axial young's modulus, G is the shear modulus, J is the section moment of inertia, ρ is the density, S is the area of section, and K is the section constant which depends on the dimension of section. K is 0.5 for the shell cylinder. The first two parts of equation (1) is the beam vibration differential equation, the third and the fourth part express the effect of the rotary inertia and shear deformation respectively.

In general, the solution of equation (1) is

$$y(x,t)=Y(x)\,\mathrm{e}^{\mathrm{i}\,(\omega_\mathrm{n}t+\varphi)} \tag{2}$$

Now substituting (2) into (1), the following equation can be got

$$EJY^{(4)}-\rho S\,\omega_n^2 Y+\rho J\,(1+\frac{E}{KG})\omega_n^2 Y''+\frac{\rho^2 J}{KG}\omega_n^4 Y=0 \tag{3}$$

Then substituting $Y(x)=e^{\lambda x}$ into (3), the equation (3) can be written as

$$EJ\lambda^4+\rho J(1+\frac{E}{KG})\omega_n^2\lambda^2+\frac{\rho^2 J}{KG}\omega_n^4-\rho S\omega_n^2=0 \tag{4}$$

Solving the quadratic equation in which λ^2 is unknown, the following results can be obtained

$$\lambda^2=\frac{\rho\omega_n^2}{2E}\left(-a_1\pm a_2\right) \tag{5}$$

Where, $a_1=1+\frac{E}{KG}$, $a_2=\sqrt{\left(1+\frac{E}{KG}\right)^2+4EJ\left(\frac{S}{\rho J^2\omega_n^2}-\frac{1}{KGJ}\right)}$

Equation (5) can be written as several expressions according to the value of a_1 and a_2. Now only one of the expressions is presented.

When $a_2 > a_1$ and $a_2 >- a_1$, the latent roots of equation (4) are

$$\pm i\,\omega_{\mathrm{n}}\sqrt{\frac{\rho(a_1+a_2)}{2E}} \quad , \quad \pm\omega_{\mathrm{n}}\sqrt{\frac{\rho(a_2-a_1)}{2E}}$$

Assuming $\lambda_1=\omega_{\mathrm{n}}\sqrt{\dfrac{\rho(a_1+a_2)}{2E}}$ and $\lambda_2=\omega_{\mathrm{n}}\sqrt{\dfrac{\rho(a_2-a_1)}{2E}}$, so the modal shape function $Y(x)$ is

$$Y(x)=A'\,\mathrm{e}^{\lambda_2 x}+B'\,\mathrm{e}^{-\lambda_2 x}+C'\,\mathrm{e}^{\mathrm{i}\,\lambda_1 x}+D'\,\mathrm{e}^{-\mathrm{i}\lambda_1 x} \tag{6}$$

Because $e^{\pm\lambda_2 x}=\mathrm{ch}\,\lambda_2 x\pm\mathrm{sh}\,\lambda_2 x$, $\mathrm{e}^{\pm\mathrm{i}\lambda_1 x}=\cos\lambda_1 x\pm i\sin\lambda_1 x$, so the formula (6) can be written as follows

$$Y(x)=A\,\mathrm{ch}\,\lambda_2 x+B\,\mathrm{sh}\,\lambda_2 x+C\cos\lambda_1 x+D\sin\lambda_1 x$$

Thereby, the solution of the differential equation (1) can be calculated to be the form

$$y(x,t)=(A\,\mathrm{ch}\,\lambda_2 x+B\,\mathrm{sh}\,\lambda_2 x+C\cos\lambda_1 x+D\sin\lambda_1 x)\,\mathrm{e}^{\mathrm{i}(\omega_n t+\varphi)} \tag{7}$$

Moreover, the expressions of angle $\theta(x,t)$, moment $M(x,t)$ and shearing force $Q(x,t)$ can be got

$$\theta(x,t)=\frac{\partial y}{\partial x},\quad M(x,t)=EJ\frac{\partial}{\partial \mathrm{x}}\left(\frac{\partial^2 y}{\partial x^2}-\frac{\rho}{KG}\frac{\partial^2 y}{\partial t^2}\right),\quad Q(x,t)=-\rho S\int\frac{\partial^2 y}{\partial t^2}\,\mathrm{d}\,x \tag{8}$$

Then, according to the free-free boundary conditions (l is the length of the cylinder),

$$M(0,t)=0,\quad M(l,t)=0,\quad Q(0,t)=0,\quad Q(l,t)=0$$

The frequency equation can be obtained as following

$$T_{31}T_{42}-T_{41}T_{32}=0 \tag{9}$$

where,

$$T_{31}=EJ\left[a_1\left(\frac{\rho}{KG}\omega_n^2+\lambda_2^2\right)\mathrm{ch}\,\lambda_2 l+c_1\left(\frac{\rho}{KG}\omega_n^2-\lambda_1^2\right)\cos\lambda_1 l\right]$$
$$T_{32}=EJ\left[b_1\left(\frac{\rho}{KG}\omega_n^2+\lambda_2^2\right)\mathrm{sh}\,\lambda_2 l+d_1\left(\frac{\rho}{KG}\omega_n^2-\lambda_1^2\right)\sin\lambda_1 l\right]$$
$$T_{41}=\rho S\omega_n^2\left(\frac{a_1}{\lambda_2}\mathrm{sh}\,\lambda_2 l+\frac{c_1}{\lambda_1}\sin\lambda_1 l\right)$$
$$T_{42}=\rho S\omega_n^2\left(\frac{b_1}{\lambda_2}\mathrm{ch}\,\lambda_2 l-\frac{d_1}{\lambda_1}\cos\lambda_1 l\right)$$

,

$$a_1=-\frac{1}{\lambda_1^2+\lambda_2^2}\left(\frac{\rho}{KG}\omega_n^2-\lambda_1^2\right)$$
$$b_1=\frac{\lambda_2}{\lambda_1^2+\lambda_2^2}$$
$$c_1=\frac{1}{(\lambda_1^2+\lambda_2^2)}\left(\frac{\rho}{KG}\omega_n^2+\lambda_2^2\right)$$
$$d_1=\frac{\lambda_1}{\lambda_1^2+\lambda_2^2}$$

The nature bending circular frequencies, ω_n, can be obtained by solving the equation (9).

2.2 The Process of Calculating the Axial Elastic Modulus

The sandwich cylinder is made of metal and one or two kinds of complex materials. The structure of the cylinder is shown in figure 1, the basic structure is metallic thin shell cylinder, and a fibre composite wraps around the outside of the metallic cylinder, then another fibre composite wraps again.

The axial elastic properties of this sandwich cylinder were emphasis in this work. In theory, the elastic properties of every material in the sandwich cylinder can be measured. But in fact, it was difficult to do because that the winding technology is wet winding and the thickness of each complex material cannot be measured exactly. So the axial elastic properties studied in this paper are equivalent elastic properties, which include the elastic properties of metal and the two complex materials.

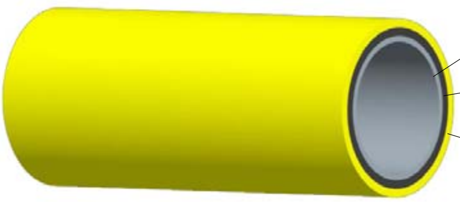

Figure 1. Sketch of the sandwich cylinder

Equation (9) can also be written as the following form

$$f(\omega_n, E_e, G_e, M, r_o, r_i, l) = 0 \tag{10}$$

Where ω_n is the nature bending circular frequency, E_e is the equivalent axial Young's modulus, G_e is the equivalent axial shear modulus, M is the mass of the sandwich cylinder, r_o, r_i, and l are the outer radius, inner radius and length of the cylinder respectively.

M, r_o, r_i, and l can be obtained accurately by measuring the cylinder. And if the two nature bending frequencies are known, the E_e and G_e can be calculated by solving the following equations.

$$\begin{cases} f(\omega_{n1}, E_e, G_e, M, r_o, r_i, l) = 0 \\ f(\omega_{n2}, E_e, G_e, M, r_o, r_i, l) = 0 \end{cases} \tag{11}$$

Two equations and two frequencies were needed in this method because E_e and G_e were unknown. If the relation of E_e to G_e is known, i.e. only one unknown number, just only one equation and one frequency are sufficient.

The equations (11) can only be solved numerically because of the nonlinearity. So the Monte Carlo Method was used to solve the equations [4], and a Fortran calculation program was programmed.

A numerical example was used for verifying the convergence of the calculation program. The example is an imaginary aluminium shell cylinder. And the first two order bending circular frequency, ω_{n1} and ω_{n2}, can be got by solving equation (10) when E_e is 68.00 GPa and G_e is 25.56 GPa.

Then substituting ω_{n1}, ω_{n2} and different initial values of E_e and G_e into (11), E_e and G_e can be obtained by solving the equation (11). The convergence results of the different initial values are shown in table 1. There were almost no errors between the calculational results and the actual values.

Table 1. The convergence results (GPa).

initial values		results		actual values	
E_{ei}	G_{ei}	E_e	G_e	E_e	G_e
80.00	30.00	68.00	25.56	68.00	25.56
50.00	25.00	68.00	25.56		
100.00	35.00	68.00	25.56		

But the convergence depended on the choice of initial values, so the appropriate initial values were important in the calculation program.

2.3 The Choice of Initial Values

If the effect of the rotary inertia and shear deformation is neglected, the equation (1) can be written as the following form

$$EJ\frac{\partial^4 y}{\partial x^4} + \rho S\frac{\partial^2 y}{\partial t^2} = 0 \tag{12}$$

For free-free boundary conditions, the frequency equation of equation (1) is [3]

$$\cos kl \operatorname{ch} kl = 0 \tag{13}$$

where,

$$k^4 = \frac{\omega_n^2}{a^2}\ ,\ a^2 = \frac{EJ}{\rho S}$$

For the first order nature frequency,

$$k_1 l \approx \frac{3}{2}\pi$$

Therefore, the first order nature circular frequency, ω_{n1}, is

$$\omega_{n1} = \left(\frac{3\pi}{2l}\right)^2\sqrt{\frac{EJ}{\rho S}} \tag{14}$$

If the first order circular frequency is known, the elastic modulus can be obtained,

$$E = \left(\frac{2l}{3\pi}\right)^4\frac{\rho S\omega_{n1}^2}{J} \tag{15}$$

The initial value for the equivalent elastic modulus of the sandwich cylinder with free-free boundary conditions could be obtained through the following steps.

Firstly, the elastic modulus and shear modulus used in the equation (1) could be given experiential values. Then substituting these values, dimension and mass values of the sandwich cylinder measured here into the equation (9), the nature circular frequency, ω_{n11}, can be obtained. Similarly, the other nature frequency, ω_{n12} can also be got by solving the equation (14). So the correctional coefficient of the testing frequency was given by

$$\gamma = \frac{\omega_{n11}}{\omega_{n12}} \tag{16}$$

Secondly, assuming that f_{exp} is the nature bending frequency measured by the modal testing, the correctional frequency, f_{cor}, can be given by the following formula

$$f_{\mathrm{cor}} = f_{\mathrm{exp}}\gamma \tag{17}$$

Finally, the initial value of elastic modulus can be obtained by solving the equations (15) and (17). But the initial value of shear modulus could only be determined through some experimental relations of elastic to shear modulus in this method.

If the cylinder is relative long, the initial value of elastic modulus can be obtained directly basing f_{exp} and equation (15) because the effect of the shear deformation can be neglected and the correctional coefficient is close to one.

3. THE MEANS OF GETTING THE NATURE FREQUENCIES

Another key of the dynamic test method is how to get the nature frequencies. In general, the modal testing is appropriate for this purpose. The MISO modal testing system was adopted to get the nature frequencies of the sandwich cylinder in this work. The sketch of the modal testing system is presented in figure 2. It mainly consists of an impact hammer (B&K8200), an accelerometer (B&K4375), two charge amplifiers (B&K2635), a signal conditioning (AZ208), a laptop with the CRAS™ modal testing software of NanJing Analyzer Software Engineering Ltd. and a testing frame.

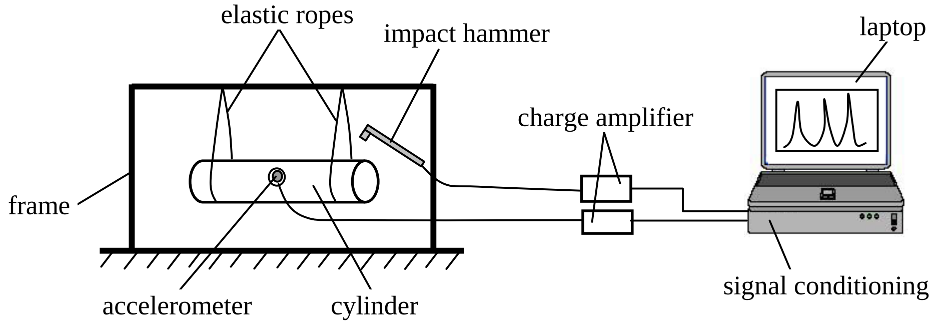

Figure 2. Sketch of the modal testing system

The testing sample, such as cylinder, was suspended from the frame by two elastic ropes to achieve the free-free boundary conditions approximatively. The impact hammer was used to make the cylinder vibrate mechanically. And the accelerometer, adhered to the surface of the cylinder, was used to pick up the response. The signal was amplified and conditioned by the charge amplifier and signal conditioning, then analyzed by the CRAS™ modal testing software. So the modal parameters and modal shapes of the cylinder can be acquired. One of the 3D modal shapes of the cylinder is shown in figure 3.

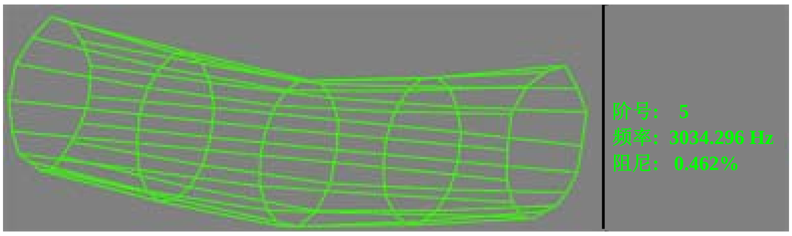

Figure 3. Modal shape of the cylinder

4. THE PROCEDURE AND APPLICATION OF THE MEASURING METHOD

4.1 The Procedure of the Measuring Method

The detailed procedure of measuring the equivalent axial elastic properties of the sandwich cylinder was

1. measure the mass of the cylinder.
2. measure the outer diameter, inner diameter and length of the cylinder.
3. obtain the first two order nature bending frequencies by the modal testing.
4. calculate the equivalent axial elastic modulus and shear modulus by solving the equations (11).

Two aluminium shell cylinders were tested by above procedure in order to evaluate the measuring method. The aluminium cylinder was chosen because its elastic properties have been well studied. The testing results indicated that the relative errors were less than 3% (see table 2). This proves that the measuring method is practicable.

Table 2. Axial elastic modulus of aluminum cylinder (GPa).

specimen	testing	actual	error (%)
1 #	70.43	70	0.6
2 #	71.64		2.3

4.2 The Application of the Measuring Method

The first kind of specimen was two relative long sandwich cylinders, which were made of two kinds of fibre composite and no metal material. Table 3 presents the first three order nature bending frequencies measured by the above modal testing system.

Table 3. Measured nature frequencies (Hz).

specimen	f_{n1}	f_{n2}	f_{n3}
1 #	453	1103	1892
2 #	436	1072	1866

The measuring results of the equivalent axial elastic and shear modulus are shown in table 4. The trend of the results is reasonable because the design of the sandwich cylinder shows that the elastic modulus of specimen 1 should be higher than specimen 2.

Table 4. The measuring results (GPa).

specimen	initial values		results	
	E_{ei}	G_{ei}	E_e	G_e
1 #	36.00	10.00	39.28	12.71
2 #	35.00	10.00	34.72	12.89

In order to evaluate this method again, the third order frequency was calculated by using the elastic modulus and shear modulus in table 4. The calculation frequencies of the two cylinders were 1883 Hz and 1848 Hz respectively. And the relative errors were both less than 1%.

The second kind of specimen was two sandwich cylinders, which were made of a metal and two kinds of fibre composite and similar to figure 1. Table 5 is the equivalent axial elastic modulus of the two cylinders measured by the above measuring method. The design of this kind of sandwich cylinder also shown that the axial elastic modulus of specimen 1 should be less than specimen 2.

Table 5. Equivalent axial elastic modulus (GPa).

specimen	initial values	results
1 #	30.00	32.39
2 #	40.00	41.01

5. SUMMARY

In this paper, a measuring method based on modal testing for identifying the equivalent axial elastic properties of the sandwich cylinder is presented. And this method is proved to be practicable by measuring some specimens. Future efforts will center on uncertainty estimation and more applications of this method.

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