ICSV14
Cairns • Australia
9-12 July, 2007
14th International Congress on Sound & Vibration

VIBRATION ANALYSIS OF A CRACKED BEAM SUBJECTED TO A TRAVELING VEHICLE

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Abstract

An analytical method is developed to present the forced responses of a cracked simply-supported beam subjected to a traveling vehicle. The cracked beam system is modeled as a two-span beam and each span of the continuous beam is assumed to obey the Euler-Bernoulli beam theory. The crack is modeled as a rotational spring with sectional flexibility and a traveling vehicle is modeled as two concentrated moving loads separated by the distance of the vehicle wheelbase. Using the analytical transfer matrix method by considering the compatibility requirements on the crack, eigensolutions of this cracked system can be obtained explicitly. The forced responses can then be determined by modal expansion theory.

1. INTRODUCTION

The dynamic behaviour of cracked structures has been studied by several analytical and numerical methods [1-9]. Many works in this field deal with cracked beams subjected to various boundary conditions. Narkis [1] has studied the inverse problem of a simply supported beam. Shen and Taylor [3] have also studied the inverse problem by structural optimization methods. Ostachowicz and Krawzuk [4] have analyzed the effect of two cracks on the fundamental frequency of a cantilever beam. Rizos *et al*. [5] have used a rotational spring to model the crack and detect the crack location through the measurement of the amplitudes of the component. Dimarogonas [8] presented a review of the dynamics of cracked structures. A complete cracked-beam vibration theory was also developed by Chondros and Dimarogonas [9] for the transverse vibration of a cracked Euler-Bernoulli beam with single-edge or double-edge open cracks. In this study, the cracked region as a local flexibility is expressed by a crack-disturbance function $f(x,z)$ which can be derived from the stress intensity factors in the theory of fracture mechanics. An inverse problem involves the determination of the crack location and extent from the measured information of the cracked beam system. Liang *et al*. [10] have studied a similar problem by finite element methods.

The effect of moving load on structures and machines is an important problem in the engineering field, for example, in the design of bridges or in the design of machining processes. A moving load will produce larger deflections and higher stresses than equivalent static load conditions. A lot of studies had also been done [12-14] on this field. However, not so many studies were reported in the previous literature on the effect of cracks on the moving load problems. Mahmoud [13] used an equivalent static load approach to determine the stress intensity factors for a crack in a beam subjected to a moving load. Mahmoud and Zaid [14] used an iterative modal analysis approach to find the response of a cracked simply supported beam with a crack. Most of the previous studies on this field have analyzed the problem numerically or hybrid numerically. The investigation in this study presents an analytical method that permits computation of the forced responses of a cracked simply-supported beam subjected to a traveling vehicle. The method is based on modelling the cracked beam as a two-span beam and each span of the continuous beam is assumed to obey the Euler-Bernoulli beam theory. The crack is modelled as a massless rotational spring with sectional flexibility and a travelling vehicle is modelled as two concentrated moving loads separated by the distance of the vehicle wheelbase. Considering the compatibility requirements on the crack, the relationships between these two spans can be obtained. By using the analytical transfer matrix method, eigensolutions of this cracked system are obtained explicitly. The forced responses can then be determined by modal expansion theory.

2. THEORETICAL MODEL

A simply-supported beam of length L with one open crack at intermediate position X_1 and a traveling vehicle with constant speed *V* is shown in Fig. 1a where X_0 and X_2 represent end points. The vehicle can be modeled as two concentrated moving loads P_1 and P_2 which are from the normal forces of the front and rear axles respectively. The vehicle has a wheelbase *D* and weight *W* with center of gravity (C.G) located at a distance a from the front axle as shown in Fig. 1b.

One major assumption in the analysis of this article is that the crack remains always open during the motions of the beam. The vibration amplitude of the transverse displacement is denoted by $Y(X,T)$. By using the Euler-Bernoulli beam theory [1, 8, 12], the equation of motion of the system, assumed to have a uniform cross section, is

$$EI\frac{\partial^4 Y(X,T)}{\partial X^4}+\rho A\frac{\partial^2 Y(X,T)}{\partial T^2}=P_1\delta(X-VT)+P_2\delta(X-(VT-D)), \tag{1}$$

where *E* is Young's modulus of the material, *I* is the moment of inertia of the beam's cross-section, ρ is the density of material, *A* is the cross-sectional area of the beam,

$\delta(X-VT)$ and $\delta(X-(VT-D))$ denote the Dirac delta distributions, V is the constant speed of the travelling vehicle, T is time and P_1, P_2 are normal forces from the front and rear axles of the vehicle and which can be expressed as:

$$P_1 = W\,\frac{D-a}{D}, \qquad P_2 = W\,\frac{a}{D},$$

where a is the distance of vehicle C.G from vehicle front axle, D is the vehicle wheelbase and W is the vehicle weight. The boundary conditions of the beam for a simply-supported case are

$$Y(0,T) = Y''(0,T) = 0, \tag{2a}$$
$$Y(L,T) = Y''(L,T) = 0, \tag{2b}$$

where $(\,'\,)$ denotes the derivative with respect to the space coordinate X. The crack is modeled as a rotational spring with sectional flexibility.

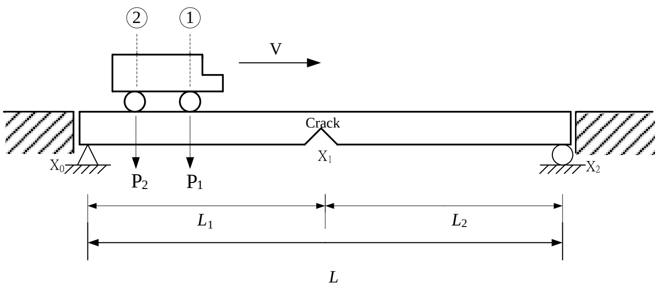

Fig. 1a: A simply-supported beam with an open crack of depth D_p located at position X_1 and a traveling vehicle with constant speed V; sub-domains L_1 and L_2 where $L_1 + L_2 = L$.

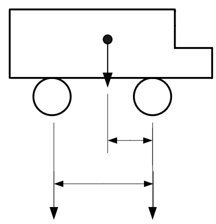

Fig. 1b: Parameters of the traveling vehicle: center of gravity C.G, weight W and wheelbase D.

The “compatibility requirements” enforce continuities of the displacement, bending moment and shear force, respectively, across the crack and can be expressed as [6, 7]

$$Y_{(1)}(X_1^-,T) = Y_{(2)}(X_1^+,T), \tag{3a}$$
$$Y''_{(1)}(X_1^-,T) = Y''_{(2)}(X_1^+,T), \tag{3b}$$
$$Y'''_{(1)}(X_1^-,T) = Y'''_{(2)}(X_1^+,T), \tag{3c}$$

where the symbols X_1^+ and X_1^- denote the locations immediately above and below the crack position X_1 and the sub-index in the parenthesis represents the segments (sub-beams) of the system. Moreover, a discontinuity in the slope of the beam across the crack exists and can be expressed as [1, 6]

$$Y'_{(2)}(X_1^+,T) - Y'_{(1)}(X_1^-,T) = \theta\, L\, Y''_{(2)}(X_1^+,T), \tag{3d}$$

where θ is the non-dimensional crack sectional flexibility, which is the function of the crack extent. [4, 6]

In the above equations, the following quantities are introduced:

$$y = \frac{Y}{L},\ x = \frac{X}{L},\ d = \frac{D}{L},\ l_1 = \frac{L_1}{L},\ l_2 = \frac{L_2}{L},\ x_i = \frac{X_i}{L},\ t = \frac{T}{\sqrt{L}},\ v = \frac{V}{\sqrt{L}}. \quad \text{(4a~4h)}$$

Thus, Eqs. (1) can then be expressed in a non-dimensional form as

$$\frac{EI}{L^3}\frac{\partial^4 y(x,t)}{\partial x^4} + \rho A\frac{\partial^2 y(x,t)}{\partial t^2} = P_1\delta(x - vt) + P_2\delta(x-(vt-d)). \quad (5)$$

The non-dimensional "compatibility requirements" from Eqs. (3a) to (3d) are

$$y_{(1)}(x_1^-,t) = y_{(2)}(x_1^+,t), \quad (6a)$$

$$y''_{(1)}(x_1^-,t) = y''_{(2)}(x_1^+,t), \quad (6b)$$

$$y'''_{(1)}(x_1^-,t) = y'''_{(2)}(x_1^+,t), \quad (6c)$$

$$y'_{(2)}(x_1^+,t) - y'_{(1)}(x_1^-,t) = \theta\, y''_{(2)}(x_1^+,t). \quad (6d)$$

3. METHOD TO FIND EIGENSOLUTIONS

Using the separable solutions: $y_{(i)}(x,t) = w_{(i)}(x)\, e^{j\omega t}$ in Eqs. (5) leads to an associated eigenvalue problem,

$$w''''_{(i)}(x) - \lambda^4 w_{(i)}(x) = 0, \qquad x_{i-1} < x < x_i,\ i = 1,2 \quad (7a)$$

where $\lambda^4 = \dfrac{\rho A\,\omega^2 L^3}{EI}$. (7b)

From Eqs. (6a) to (6d), the corresponding compatibility requirements across the crack lead to

$$w_{(1)}(x_1^-) = w_{(2)}(x_1^+), \quad (8a)$$

$$w''_{(1)}(x_1^-) = w''_{(2)}(x_1^+), \quad (8b)$$

$$w'''_{(1)}(x_1^-) = w'''_{(2)}(x_1^+), \quad (8c)$$

$$w'_{(2)}(x_1^+) - w'_{(1)}(x_1^-) = \theta\, w''_{(2)}(x_1^+). \quad (8d)$$

A closed-form solution to this eigenvalue problem can be obtained by employing transfer matrix methods [15]. The general solution of Eqs. (7a), for each segment, is

$$w_{(i)}(x) = A_i \sin\lambda(x - x_{i-1}) + B_i\cos\lambda(x - x_{i-1}) + C_i\sinh\lambda(x - x_{i-1}) + D_i\cosh\lambda(x - x_{i-1})$$
$$x_{i-1} < x < x_i, \qquad i = 1,2 \quad (9)$$

where A_i, B_i, C_i and D_i are constants associated with the *i-th* segment ($i = 1,2$). These constants in the second segment (A_2, B_2, C_2 and D_2) are related to those in the first segment (A_1 , B_1 , C_1 and D_1) through the compatibility requirements in Eqs. (8a) to (8d) and can be expressed as [6, 7]

$$\begin{Bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{Bmatrix} = \begin{bmatrix} t_{11}\ t_{12}\ t_{13}\ t_{14} \\ \vdots \\ \\ \ldots\ldots\ldots\ldots t_{44} \end{bmatrix} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \underline{T}_{4\times 4} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix}, \tag{10}$$

where $\underline{T}_{4\times4}$ is a 4×4 transfer matrix which depends on eigenvalue λ and the elements of which are derived in [6] and rewritten here:

$t_{11} = \cos\lambda l_1 - (1/2)\theta\ \lambda \sin\lambda l_1$, $\quad t_{12} = -\sin\lambda l_1 - (1/2)\theta\ \lambda\cos\lambda l_1$,

$t_{13} = (1/2)\theta\ \lambda\sinh\lambda l_1$, $\quad t_{14} = (1/2)\theta\ \lambda\cosh\lambda l_1$,

$t_{21} = \sin\lambda l_1$, $\quad t_{22} = \cos\lambda l_1$,

$t_{23} = 0$, $\quad t_{24} = 0$,

$t_{31} = -(1/2)\ \theta\ \lambda\sin\lambda l_1$, $\quad t_{32} = -(1/2)\ \theta\ \lambda\cos\lambda l_1$,

$t_{33} = \cosh\lambda l_1 + (1/2)\ \theta\ \lambda\sinh\lambda l_1$, $\quad t_{34} = \sinh\lambda l_1 + (1/2)\ \theta\ \lambda\cosh\lambda l_1$,

$t_{41} = 0$, $\quad t_{42} = 0$,

$t_{43} = \sinh\lambda l_1$, $\quad t_{44} = \cosh\lambda l_1$.

After applying the boundary conditions, the following equation can be obtained as:

$$\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \underline{B}_{2\times4} \begin{Bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{Bmatrix} = \underline{B}_{2\times4}\underline{T}_{2\times4} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \underline{R}_{2\times4} \begin{Bmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{Bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \end{bmatrix} \begin{Bmatrix} A_1 \\ 0 \\ C_1 \\ 0 \end{Bmatrix}, \tag{11a}$$

where $$\underline{B}_{2\times4} = \begin{bmatrix} \sin\lambda l_2 & \cos\lambda l_2 & \sinh\lambda l_2 & \cosh\lambda l_2 \\ -\sin\lambda l_2 & -\cos\lambda l_2 & \sinh\lambda l_2 & \cosh\lambda l_2 \end{bmatrix}, \tag{11b}$$

$$\underline{R}_{2\times4} = \underline{B}_{2\times4}\underline{T}_{4\times4} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \end{bmatrix}. \tag{11c}$$

Thus, the existence of non-trivial solutions requires

$$\det \begin{vmatrix} r_{11}(\lambda) & r_{13}(\lambda) \\ r_{21}(\lambda) & r_{23}(\lambda) \end{vmatrix} = 0. \tag{12}$$

This determinant provides the single (characteristic) equation for the solution of the eigenvalues λ_n. This is a matrix of only 2×2 dimensions: therefore, it is possible to obtain the corresponding characteristic equation explicitly. After the expansion of Eq. (12), the following symbolic characteristic equation can be obtained explicitly:

$$\begin{aligned} &4\sin\lambda_n \sinh\lambda_n + \theta\lambda_n\ (\cos\lambda_n \sinh\lambda_n + \sin\lambda_n\cosh\lambda_n) \\ &\quad - \theta\lambda_n\ [\cos(\lambda_n(1-2l_1))\sinh\lambda_n + \sin\lambda_n\cosh(\lambda_n(1-2l_1))] = 0. \end{aligned} \tag{13}$$

where λ_n is the eigenvalue of the system, $l_1(\equiv \frac{L_1}{L})$ is the non-dimensional crack length of the first span and θ is the non-dimensional crack sectional flexibility which can be obtained described above for double and single-sided open cracks, respectively. The coefficients of the eigenfunctions, $w_n(x)$, are obtained by back substitution into Eqs. (11a), (10) and then Eq.(9).

4. FORCED RESPONSES

The original equation of motion (Eq.(5)) can be expressed as

$$\frac{\partial^4 y(x,t)}{\partial x^4}+\frac{\rho A L^3}{EI}\frac{\partial^2 y(x,t)}{\partial t^2}=\frac{P_1 L^3}{EI}\delta(x-vt)+\frac{P_2 L^3}{EI}\delta(x-(vt-d)). \tag{14}$$

Using the modal expansion theory, the forced response $y(x,t)$ can be expressed as:

$$y(x,t)=\sum_{k=1}^{N} w_k(x)q_k(t), \tag{15}$$

where $w_k(x)$ are normalized eigenfunctions of the cracked system and which are obtained from the above section (**section 3**), $q_k(t)$ are generalized coordinates and N is the number of terms used to approximate the solution.

Substitute Eq.(15) into Eq. (14), multiplying by $w_j(x)$, and integrating from 0 to 1 leads to

$$\ddot{q}_k(t)+\omega_k^2 q_k(t)=\frac{P_1}{\rho A}\int_0^1 w_k(x)\delta(x-vt)dx+\frac{P_2}{\rho A}\int_0^1 w_k(x)\delta(x-(vt-d))dx$$
$$=\frac{P_1}{\rho A}w_k(vt)+\frac{P_2}{\rho A}w_k(vt-d)=Q_k(t), \quad k=1,2,...,N. \tag{16}$$

The generalized coordinate $q_k(t)$ are solved from Eq. (16) as:

$$q_k(t)=q_k(0)\cos\omega_k t+\frac{\dot{q}_k(0)}{\omega_k}\sin\omega_k t+\frac{1}{\omega_k}\int_0^t \sin\omega_k(t-\tau)Q_k(\tau)d\tau, \tag{17}$$

where $q_k(0)=\int_0^1 y_0(x)w_k(x)dx$, $\quad \dot{q}_k(0)=\int_0^1 \dot{y}_0(x)w_k(x)dx$, $\quad k=1,2,...,N$,

and $y_0(x)=y(x,0)$, $\dot{y}_0(x)=\dot{y}(x,0)$ are initial conditions of the system.

The eigenfunctions $w_k(x)$ used in Eq. (17) are from Eq.(9) and can be expressed as:

$$w_k(x)=\begin{cases} f_{k1}(x)=A_{k1}\sin\lambda_k(x-x_0)+B_{k1}\cos\lambda_k(x-x_0)+C_{k1}\sinh\lambda_k(x-x_0)+D_{k1}\cosh\lambda_k(x-x_0), \\ \hfill x\le x_1 \\ f_{k2}(x)=A_{k2}\sin\lambda_k(x-x_1)+B_{k2}\cos\lambda_k(x-x_1)+C_{k2}\sinh\lambda_k(x-x_1)+D_{k2}\cosh\lambda_k(x-x_1), \\ \hfill x> x_1. \end{cases}$$

Thus, the generalized forcing term $Q_k(t)$ in Eq.(16) can be written as

$$Q_k(t)=\frac{P_1}{\rho A}w_k(vt)+\frac{P_2}{\rho A}w_k(vt-d)=\begin{cases}\frac{P_1}{\rho A}f_{k1}(vt), & 0<vt\le d \quad \text{(18a)}\\ \frac{P_1}{\rho A}f_{k1}(vt)+\frac{P_2}{\rho A}f_{k1}(vt-d), & d<vt\le x_1 \quad \text{(18b)}\\ \frac{P_1}{\rho A}f_{k2}(vt)+\frac{P_2}{\rho A}f_{k1}(vt-d), & x_1<vt\le x_1+d \quad \text{(18c)}\\ \frac{P_1}{\rho A}f_{k2}(vt)+\frac{P_2}{\rho A}f_{k2}(vt-d), & vt>x_1+d \quad \text{(18d)}\end{cases}$$

In the range $0<vt\le d$, the rear axle does not enter the beam, the only moving load is the load of the front axle P_1 as in Eq. (18a). The term $\frac{1}{\omega_k}\int_0^t \sin\omega_k(t-\tau)Q_k(\tau)d\tau$ in Eq. (17) can thus be expressed as

$$\frac{1}{\omega_k}\int_0^t \sin\omega_k(t-\tau)Q_k(\tau)d\tau=\frac{P_1}{\rho A}\frac{1}{\omega_k}\int_0^t \sin\omega_k(t-\tau)f_{k1}(v\tau), \qquad 0<\tau\le\frac{d}{v}, \tag{19a}$$

$$\frac{P_1}{\rho A}\frac{1}{\omega_k}\int_0^t \sin\omega_k(t-\tau)f_{k1}(v\tau)d\tau+\frac{P_2}{\rho A}\frac{1}{\omega_k}\int_{d/v}^t \sin\omega_k(t-\tau)f_{k1}(v\tau-d)d\tau, \qquad \frac{d}{v}<\tau\le\frac{x_1}{v} \tag{19b}$$

$$\frac{P_1}{\rho A}\frac{1}{\omega_k}[\int_0^{x_1/v}\sin\omega_k(t-\tau)f_{k1}(v\tau)d\tau+\int_{x_1/v}^t \sin\omega_k(t-\tau)f_{k2}(v\tau)d\tau\ ]$$
$$+\frac{P_2}{\rho A}\frac{1}{\omega_k}\int_{d/v}^t \sin\omega_k(t-\tau)f_{k1}(v\tau-d)d\tau, \qquad \frac{x_1}{v}<\tau\le\frac{x_1+d}{v} \tag{19c}$$

$$\frac{P_1}{\rho A}\frac{1}{\omega_k}[\int_0^{x_1/v}\sin\omega_k(t-\tau)f_{k1}(v\tau)d\tau+\int_{x_1/v}^t \sin\omega_k(t-\tau)f_{k2}(v\tau)d\tau\ ]$$
$$+\frac{P_2}{\rho A}\frac{1}{\omega_k}[\int_{d/v}^{x_1/v}\sin\omega_k(t-\tau)f_{k1}(v\tau-d)d\tau+\int_{x_1/v}^t \sin\omega_k(t-\tau)f_{k2}(v\tau-d)d\tau], \qquad \tau>\frac{x_1+d}{v}. \tag{19d}$$

After the generalized coordinates $q_k(t)$ in Eq. (17) are obtained, the forced response solutions $y(x,t)$ can then be reconstructed from Eq. (15).

5. CONCLUSIONS

An analytical method is developed to present the dynamic responses of a cracked simply-supported beam subjected to a traveling vehicle load. The cracked beam system is modeled as a two-span beam and each span of the continuous beam is assumed to obey Euler-Bernoulli beam theory. The crack is modeled as a rotational spring with sectional flexibility and a traveling vehicle is modeled as two concentrated moving loads separated by the distance of the vehicle wheelbase. Considering the compatibility requirements on the crack, the relationships between these two spans can be obtained. By using the analytical transfer matrix method, eigensolutions of this cracked system are obtained explicitly. The eigenfunctions obtained in this article are analytical solutions and forced responses can be obtained by the modal expansion of eigenfunctions.

ACKNOWLEDGMENT

The author gratefully acknowledges the support of the National Science Council in Taiwan R.O.C. under grant number NSC 93-2745-E-212-006-URD.

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