

DETERMINATION OF MECHANICAL PROPERTIES OF A NON-UNIFORM BEAM USING THE MEASUREMENT OF THE EXCITED LONGITUDINAL ELASTIC VIBRATIONS.

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Abstract

The purpose of this paper is to present the statement of the inverse problem for longitudinal elastic vibrations of a non-uniform beam and the approximate analytical and numeric solutions of this problem under the condition of weak heterogeneity. To solve the inverse problem, certain longitudinal vibrations are excited in the beam and mechanical properties of the non-uniform beam (Young's module and density) have to be determined from the measured vibration in a certain point of the beam. Under the condition of weak heterogeneity the problem doesn't lose its importance. Weak heterogeneity can be found in many natural and produced materials. In this case it's possible to linearize the problem, expressing the difference between the properties of the base homogeneous beam and the properties of the weak heterogeneous beam through the difference between their vibrations and solve the problem. The problem is decomposed into 2 sub-problems – finding the difference between vibration of the uniform and non-uniform beams (solving the partial differential equation of the 2^{nd} order with constant coefficients) and finding the difference between the mechanical properties of the beams (solving the system of linear ordinary differential equations of the 1^{st} order).

1. INTRODUCTION

The equation describing distribution of longitudinal elastic waves in a non-uniform beam (in a dimensionless form):

$$\frac{\partial}{\partial\xi} \left(E(\xi) \frac{\partial w}{\partial\xi} \right) - \rho(\xi) \frac{\partial^2 w}{\partial\tau^2} = f(\xi, \tau), \tag{1}$$

where $E(\xi) = \tilde{E}(\xi)\tilde{A}(\xi), \rho(\xi) = \tilde{\rho}(\xi)\tilde{A}(\xi);$

 $\xi\,$ - distance from an end face of the beam ($\xi\,{=}\,0$) to the given section, $\,0\,{\leq}\,\xi\,{\leq}\,1\,;$

 τ - dimensionless time;

 $\tilde{E}(\xi)$ - Young's module;

 $\tilde{\rho}(\xi)$ - beam's density;

 $\tilde{A}(\xi)$ - the area of cross-section of the beam;

 $f(\xi,\tau)$ - longitudinal distributed force;

 $w = w(\xi, \tau)$ - longitudinal displacement of the beam's cross-section.

The equation (1) should satisfy boundary

$$w(0,\tau) = w(1,\tau) = 0,$$
(2)

and initial

$$w(\xi,0) = \varphi(\xi), \frac{\partial w}{\partial \tau}\Big|_{(\xi,0)} = \psi(\xi)$$
(3)

conditions.

In this study, we investigate the inverse coefficient identification problem which requires the identification of unknown non-uniform coefficients $\{E(\xi), \rho(\xi)\}$ of the equation (1) satisfying (2) and (3) with known functions $\{f(\xi,\tau), \varphi(\xi), \psi(\xi)\}$, and also some data known on function $w(\xi,\tau)$ (the experimental data received from the measurement of the beam vibrations initiated by the given initial conditions and the distributed force). We use the data measurement of longitudinal displacement of the fixed beam's cross section in time:

$$w(a,\tau) = \chi_a(\tau), w(b,\tau) = \chi_b(\tau), \frac{\partial w}{\partial \xi}\Big|_{(a,\tau)} = \gamma_a(\tau),$$
(4)

where a, b – distances from an end face of the beams to the studied cross-sections (0 < a, b < 1).

We shall restrict ourselves to determination of only one coefficient, considering the other one known and constant. We shall consider, that $\rho(\xi) = \rho = const$ is a known value. Thus the problem is reduced to determination of only $E(\xi)$.

In this study, the approximate analytical and the numeric solutions are investigated under the condition of weak heterogeneity of the beam.

2. ANALYTICAL SOLUTION

We shall solve the stated inverse problem analytically, based on a method of inverse problems solution suggested in [1]. We shall assume weak heterogeneity of the beam's properties, i.e.

$$E(\xi) = E^{0} + E^{\varepsilon}(\xi), \left| E^{\varepsilon}(\xi) \right| \ll 1,$$
(5)

where $E^0 = const$ - is the property of the base homogeneous beam.

Further we consider E^0 and ρ to be known.

We can express $w = w(\xi, \tau)$ as:

$$w(\xi,\tau) = w^{0}(\xi,\tau) + w^{\varepsilon}(\xi,\tau), \left|w^{\varepsilon}(\xi,\tau)\right| \ll 1.$$
(6)

Substituting expressions (5) and (6) in the equation (1), we receive:

$$F_{1}(\xi,\tau) - f(\xi,\tau) = -F_{2}(\xi,\tau) + F_{3}(\xi,\tau),$$

$$F_{1}(\xi,\tau) = E^{0} \frac{\partial^{2} w^{0}}{\partial \xi^{2}} - \rho \frac{\partial^{2} w^{0}}{\partial \tau^{2}},$$

$$F_{2}(\xi,\tau) = E^{0} \frac{\partial^{2} w^{\varepsilon}}{\partial \xi^{2}} - \rho \frac{\partial^{2} w^{\varepsilon}}{\partial \tau^{2}} + \frac{\partial}{\partial \xi} \left(E^{\varepsilon} \frac{\partial w^{0}}{\partial \xi} \right),$$

$$F_{3}(\xi,\tau) = \frac{\partial}{\partial \xi} \left(E^{\varepsilon} \frac{\partial w^{\varepsilon}}{\partial \xi} \right).$$
(7)

The analytical solution of the problem is possible, when the following conditions are met (we do not affirm that the problem can't be solved analytically if these conditions are not met, however these conditions are necessary for the solution offered below):

- only natural vibrations are considered, i.e. $f(\xi, \tau) \equiv 0$;
- initial conditions are selected in such a way, that in a corresponding base homogeneous beam the same initial conditions excite harmonic vibrations;
- $|F_2(\xi,\tau)| \ll |F_1(\xi,\tau)|$ and $|F_3(\xi,\tau)| \ll |F_2(\xi,\tau)|$ (these conditions are provided by the fact that the functions w^{ε} and E^{ε} are small by absolute value, as given in (5) and (6)).

If all specified conditions are met, the equation (7) splits up into two:

$$E^{0}\frac{\partial^{2}w^{0}}{\partial\xi^{2}} - \rho\frac{\partial^{2}w^{0}}{\partial\tau^{2}} = 0,$$

with boundary and initial conditions given by:

$$w^{0}(0,\tau) = w^{0}(1,\tau) = 0, w^{0}(\xi,0) = \varphi(\xi), \frac{\partial w^{0}}{\partial \tau}\Big|_{(\xi,0)} = \psi(\xi)$$
(8)

and

$$E^{0}\frac{\partial^{2}w^{\varepsilon}}{\partial\xi^{2}} - \rho\frac{\partial^{2}w^{\varepsilon}}{\partial\tau^{2}} = -\frac{\partial}{\partial\xi}\left(E^{\varepsilon}\frac{\partial w^{0}}{\partial\xi}\right),\tag{9}$$

with homogeneous boundary and initial conditions:

$$w^{\varepsilon}(0,\tau) = w^{\varepsilon}(1,\tau) = 0, w^{\varepsilon}(\xi,0) = 0, \frac{\partial w^{\varepsilon}}{\partial \tau}\Big|_{(\xi,0)} = 0.$$
(10)

Substituting expression (5) in experimental data (4), we obtain the additional source data for the solution:

$$w^{\varepsilon}(a,\tau) = \chi^{\varepsilon}_{a}(\tau) = \chi_{a}(\tau) - w^{0}(a,\tau), \frac{\partial w^{\varepsilon}}{\partial \xi}\Big|_{(a,\tau)} = \gamma^{\varepsilon}_{a}(\tau) = \gamma_{a}(\tau) - \frac{\partial w^{0}}{\partial \xi}\Big|_{(a,\tau)}.$$
(11)

To meet the condition of the harmonic vibrations in the base homogeneous beam, the displacement $w^0(\xi,\tau)$ should take the form:

$$w^{0}(\xi,\tau) = \sin(\alpha k\xi) (B\sin\alpha\tau + C\cos\alpha\tau), \alpha = \frac{\pi n}{k}, k = \sqrt{\frac{\rho}{E^{0}}}, n \in N,$$
(12)

which means that the initial conditions are restricted to be in the form:

$$w^{0}(\xi,0) = \varphi(\xi) = C\sin(\alpha k\xi), \frac{\partial w^{0}}{\partial \tau}\Big|_{(\xi,0)} = \psi(\xi) = \alpha B\sin(\alpha k\xi).$$

We consider all parameters of the initial conditions $\{B, C, n\}$ to be known. Substituting

them in (12) we obtain $w^0(\xi,\tau)$.

Now we should obtain $w^{\varepsilon}(\xi,\tau)$ from the equation (9), boundary and initial conditions (10) and measurement data (11). Substituting (12) in (9), we get:

$$\frac{\partial^2 w^{\varepsilon}}{\partial \xi^2} - k^2 \frac{\partial w^{\varepsilon}}{\partial \tau} = -\frac{\alpha k}{E^0} \frac{\partial}{\partial \xi} \Big(E^{\varepsilon}(\xi) \cos(\alpha k\xi) \Big) \Big(B \sin \alpha \tau + C \cos \alpha \tau \Big).$$
(13)

Applying operator $\partial^2 + \alpha^2 I (I - \text{identity operator})$ to the equation (13), we get the system of the differential equations:

$$\begin{cases} \frac{\partial^2 v(\xi,\tau)}{\partial \xi^2} - k^2 \frac{\partial^2 v(\xi,\tau)}{\partial \tau^2} = 0, \\ v(\xi,\tau) = \frac{\partial^2 w^{\varepsilon}(\xi,\tau)}{\partial \tau^2} + \alpha^2 w^{\varepsilon}(\xi,\tau). \end{cases}$$
(14)

Substituting the measurement data (11) in the second equation of the system (14), we obtain:

$$v(a,\tau) = \chi_a^v(\tau) = \frac{d\chi_a^\varepsilon(\tau)}{d\tau} + \alpha^2 \chi_a^\varepsilon(\tau), \frac{\partial v}{\partial \xi}\Big|_{(a,\tau)} = \gamma_a^v(\tau) = \frac{d\gamma_a^\varepsilon(\tau)}{d\tau} + \alpha^2 \gamma_a^\varepsilon(\tau).$$
(15)

Solving the first equation of the system (14) and using the (15), we obtain $v(\xi, \tau)$:

$$v(\xi,\tau) = \frac{\chi_a^v(\tau - ka + k\xi) + \chi_a^v(\tau + ka - k\xi)}{2} + \frac{\int_{\tau + ka - k\xi}^{\tau + ka - k\xi} \gamma_a^v(z)dz}{2k}$$

Solving the second equation of the system (14) with respect to $w^{\varepsilon}(\xi,\tau)$ and homogeneous initial conditions (10), we find:

$$w^{\varepsilon}(\xi,\tau) = -W(\xi,0)\cos\alpha\tau - \frac{1}{\alpha}\frac{\partial W}{\partial\tau}\Big|_{(\xi,0)}\sin\alpha\tau + W(\xi,\tau),$$

$$W(\xi,\tau) = \frac{\sin\alpha\tau\int v(\xi,\tau)\cos\alpha\tau d\tau - \cos\alpha\tau\int v(\xi,\tau)\sin\alpha\tau d\tau}{\alpha}.$$
(16)

Substituting (16) in (13) and solving the ordinary differential equation with respect to $E^{\varepsilon}(\xi)$, we find the solution to the inverse problem:

$$E^{\varepsilon}(\xi) = -\frac{E^{0}}{\alpha k \cos \alpha k \xi} \left[D + \frac{1}{\left(B \sin \alpha \tau + C \cos \alpha \tau\right)} \int_{0}^{\xi} \left(\frac{\partial^{2} w^{\varepsilon}(z,\tau)}{\partial z^{2}} - k^{2} \frac{\partial^{2} w^{\varepsilon}(z,\tau)}{\partial \tau^{2}} \right) dz \right],$$

where *D* is arbitrary constant subject to determination. We can define it, having set the value of E^{ε} at $\xi = 0$:

$$E^{\varepsilon}(0) = -\frac{E^{0}D}{\alpha k} = E_{0}^{\varepsilon}, D = \frac{E_{0}^{\varepsilon}\alpha k}{E^{0}}.$$

3. NUMERIC SOLUTION

The stated inverse problem can also be solved using the regularization method of Tikhonov & Arsenin [2], which requires finding the minimum of the functional with respect to E^{ε} :

$$\Phi 0_{\lambda} \left(E^{\varepsilon} \right) = \int_{0}^{1} \left[\left(w_{E^{\varepsilon}}^{\varepsilon} \left(a, \tau \right) - \chi_{a}^{\varepsilon} \left(\tau \right) \right)^{2} + \lambda \left(E^{\varepsilon} \left(\tau \right) \right)^{2} \right] d\tau,$$
(17)

where $\lambda > 0$ - regularization parameter;

 $w_{E^{\varepsilon}}^{\varepsilon}$ - solution of the equation (9) with boundary and initial conditions (10) and given $E^{\varepsilon}(\xi)$;

 $\chi_a^{\varepsilon}(\tau)$ - measurement data (11).

Contrary to analytical solution, the regularization method can be used with arbitrary initial conditions and longitudinal distributed force. The following conditions remain unchanged (from analytical solution): $|F_2(\xi,\tau)| \ll |F_1(\xi,\tau)|$; $|F_3(\xi,\tau)| \ll |F_2(\xi,\tau)|$.

The solution of the equation (9) with boundary and initial conditions (10) is given by:

$$w_{E^{\varepsilon}}^{\varepsilon}\left(\xi,\tau\right) = \frac{\int\limits_{0}^{\tau} \left[E^{\varepsilon}\left(\xi + \frac{\tau - z}{k}\right) \frac{\partial w^{0}}{\partial \xi} \left(\xi + \frac{\tau - z}{k}, \tau\right) - E^{\varepsilon}\left(\xi - \frac{\tau - z}{k}\right) \frac{\partial w^{0}}{\partial \xi} \left(\xi - \frac{\tau - z}{k}, \tau\right) \right] dz}{2\sqrt{\rho E^{0}}},$$

where $w^0(\xi,\tau)$ is the solution of the equation

$$E^{0}\frac{\partial^{2}w^{0}}{\partial\xi^{2}}-\rho\frac{\partial^{2}w^{0}}{\partial\tau^{2}}=f(\xi,t),$$

with boundary and initial conditions (8).

Any solution $E^{\varepsilon}(\xi)$ found as the result of minimization of the functional (17) may not be unique. If $E_1^{\varepsilon}(\xi)$ satisfies

$$w_{E^{\varepsilon}}^{\varepsilon}\left(a,\tau\right) = \frac{\int\limits_{0}^{\tau} \left[E^{\varepsilon} \left(a + \frac{\tau - z}{k}\right) \frac{\partial w^{0}}{\partial \xi} \left(a + \frac{\tau - z}{k}, \tau\right) - E^{\varepsilon} \left(a - \frac{\tau - z}{k}\right) \frac{\partial w^{0}}{\partial \xi} \left(a - \frac{\tau - z}{k}, \tau\right) \right] dz}{2\sqrt{\rho E^{0}}},$$
(18)

and $w^0(\xi,\tau)$ can be expressed as $w^0(\xi,\tau) = w^0_{\xi}(\xi)w^0_{\tau}(\tau)$, then

$$E_2^{\varepsilon}(\xi) = E_1^{\varepsilon}(\xi) + \beta w_{\xi}^0(2a - \xi),$$

where β is an arbitrary constant, also satisfies (18). This means that the same vibrations at the fixed cross-section can be satisfied by infinitively many properties $E^{\varepsilon}(\xi)$. This should have been expected, as for any second-order differential equation we must define two functions to remove ambiguity from the solution. The analytical solution to the inverse problem also uses two functions (instead of one) to remove ambiguity. To avoid ambiguity in the regularization solution, we should modify the functional (17) to include two functions from the measurement results. We shall use the displacement in time of another cross-section at the distance $\xi = b$. Modified functional will take the form:

$$\Phi 1_{\lambda} \left(E^{\varepsilon} \right) = \int_{0}^{1} \left[\left(w_{E^{\varepsilon}}^{\varepsilon} \left(a, \tau \right) - \chi_{a}^{\varepsilon} \left(\tau \right) \right)^{2} + \left(w_{E^{\varepsilon}}^{\varepsilon} \left(b, \tau \right) - \chi_{b}^{\varepsilon} \left(\tau \right) \right)^{2} \right] d\tau + \lambda \int_{0}^{1} \left(E^{\varepsilon} \left(\tau \right) \right)^{2} d\tau,$$

$$\tag{19}$$

where $\chi_a^{\varepsilon}(\tau) = w^{\varepsilon}(a,\tau) = \chi_a(\tau) - w^0(a,\tau), \quad \chi_b^{\varepsilon}(\tau) = w^{\varepsilon}(b,\tau) = \chi_b(\tau) - w^0(b,\tau).$ To minimize functional (10) we shall use Pitz method, representing:

To minimize functional (19) we shall use Ritz method, representing:

$$E^{\varepsilon}\left(\xi\right) = \sum_{i=0}^{n} d_i \xi^i.$$
(20)

To solve the minimization problem we now have to determine the set of coefficients $\{d_0, d_1, ..., d_n\}$ of the (20), which will minimize the function of the n+1 variables:

$$\Phi l^{n}_{\lambda}(d_{0},d_{1},...,d_{n}) = \lambda \int_{0}^{1} \left(\sum_{i=0}^{n} d_{i}\tau^{i}\right)^{2} d\tau - \frac{\int_{0}^{1} \left[\left(\sum_{i=0}^{n} d_{i}I_{i}(a,\tau) - \chi^{\varepsilon}_{a}(\tau)\right)^{2} + \left(\sum_{i=0}^{n} d_{i}I_{i}(b,\tau) - \chi^{\varepsilon}_{b}(\tau)\right)^{2}\right] d\tau}{2\sqrt{\rho E^{0}}}, (21)$$

$$I_{i}(\xi,\tau) = \int_{0}^{\tau} \left[\left(\xi + \frac{\tau - z}{k}\right)^{i} \frac{\partial w^{0}}{\partial \xi} \left(\xi + \frac{\tau - z}{k}, z\right) - \left(\xi - \frac{\tau - z}{k}\right)^{i} \frac{\partial w^{0}}{\partial \xi} \left(\xi - \frac{\tau - z}{k}, z\right)\right] dz.$$

To determine $\{d_0, d_1, ..., d_n\}$ we shall find extrema of the function (21). For this purpose we differentiate $\Phi l_{\lambda}^n(d_0, d_1, ..., d_n)$ in respect to each of it's variables $d_0, d_1, ..., d_n$ and, having equated to zero, we shall receive the system of n+1 equations and n+1 variables:

$$\begin{cases} \frac{\partial \Phi I_{\lambda}^{n}}{\partial d_{j}} = 0, j = \overline{0, n}. \end{cases}$$
(22)

Each equation of the system will be of the following form:

$$\frac{\partial \Phi I_{\lambda}^{\alpha}}{\partial d_{j}} = R_{j} + \sum_{i=0}^{n} \theta_{ji} d_{i} = 0,$$

$$\theta_{ji} = \frac{2\lambda}{i+j+1} + \frac{1}{2\rho E^{0}} \int_{0}^{1} (I_{i}(a,\tau)I_{j}(a,\tau) + I_{i}(b,\tau)I_{j}(b,\tau)) d\tau,$$

$$R_{j} = -\frac{1}{\sqrt{\rho E^{0}}} \int_{0}^{1} (I_{j}(a,\tau)\chi_{a}^{\varepsilon}(\tau) + I_{j}(b,\tau)\chi_{b}^{\varepsilon}(\tau)) d\tau.$$
(23)

As can be seen from (23), the system (22) is linear. Solving the system (22) and substituting in (20), we shall obtain the solution $E^{\varepsilon}(\xi)$ of the inverse problem. Because system (22) is linear, it has exactly one solution if the determinant of the matrix θ is not zero.

4. EXAMPLE OF THE INVERSE PROBLEM SOLUTION

In this section we shall show analytical and numeric solutions of the same inverse problem under the condition of weak heterogeneity.

In order to solve the inverse problem, first we have to determine the results of the measurement (4). In this study we shall obtain these functions as the result of the solution of the direct problem with known heterogeneous beam properties. We shall use the following properties:

$$E(\xi) = \frac{160}{40 + \xi}, \rho = 9.$$
(24)

Usage of the specified properties allows us to solve the problem of determining vibrations of such beam exactly. These properties also correspond to the base homogeneous beam with the properties $E = 4, \rho = 9$. We shall excite the harmonic vibrations in the base homogeneous beam by setting the initial conditions to:

$$\varphi(\xi) = 0, \psi(\xi) = 3\sin \pi \xi. \tag{25}$$

Solving the direct problem, we obtain the vibrations of the weakly heterogeneous beam at the points $a = \frac{1}{2}, b = \frac{1}{3}$. The difference between the vibrations of the base homogeneous and weakly heterogeneous beams are shown in Fig. 1. These functions will be used as the measurement data for the inverse problem.



Figure 1. Charts of the measurement data (difference between vibrations of the homogeneous and corresponding weakly heterogeneous beams): $a - \chi_a^{\varepsilon}(\tau)$; $b - \gamma_a^{\varepsilon}(\tau)$; $c - \chi_b^{\varepsilon}(\tau)$

We shall now solve the inverse problem analytically – i.e. we should find $E^{\varepsilon}(\xi)$ by known initial conditions (25), properties of the base homogeneous beam $(E = 4, \rho = 9)$ and measurement data $\chi_a^{\varepsilon}(\tau), \gamma_a^{\varepsilon}(\tau)$. Carrying out all sequence of actions described in the second section of this study, we find $E^{\varepsilon}(\xi)$ taking into account that $E^{\varepsilon}(0)=0$. The chart of this function is given in Fig. 2. Maximal deviation of the found function from the original function is 0,00532 or 5,32%.



Figure 2. Chart of the function $E^{\varepsilon}(\xi)$, determined from the analytical inverse problem solution (——), original (24) (----)

Now, solving the same problem using the regularization method, setting n=6 and $\lambda = 0$, we obtain:

$$E^{\varepsilon}(\xi) = 0.37\xi^{6} - 0.59\xi^{5} - 0.1\xi^{4} + 0.63\xi^{3} - 0.37\xi^{2} - 0.25\xi - 0.0044.$$

The chart of this function is given in Fig. 3. Maximal deviation of the found function from the original function is 0,00428 or 4,28%.



Figure 3. Chart of the function $E^{\varepsilon}(\xi)$, determined from regularization method inverse problem solution (-----), original (24) (----)

As seen from the data and the charts, the results of both approaches to solution differ insignificantly. However, when solving the inverse problem we often have to deal with the

errors of measurement and also possible influence of unknown factors, which introduce some random noise into the measurement data. In this case the regularization method is more preferable than the analytical solution.

To take the random noise into account, we shall add the random functions to the measurement data:

$$\chi_{a}^{*}(\tau) = \chi_{a}(\tau) + e_{1}(\tau), \chi_{b}^{*}(\tau) = \chi_{b}(\tau) + e_{2}(\tau), \gamma_{a}^{*}(\tau) = \gamma_{a}(\tau) + e_{3}(\tau),$$
(26)

where $e_1(\tau), e_2(\tau), e_3(\tau)$ - the generated random functions not exceeding 10% of the measurement data by absolute value.

Solving the inverse problem with the new measurement data (26) analytically, we obtain the results given in Fig. 4. Maximal deviation of the found function from the original function is 1,8 or 1800%. I.e. the measurement error in 10% results in solution error in 1800%.



Figure 4. Chart of the function $E^{\varepsilon}(\xi)$, determined from the analytical inverse problem solution with the random noise added to measurement data (-----), original (24) (----)

Solving the problem using the regularization method, we obtain the results given in Fig. 5. Maximal deviation of the found function from the original function is 0,0082 or 8,2%. I.e. the measurement error in 10% results in solution error in 8,2%.



Figure 5. Chart of the function $E^{\varepsilon}(\xi)$, determined from regularization method inverse problem solution with random noise added to measurement data (-----), original (24) (-----)

As can be seen from the given results, the solution using the regularization method is much more stable to measurement errors than the analytical solution.

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