

THE ANALYTICAL CALCULATIONS OF THE PARAMETERS OF THE FLUCTUATIONS OF THE BALLISTIC MISSILES

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Abstract

The analytical calculations of natural frequencies and forms of the longitudinal and flexural vibrations of the heterogeneous rods, which simulate ballistic missiles, are proposed. These objects are characteristic by significant drops in the distribution of stiffness and density along the length. The indicated dynamical characteristics knowledge is necessary for the design of the system of control and evaluation of the strength of articles.

1. INTRODUCTION

Traditionally [4] calculation of frequencies and forms of vibration of missiles is produced by the very labor-consuming method of sequential approximations.

It is proposed to use the analytical method, operational and effective for the arbitrary laws of distribution of stiffness and densities, for the purpose of the reduction of time and cost of dynamic calculations at the stage of preliminary design (selection of layout, sizes and materials).

The mathematical model of the dynamics of ballistic missile are received differential equations second and fourth – orders with the variable coefficients and the corresponding boundary and initial conditions, and also different assumptions and limitation. The essence of the proposed method is based on the ideas of the asymptotic phase integral method of Wentzel-Kramers-Brillouin and method of Liouville-Steklov [1-2]. According to the proposed method the calculation of the natural frequencies of oscillation is reduced to the calculation of several ten simplest integrals. The comparison of the executed calculations with the results, obtained by approval traditional method shows the high accuracy of the proposed method (divergence < 2%). Advantages of the method: analyticity, small labor expense, clarity, universality. For these reasons it is recommended for putting into practice of design.

2. CONSTRUCTION OF THE APPROXIMATE SOLUTION EQUATION

The equation of natural bending elastic vibrations of a rod with the any laws of distribution of the Young's module E = E(x), axial moment of inertia J = J(x) and area A = A(x) of cross section and density $\rho = \rho(x)$ along an axis of a rod x looks like

$$\frac{\partial^2}{\partial x^2} \left[E(x)J(x)\frac{\partial^2 w}{\partial x^2} \right] + \rho(x)A(x)\frac{\partial^2 w}{\partial t^2} = 0, \tag{1}$$

where x – axial co-ordinate, t - time, w = w(x, t) - deflection. After entering parameters

$$\xi = \frac{x}{l}, \quad \tau = \frac{t}{l^2} \sqrt{\frac{E_* J_*}{\rho_* A_*}}, \quad \overline{w} = \frac{w}{l}, \quad G = \frac{EJ}{E_* J_*}, \quad S = \frac{\rho A}{\rho_* A_*},$$

where l – length of a rod, E_* , J_* , ρ_* , A_* some meanings of the mechanical (E_* , ρ_*) and geometrical (J_* , A_*) characteristics of a rod, the equation (1) is resulted in a dimensionless kind

$$\frac{\partial^2}{\partial\xi^2} \left[G(\xi) \frac{\partial^2 \overline{w}}{\partial\xi^2} \right] + S(\xi) \frac{\partial^2 \overline{w}}{\partial\tau^2} = 0.$$
⁽²⁾

The solutions of equation (2) should satisfy certain boundary and entry conditions. Further the stationary vibrations are studied, therefore entry conditions do not formulate. The boundary conditions are entered at a conclusion eigenvalue equations.

Believing harmonic vibrations, make separation of variables in the equation (2) (and in boundary conditions) with the help of representation

$$\overline{w} = W(\xi) \exp(i\rho\tau), \tag{3}$$

where $W=W(\xi)$ and p – eigenfunction and circular frequency of vibrations. Substituting (3) in (2) we shall receive the equation

$$[G(\xi)W'']'' - S(\xi)p^2W = 0, \qquad (4)$$

where stroke means differentiation on ξ . For approximate solution of equation (4) we use procedure of matrix variant of a method of phase integrals [2].

Let's present the equation (4) as system of four equations of 1-st order, for what we shall enter unknowns

$$y_1 = W, \quad y_2 = W', \quad y_3 = GW'', \quad y_4 = (GW'')',$$
 (5)

In the matrix form the system of the equations connecting entered unknowns (5), looks like

$$Y' = TY, (6)$$

where Y – column-vector unknowns y_j (j = 1, 2, 3, 4), T – square (4x4) matrix containing coefficients of equation (4)

Let's enter transformation of unknowns (5) kinds

$$Y = UF, \tag{7}$$

where U-square (4 x 4), nondegenerate matrix $U = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \beta & -\beta & i\beta & -i\beta \\ \alpha & \alpha & -\alpha & -\alpha \\ \alpha\beta & -\alpha\beta & -i\alpha\beta & i\alpha\beta \end{pmatrix}$, $\alpha = \frac{S}{\beta^2} p^2$,

$$\beta^4 = \frac{S}{G} p^2 \,.$$

Columns of matrix U are formed by components of latent vectors U_k (k = 1, 2, 3, 4) matrix T, F – column-vector new unknowns f_j (j = 1, 2, 3, 4). Substituting (7) in (6), we shall receive system of the equations concerning functions f_j in matrix

$$F' = U^{-1}TUF - U^{-1}U'F.$$
(8)

Structure of system of the equations (8) differs from structure of initial system of the equations (6) essentially.

Assuming a little of collateral elements of matrix $U^{-1}U'$, that is neglecting interaction of the equations of system (8), we shall split system (8) up into four independent equations for functions f_i

$$f'_j = \varepsilon_j \beta f_j - \frac{1}{2} \left(\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta} \right) f_j .$$

Whence, integrating, we shall receive

$$f_j = (\alpha\beta)^{-\frac{1}{2}} \exp\varepsilon_j \omega(\xi, 0), \qquad (9)$$

where $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, $\varepsilon_3 = i$, $\varepsilon_4 = -i$, $\omega(\xi, 0) = \int_0^{\xi} \beta(s) ds = p^2 \int_0^{\xi} \left(\frac{S}{G}\right)^{\frac{1}{4}} ds$.

Taking into account (7) and (5), shall receive function of deflections as

$$W = q[C_1 \exp \varepsilon_1 \omega(\xi, 0) + C_2 \exp \varepsilon_2 \omega(\xi, 0) + C_3 \exp \varepsilon_3 \omega(\xi, 0) + C_4 \exp \varepsilon_4 \omega(\xi, 0)], \quad (10)$$

where $q = (\alpha \beta)^{-1/2}$.

Thus, we accept functions (9) as approximate partial solutions of the initial equation (4), and their sum (10) – as good-enough general solution.

3. EQUATIONS AND FORMULAS FOR FREQUENCIES

Having the partial solution (9) equations of bending vibrations of a rod (4) consecutive differentiation are possible to receive kinematic (ω, ω') and force (M, Q) factors of a task

$$W(\xi) = \sum_{j=1}^{4} C_{j} W_{j}(\xi), \ W'(\xi) = \sum_{j=1}^{4} C_{j} W_{j}'(\xi), \ M(\xi) = G(\xi) \sum_{j=1}^{4} C_{j} W_{j}''(\xi),$$
$$Q(\xi) = G(\xi) \sum_{j=1}^{4} C_{j} \left(\frac{G'(\xi)}{G(\xi)} W_{j}''(\xi) + W_{j}'''(\xi) \right), \ W_{j} \approx f_{j}.$$
(11)

The standard procedure of satisfaction to boundary conditions results in homogeneous system of the algebraic equations concerning constant C_i (j = 1, 2, 3, 4). Existence condition of non-trivial solution for this system gives the eigenfrequencies. The non-trivial solution defines the form eigenfunctions.

By us is shown [3], that for various types of support can be used asymptotic representation of the equations frequency of a kind

$$\Pi_0(p^{1/2}) + \frac{1}{p^{1/2}} \Pi_1(p^{1/2}) + O(p^{-1}) = 0, \quad p \to \infty.$$
(12)

Let's result some private kinds of functions $\Pi_0 = \Pi_0(p^{1/2}), \ \Pi_1 = \Pi_1(p^{1/2})$:

1. The cantilever
$$(W(0) = W'(0) = M(1) = Q(1) = 0)$$
,
 $\Pi_0 = 1 + \cos \omega(1,0)ch\omega(1,0), \ \Pi_1 = \frac{\gamma_4(1)}{\gamma_3(1)}(\sin \omega(1,0)ch\omega(1,0) + sh\omega(1,0)\cos \omega(1,0)),$
where $\gamma_4 = \gamma_2(\xi)\beta(\xi) + \gamma_1'(\xi), \ \gamma_3 = \gamma_1(\xi)\left(\frac{S(\xi)}{G(\xi)}\right)^{\frac{1}{4}}, \ \gamma_2 = q'(\xi), \ \gamma_1 = q(\xi)\left(\frac{S(\xi)}{G(\xi)}\right)^{\frac{1}{4}},$

$$q(\xi) = \left(S^{3}(\xi)G(\xi)\right)^{-\frac{1}{8}}.$$

2. The free ends $(M(0) = \varphi(0) = M(1) = Q(1) = 0)$
 $\Pi_{0} = 1 - \cos \omega(1, 0)ch\omega(1, 0),$
 $\Pi_{1} = \left(\frac{\gamma_{4}(0)}{\gamma_{3}(0)} - \frac{\gamma_{4}(1)}{\gamma_{3}(1)}\right)(\sin \omega(1, 0)ch\omega(1, 0) + sh\omega(1, 0)\cos \omega(1, 0)).$

The similar correlations for hinge and restrained rods are received.

For account of own frequencies is applicable to the equation (12) general theorem about nulls of function with known asymptotic representation [3]: if the functions $\Pi_0(p^{1/2})$ $\Pi_1(p^{1/2})$ и $\Pi'_1(p^{1/2})$ are bounded $|\Pi_0(p^{1/2})| + |\Pi'_0(p^{1/2})| \ge r \ge 0$, (r-some constant) and nulls of function $\Pi_0(p^{1/2})$ at $p_0^{1/2} \le p^{1/2} \le \infty$ form an infinite sequence, nulls of function

$$\Pi = \Pi_0 \left(p^{1/2} \right) + \frac{1}{p^{1/2}} \Pi_1 \left(p^{1/2} \right) + O\left(p^{-1} \right)$$

also form an infinite sequence

$$(p^{1/2})_{v}^{*} < (p^{1/2})_{v+1}^{*} < (p^{1/2})_{v+2}^{*} < \dots,$$

and

$$\left(P^{1/2}\right)_{n}^{*} = P_{n}^{1/2} - \frac{\Pi_{1}\left(P_{n}^{1/2}\right)}{P_{n}^{1/2}\Pi_{1j}\left(P_{n}^{1/2}\right)} + O\left(P_{n}^{-1}\right), \tag{13}$$

The application to the equation (12) theorem of nulls of function with known asymptotic representation gives uniform asymptotic expression for frequencies for several types support of a kind

$$P_{nj}^{1/2} = \lambda_{nj} \left(\frac{1}{B} + \frac{1}{\lambda_{nj}^2} \left(A_j V_{nj} + C_j \right) \right) + O\left(\lambda_{nj}^{-2}\right), \tag{14}$$

where *n* – number of frequency, $B = \int_{0}^{1} (SG^{-1})^{1/4} d\xi$; A_j, V_{nj}, C_j – known functions dependent on a type support, λ_{nj} – frequencies of null approach determined from the appropriate equations of a classical kind (for example, for the cantilever $\cos \lambda_2 ch \lambda_2 - 1 = 0$).

The index j characterizes a type support: 1 – restrained, 2 – cantilever, 3 – hinges on the ends, 4 – free ends. The asymptotic representation for the forms of own vibrations in a case of bending vibrations looks like

$$W_{nj} = q(\xi) \left(W_{nj}^* + \frac{1}{P_{nj}^{1/2}} \left(A_{nj} ch \omega_{nj}(\xi) + B_{nj} sh \omega_{nj}(\xi) + C_{nj} \cos \omega_{nj}(\xi) + D_{nj} \sin \omega_{nj}(\xi) \right) \right) + O(P_{nj}^{-1}),$$

where $W_{nj}^* = W_{nj}^*(\xi)$ – form of own vibrations of the appropriate homogeneous rods with frequencies $P_{nj}^{1/2}$, that is WKB – approach, $A_{nj}, B_{nj}, C_{nj}, D_{nj}$ – constant, determined by a kind support of a rod.

4. NUMERICAL RESULTS

4.1. Longitudinal Vibrations

For the accuracy estimation of the offered technique for the calculation of eigen forms and vibrations frequencies of rods with a variable rigidity and density we'll calculate the main frequency of longitudinal vibrations of the bar with the staged change of the rigidity and density along the axis.

Distribution diagrams in Figure 1 [4]. characterize rigidity distribution for stretchingpressure \overline{EF} and a mass \overline{m} ballistic missile and taken from the work [4], where it is shown a

value of the main frequency for longitudinal vibrations $p_1 = 0.2811 \sqrt{\frac{\overline{E}\overline{F_0}}{\overline{m}_0}} \frac{1}{\sec}$, found with the

method of consistent approximations, usually used in designing calculations. It is shown here a calculation technique basic results of which are in the table 19.1 [4].

In short, the calculation algorithm of frequencies for own vibrations according to the method of sequent approximations consists of the following actions.

One presets the main form of eigen vibrations for homogeneous bar simulating a missile of homogeneous bar with free ends f(x); one divides a bar into N parts of equal length Δ and with the aid of tables taking into account boundary conditions one formulates function $f_1(x)$, considered further as initial for the second approximation and so on; one calculates the main frequency of eigen vibrations. At the calculation of the second and next forms and frequencies one uses a property of form orthogonality. In columns (19), (20), and (21) of the table 19.1 [4] the values of the first, the second and the third approximations of the main form of vibrations are shown accordingly. We'll present a formula for the calculation of the main frequency and a concrete result, corresponding to the distribution diagram in Figure 1.



Figure 1. Distribution of rigidity for stretching-pressure \overline{EF} and the mass \overline{m} along length of the ballistic missile

$$p_{1} = \sqrt{\frac{EF_{0}\sum_{0}^{N}\overline{m}f_{1}^{2}}{m_{0}\Delta_{1}^{2}\sum_{0}^{N}\frac{N_{1x}^{2}}{\overline{EF}}}} = \sqrt{\frac{31,176}{1579,19\cdot0.5^{2}}\frac{\overline{EF_{0}}}{\overline{m_{0}}}} = 0,2811\sqrt{\frac{\overline{EF_{0}}}{\overline{m_{0}}}}\frac{1}{sec}, \ \Delta_{1} = \frac{\Delta}{2}, \ \Delta = 1m, \ l = 10m, \ N = 10.$$

$$(15)$$

The solution of this problem is based on correlations for equations of the second order, analogous shown in this paper, describing longitudinal fluctuations, gained by S.N. Ovsyannikova [5], and has a kind for a dimensionless frequency.

$$\overline{p_1} = \frac{\pi}{\int\limits_0^1 \sqrt{\frac{S}{G} d\xi}} - \frac{1}{16\pi} \int\limits_0^1 \sqrt{\frac{G(\xi)}{S(\xi)}} \left(\left(\frac{G'(\xi)}{G(\xi)} - \frac{S'(\xi)}{S(\xi)} \right)^2 - 4 \left(\frac{S'(\xi)}{S(\xi)} \right)^2 \right) d\xi$$

Further one uses data of the distribution diagrams on Figure 1, the integration interval is divided into parts with the constant laws of change \overline{EF} and \overline{m} within every part. After this the problem is reduced to the calculation of twenty simplest integrals.

$$\overline{p_1} = \pi \left(\int_{0}^{0,1} \sqrt{\frac{2}{4}} d\xi + \dots\right)^{-1} - \frac{1}{16\pi} \left(\int_{0,4}^{0,45} \left(\frac{-20}{20(\xi - 0,4) + 2} - \frac{20}{20(\xi - 0,4) + 2}\right)^2 + \dots\right).$$

After dimension parameters introduction it is gained

$$p_1 = 0.29 \sqrt{\frac{EF_0}{m_0}} \frac{1}{\sec}$$
 (16)

Comparison of results (15) and (16) depicts, that the difference of the values determined does not exceed 3%.

4.2. Bending vibrations

We'll perform the analogous comparison for the case of bending vibrations of the bar, modeling missile with the length 15 *M* and rigidity \overline{EJ} and mass characteristics \overline{m} , shown in





Figure 2. Distribution of bending rigidity \overline{EJ} and the mass \overline{m} along length of missile

The usual calculation according to the method of sequent approximations is fulfilled through the analogous algorithm and for the case, corresponding to distributions on Figure 2, in the third approximation gives the main frequency of bending vibrations [4]

$$\omega_1 = 0.1136 \sqrt{\frac{EJ}{m}} \frac{1}{\text{sec}} \,. \tag{17}$$

In our case for the calculation of bending vibrations eigen frequencies it is convenient to use the unified asymptotic formula for the parameter of the eigen frequency of heterogeneous bar (14), where λ_{nj} – parameter of eigen frequency of the corresponding homogeneous bar

(for example,
$$\lambda_{11}=4.73$$
, $\lambda_{12}=7.86$, ..., $\lambda_{13}=\lambda_{11}$, $\lambda_{23}=\lambda_{21}$, ...); $\lambda_{nj}\frac{1}{B} = \left(\int_{0}^{1} \left(\frac{S}{G}\right)^{\frac{1}{4}} d\xi\right)^{-1} - \frac{1}{2}\left(\frac{S}{G}\right)^{\frac{1}{4}} d\xi$

parameter of eigen frequency, corresponding approximation WKB.

Functions A_j, V_{nj}, C_j are determined by use of dependences (11), (10) and have a kind in particular

$$A_{3} = \frac{1}{2l} \left(\left(\frac{G(1)}{S(1)} \right)^{\frac{1}{4}} \left(\frac{S'(1)}{S(1)} + \frac{G'(1)}{G(1)} \right) - \left(\frac{G(0)}{S(0)} \right)^{\frac{1}{4}} \left(\frac{S'(0)}{S(0)} + \frac{G'(0)}{G(0)} \right) \right)$$
$$C_{3} = 0, \ V_{13} = \frac{ch\lambda_{13}\sin\lambda_{13} + sh\lambda_{13}\cos\lambda_{13}}{sh\lambda_{13}\cos\lambda_{13} - ch\lambda_{13}\sin\lambda_{13}}.$$

We'll show values of the calculation parameters for the example under consideration in Figure 2

$$V_{13} = \frac{ch4,73\sin4,73 + sh4,73\cos4,73}{sh4,73\cos4,73 - ch4,73\sin4,73} = \frac{56,65(-0,9998) + 56,643 \cdot 0,0176}{56,643 \cdot 0,0176 - 56,65(-0,9998)} = -0,965,$$

$$G(0) = 4$$
, $S(0) = 1$; $G'(0) = 10.8$, $S'(0) = 20$; $G'(1) = S'(1) = 0$.

$$A_3 = -\frac{1}{2 \cdot 15} \left(\frac{G(0)}{S(0)} \right)^{\frac{1}{4}} \left(\frac{S'(0)}{S(0)} + \frac{G'(0)}{G(0)} \right) = -\frac{1}{30} \left(\frac{4}{1} \right)^{\frac{1}{4}} \left(\frac{20}{1} + \frac{10,8}{4} \right) = -1,07 .$$

Further

$$\frac{1}{B} = \left(\int_{0}^{1} \left(\frac{S}{G}\right)^{\frac{1}{4}} d\xi\right)^{-1} = \left(\int_{0}^{0,1} \left(\frac{20\xi + 1}{10,8\xi + 4}\right)^{\frac{1}{4}} d\xi + \int_{0,1}^{0,185} \left(\frac{1}{10,8\xi + 4}\right)^{\frac{1}{4}} d\xi + \int_{0,185}^{0,26} \left(\frac{1}{2,2}\right)^{\frac{1}{4}} d\xi + \dots\right)^{-1} = 1.019$$

Then from the formula (14) and taking into account the first two items we'll gain a parameter of the main frequency

$$p_1 = \frac{p_{13}}{l^2} \sqrt{\frac{\overline{EJ_0}}{\overline{m_0}}} = \left(\frac{\frac{\lambda_{11}}{B} + \frac{A_3 V_{13}}{\lambda_{11}}}{l}\right)^2 \sqrt{\frac{\overline{EJ_0}}{\overline{m_0}}} = 0.1128 \sqrt{\frac{\overline{EJ_0}}{\overline{m_0}}} \frac{1}{\text{sec}}.$$
 (18)

Comparing formulae (17) and (18) we see that they practically coincide (the error ≈0,7%). The examples shown depict the efficiency of the technique offered, high accuracy and its advantage regarding usual difficult methods [4].

5. CONCLUSION

The received results of a method are based on ideas asymptotical method of phase integrals WKB and method Liouville-Steklov. The offered method with the specified classical methods unites the item of information of tasks to systems of the integral equations with their subsequent solution by iterations and representation of the solution through resolventa.

Advantage of a method is that fact, that for its application it is not required, that the coefficients of the initial equation poorly should differ from constant, there is enough, that they were bounded. However, the method is especially effective, if the coefficients of the initial equation really describe weak heterogeneity.

Offered approach to the search of the approximate solutions of the differential equation with variable coefficients, when the exact solution is not found, is rational. There are exact solutions of the initial problem, that was specifically changed and this change is easier to explain, than any approximation made during numerical or any other approximate solution.

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