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STOCHASTIC AND CHAOTIC ANALYSIS OF SUBHARMONIC RESPONSE OF A SHALLOW CABLE

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Abstract

The paper deals with the subharmonic response of a shallow cable due to random chord length variations, caused by time varying motions of the support points of the cable. Under deterministic harmonic support point motions the stable subharmonic motion of order 2 consists of a harmonically varying component in the equilibrium plane and a large subharmonic out-of-plane component with a fixed phase lag producing a trajectory of the midpoint with a shape like the symbol used for infinity. A more realistic excitation is obtained by replacing the harmonic chord length variation by a narrow-banded process with the same variance and centre frequency. In this case a very different response pattern is observed even for a very small band width of the excitation process. The phase between the in- and out-of-plane displacements is no longer locked at a fixed value, causing the trajectory to rotate slowly around the chord line. As a consequence a substantial in-plane subharmonic response component is brought forward. Further, the time-varying amplitudes of the elongation variations tend to enhance chaotic behaviour of the response, which is detectable via extreme sensitivity on the initial conditions or via the sign of a numerical calculated Lyapunov exponent. The dependence of these findings on the specific stochastic modelling is investigated by analysing two chord elongation processes with almost identical auto-spectral densities, i.e. the statistical second moment properties of the processes are almost identical, whereas higher order moments differ significantly. In one case, the chord elongation is modelled as a filtration of a Gaussian white noise through a linear second order differential filter. The Gaussian output process has realizations with slowly varying phases and amplitudes slowly varying around the amplitude of the comparable harmonic excitation. In the other case the excitation process is modelled by a harmonic zero time-lag transformation of a Wiener process. All realizations have constant amplitudes equal to the purely harmonic excitation, but with a slowly varying phases. The two stochastic models provide qualitatively and quantitatively identical results. The conclusion is that the chaotic response caused by stochastic excitation mainly is due to time variation of the phases, whereas the amplitude variations are of minor importance.

1. INTRODUCTION

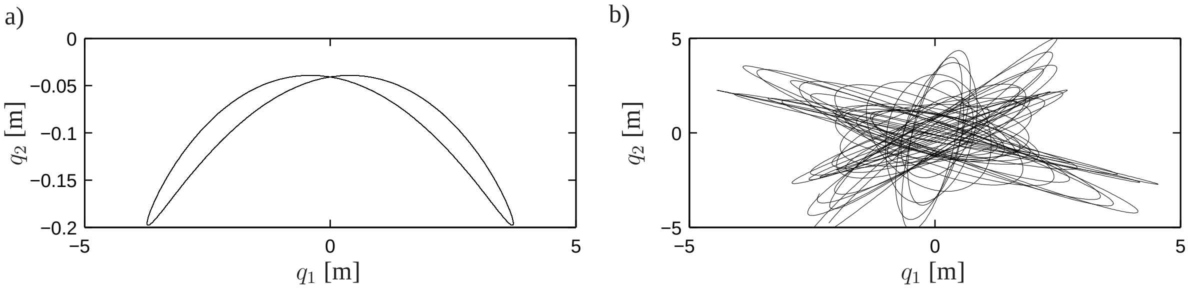

Figure 1. Trajectory at the midpoint of the cable. (a) Harmonic chord elongation. (b) Stochastic chord elongation, unstable response.

Cable systems are of great interest in a wide range of applications in civil engineering to supply both support and stability to large structures. Typically, cables used as support of cable-stayed bridges, masts and TV-towers are characterized by a sag-to-chord-length ratio below say 0.01, which means that the natural frequencies for the in-plane eigenvibrations $\omega_2, \omega_4, \ldots$, and the out-of-plane eigenvibrations $\omega_1, \omega_3, \ldots$ are pairwise close. The primarily external excitation of such cables is caused by the motion of the support points of the cable rather than by external distributed dynamic wind or aeroelastic loads. Especially, the component of the support point motion along the chord of the equilibrium suspension introduces both additive and parametric excitation terms in the nonlinear modal equations of motion due to the elongation and shortening of the chord length.

Dangerous situations arise when the chord elongation is harmonically varying with a circular frequency ω_0 in certain disjoint intervals. Especially, when ω_0 is about twice the fundamental out-of-plane circular eigenfrequency ω_1, large subharmonic vibrations with the circular frequency $\omega_0/2$ may take place. It turns out that the single mode in-plane subharmonic of the order 2 is unstable for arbitrarily small excitation amplitudes. Instead a coupled vibration occurs, in which large subharmonic vibrations out of the static equilibrium plane take place with the circular frequency $\omega_0/2$, whereas the in-plane vibrations are harmonically varying with the circular frequency ω_0, and with a relatively small amplitude. The out-of-plane displacement is brought forward by nonlinear couplings, and has a well-defined phase leading to the in-plane harmonic component. The indicated phase locking between the two vibration components produces a trajectory of shape like an infinity sign as shown in Figure 1a.

In reality the supported structure is performing narrow-banded stochastic vibrations with a centre frequency ω_0 close to ω_1. Hence, the chord elongation will also be narrow-banded stochastic varying. The resulting subharmonic response is qualitatively very different from the comparable harmonic excitation as shown in Figure 1b. In this case the phase between the two displacement components is no longer locked at a certain value but becomes slowly varying with time. This phase variation causes the trajectory of the response to rotate slowly around the chord line, introducing a large subharmonic response component also in the static equilibrium plane. This phenomena was investigated by Zhou et al. [1] based on extensive Monte Carlo simulations, modelling the stochastic variation of the chord elongation as a filtered white noise process. It was concluded that the unlocking of the phases, and hence the rotation of the trajectory, was caused by the slowly varying amplitudes of the narrow-banded excitation process. Another finding was that the stochastic variations of the chord elongation enhanced the tendency to chaotic response relative to the comparable harmonic excitation. Still, it is open whether these findings are entirely caused by the variation of the amplitudes of the chord elongation process, or similar effects may occur due to the simultaneous slowly variation of the phase of the excitation process. In the present paper this is investigated by comparing with an alternative Monte Carlo simulation approach due to Griesbaum [2]. All realizations of the underlying stochastic process have constant amplitude equal to the referential harmonic excitation, and the randomness is completely caused by a slowly varying phase, which is modelled as a Wiener process.

2. THEORY

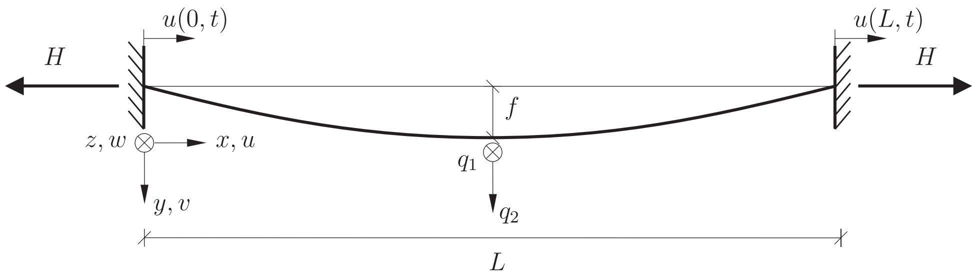

Figure 2. Cable in static equilibrium configuration

Figure 2 shows a cable in the static equilibrium state with the chord length L and the sag f. Although depicted in a horizontal position the cable should be thought of as inclined with the chord making an angle θ with a horizontal line. The sag is caused by the component $g\cos\theta$ of the acceleration of gravity in the orthogonal direction of the chord. The plane equilibrium state is maintained by a prestress force H along the chord line. The sag-to-chord-length ratio f/L is assumed to be sufficiently small that a parabolic approximation may be used for the suspension. The dynamic displacement components $u(x,t), v(x,t), w(x,t)$ of a material point of the cable along the axes of the indicated (x,y,z)-coordinate system are caused by the chord elongation $u(L,t)-u(0,t)$ induced by the motion of the two support points. Conveniently, the chord elongation may be described by the following non-dimensional parameter

$$e(t) = \frac{EA}{H}\frac{u(L,t)-u(0,t)}{L} \tag{1}$$

where E is the elasticity modulus and A is the cross-sectional area. The in-plane and out-of-plane displacement components $v(x,t)$ and $w(x,t)$ are dominated by the fundamental eigenmodes. The second in-plane and second out-of-plane modes are not exposed by the chord elongation, because the corresponding modes are antisymmetric. Then, the following single mode expansions of the in-plane and out-of-plane displacements turns out to be appropriate

$$v(x,t) \simeq \Phi_2(x)q_2(t) \quad , \quad w(x,t) \simeq \Phi_1(x)q_1(t) \tag{2}$$

The eigenmodes $\Phi_1(x)$ and $\Phi_2(x)$ are normalized to 1 at the midpoint, so that the corresponding modal coordinates $q_1(t)$ and $q_2(t)$ are measures of the actual displacements at this position. The related out-of-plane circular eigenfrequency ω_1 is slightly smaller than the in-plane eigenfrequency ω_2 due to effect of the suspension. Retaining up to cubic non-linear terms in the equations of motion the following highly reduced 2 degrees-of-freedom system may be formulated, Zhou et al. [1]

$$\left.\begin{aligned} &\ddot{q}_1 + 2\zeta_1\omega_1\dot{q}_1 + \omega_1^2\big(1+e(t)\big)q_1 + \beta_1 q_1 q_2 + q_1\big(\gamma_1 q_1^2 + \gamma_2 q_2^2\big) = 0 \\ &\ddot{q}_2 + 2\zeta_2\omega_2\dot{q}_2 + \omega_2^2\big(1+\alpha e(t)\big)q_2 + \beta_2 q_1^2 + \beta_3 q_2^2 + q_2\big(\gamma_3 q_1^2 + \gamma_4 q_2^2\big) = -\eta e(t) \end{aligned}\right\} \tag{3}$$

ζ_1 and ζ_2 are the modal damping ratios, and $\alpha,\ \beta_1,\ \beta_2,\ \beta_3,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \eta$ are all non-dimensional parameters of magnitude 1, which depends on the eigenmodes $\Phi_1(x)$ and $\Phi_2(x)$. As seen the chord elongation $e(t)$ is exposing the system both to external and parametric excitation. Based on simulations with a full non-linear finite difference model it was demonstrated by Zhou et al. [1] that the indicated two degree-of-freedom model was adequate in predicting qualitatively and quantitatively as well as the dynamic response and stability as the chaotic response of the cable.

The referential harmonic varying chord elongation is given as

$$e(t) = e_0 \cos(\omega_0 t + W_0) \tag{4}$$

where ω_0 denotes the circular frequency, e_0 is a non-dimensional amplitude of magnitude 1, and W_0 is a constant deterministic phase. The corresponding chord elongation obtained by the filtration of a unit intensity white noise through a second order filter is given

$$\ddot{e} + 2\mu\dot{e}\omega_0 + \omega_0^2 e = \sqrt{2\mu\omega_0^3}\, e_0 w(t) \tag{5}$$

where μ is the band width (damping ratio) of the filter, and $w(\tau)$ is a zero mean unit intensity Gaussian white noise with the auto-covariance function

$$\kappa_{ww}(\tau) = E[w(t)w(t+\tau)] = \delta(\tau) \tag{6}$$

The auto-covariance function of the out-put process becomes, Lin [3]

$$\kappa_{ee}(\tau) = \frac{1}{2}e_0^2\, \mathrm{e}^{-\mu\omega_0|\tau|}\left(\cos(\omega_d\tau) + \frac{\mu}{\sqrt{1-\mu^2}}\sin\big(\omega_d|\tau|\big)\right) \quad , \quad \omega_d = \omega_0\sqrt{1-\mu^2} \tag{7}$$

Alternatively, the chord elongation may be modelled as a harmonic zero-time lag transformation of a Wiener process (the integral of a white noise process) modelled by Griesbaum [2] where ω_d signifies the damped circular eigenfrequency of the filter,

$$e(t) = e_0 \cos\big(\omega_0 t + W(t)\big) \tag{8}$$

$$W(t) = \sqrt{2\mu\omega_0}\int_0^t w(\tau)d\tau \tag{9}$$

After a transient phase the auto-covariance function of (8) can be shown to approach the stationary value

$$\kappa_{ee}(\tau) = \frac{1}{2} e_0^2 \, \mathrm{e}^{-\mu\omega_0|\tau|} \cos(\omega_0 \tau) \tag{10}$$

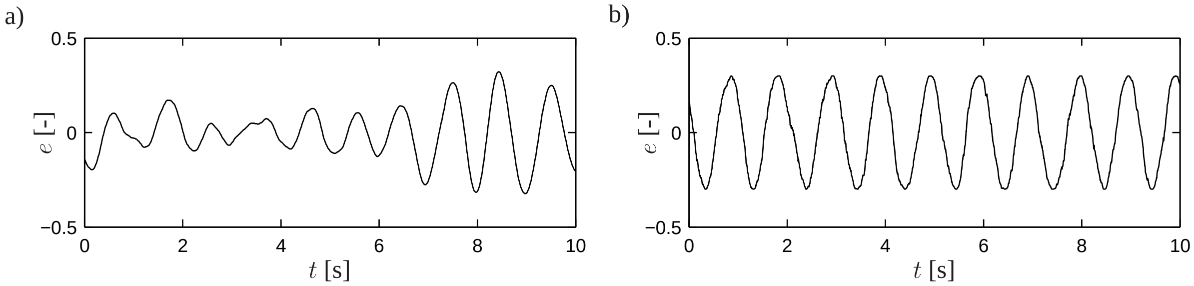

Figure 3. (a) Filtered white noise process. (b) Varying phase process. $\mu = 0.01, e_0 = 0.3$.

The models (4), (5) and (8) all have zero mean, variance $\frac{1}{2}\,e_0^2$, and the dominating circular frequency ω_0. The amplitudes of (4) and (8) are both constant and equal to e_0, whereas the amplitude of (5) is slowly varying around e_0.

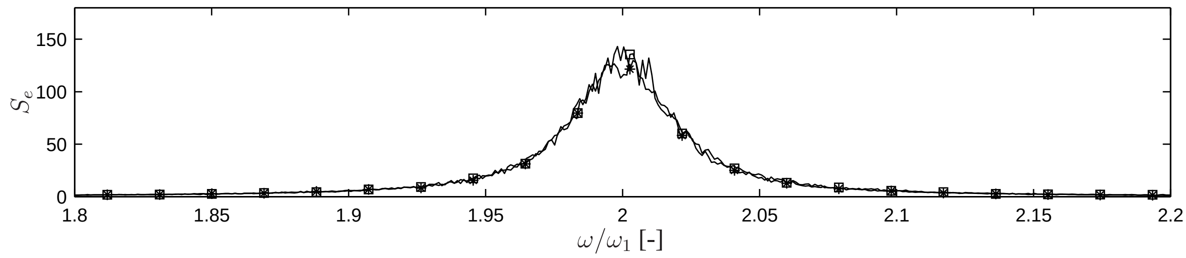

Figure 4. One-sided auto-spectral density function for the two stochastic chord elongation processes. (*) Filtered white-noise process (□) Varying phase process. $\mu = 0.1, e_0 = 0.3$.

Despite the apparently completely different realizations of (5) and (8) as shown on Figures 3a and 3b the corresponding auto-covariance functions (7) and (8) are identical within an error of magnitude μ. This has been illustrated in Figure 4 by a numerical calculation of the corresponding auto-spectral densities based on obtained realization of the processes.

3. NUMERICAL EXAMPLE

Figure 5 shows the variation of the variances $E[q_1^2]$ and $E[q_1^2]$ of the out-of-plane and in-plane modal coordinates as a function of the band width parameter μ. The amplitude parameter is fixed at the value $e_0 = 0.3$, and the central circular frequency is $\omega_0 = 2\omega_1$. As seen $E[q_1^2]$ decreases, and $E[q_1^2]$ increases with μ. Especially, the out-of-plane response ceases for $\mu > 0.11$. Hence, a bifurcation in the stochastic response takes place at this point. Within the sampling error of the monte Carlo simulations the results produced by the two models for the chord elongation are quite identical.

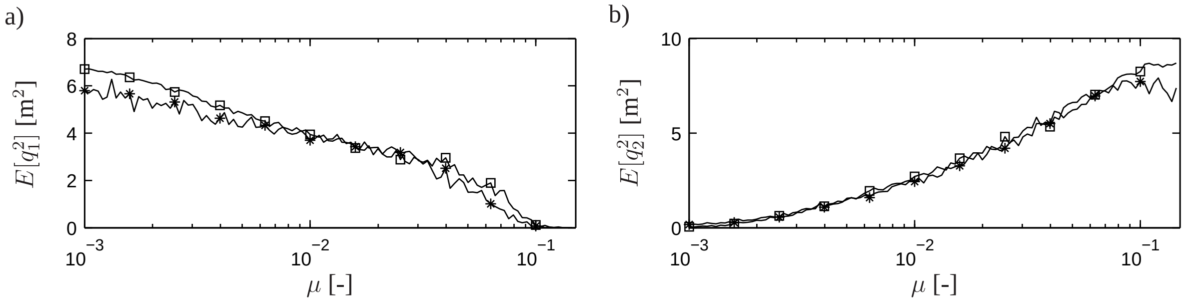

Figure 5. Varians variation with μ. (*) Filtered white-noise process (☐) Varying phase process. $e_0 = 0.3$, $\omega_0 = 2\,\omega_1$.

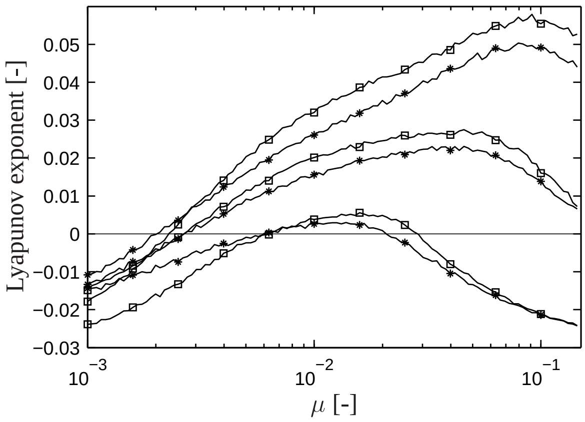

Figure 6. Lyapunov exponent variation with μ, , $\omega_0 = 2\,\omega_1$. (*) Filtered white-noise process (☐) Varying phase process. (bottom) $e_0 = 0.1$. (middle) $e_0 = 0.2$. (top) $e_0 = 0.3$.

Chaotic behaviour is characterized by exponential growth between two neighboring dynamic states exposed to the same external excitation. This can be analysed by the sign of the so-called Lyapunov exponents, where a positive value indicates exponential growth of the distance between the two states, whereas a negative value indicates exponential approach of the states with time. Hence, predictability, is related with negative Lyapunov exponents, whereas positive Lyapunov exponents indicate chaotic behaviour. Numerically, the Lyapunov exponent may be sampled by the algorithm of Wolf et al. [4]. The designation "stochastic chaos" means that the response behaves chaotic for almost all realizations of the response (chaotic behaviour with probability 1). The chaotic behaviour of the two chord elongation processes have been compared by comparing the Lyapunov exponent of the responses calculated by exactly the same realization of the underlying unit intensity white noise process $w(\tau)$. Figure 6 shows the estimated Lyapunov exponents as a function of μ and discrete values of the amplitude parameter e_0. (*) indicates results obtained from the filtered white noise, (☐) those obtained from the varying phase process. As seen, no noticeable differences between these curves are obtained. The results for the referential harmonic chord elongation process is obtained in the limit $\mu \rightarrow 0$. As seen, these are all negative. Hence, the response is predictable at least for the non-dimensional chord elongation amplitude $e_0 \leq 0.3$. By contrast, the stochastic response looses predictability at a critical value of μ even for the relative small amplitude value $e_0 = 0.1$. If e_0 is not too large predictability is eventually recovered at sufficiently large values of the band width parameter.

4. CONCLUSIONS

The stochastic response and chaotic behaviour of a shallow cable have been analysed by two comparable stochastic models for the chord elongation. One model the chord elongation process is obtained by linear filtration of Gaussian white noise through a second order filter. The other model is based on a zero-time lag harmonic transformation of a Wiener process. The processes have zero mean value, and almost identical auto-covariance functions. The former process has both slowly varying amplitudes and phases, whereas the amplitudes of the latter is constant, and only the phases are varying. Based on Monte Carlo simulations with the two models almost identical results were obtained for the variance response of the modal coordinates. Neither, did the Lyapunov exponents obtained by numerical sampling of the response, using the same input white noise in the two models, show much difference. From this it is concluded that the completely different subharmonic response compared to that of the referential harmonic excitation, is primarily caused by the phase variation. Since, the two stochastic chord elongation models have completely different higher order statistical moments, it is further concluded that the indicated phenomena is embedded in the second order statistical moments, i.e. the auto-variance function or auto-spectral density function.

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