

AN ACOUSTIC FINITE ELEMENT INCLUDING VISCOTHERMAL EFFECTS

M.J.J. Nijhof, Y.H. Wijnant and A. de Boer

Chair of Structural Dynamics and Acoustics, University of Twente P.O. box 217, 7500AE Enschede, the Netherlands <u>m.j.j.nijhof@ctw.utwente.nl</u>

Abstract

In acoustics it is generally assumed that viscous- en thermal boundary layer effects play a minor role in the propagation of sound waves. Hence, these effects are neglected in the basic set of equations describing the sound field. However, for geometries that include small confinements of air or thin air layers, this assumption is not valid. Special models that include viscous and thermal effects are available (for example the Low Reduced Frequency model) but only for a limited number of geometries. To overcome these limitations and provide a solution that can be used for arbitrary geometries, an acoustic finite (2D) element that includes viscous and thermal effects is developed. The model is based on the linearized Navier stokes equations (including shear), the equation of continuity, the equation of state for an ideal gas and the energy equation. The method of weighed residuals is used in combination with a mixed formulation of pressure, temperature and particle velocity degrees of freedom. The results of the developed element code are compared with the results of an existing (analytical) Low Reduced Frequency solution and a viscothermal element that was found in literature.

1. INTRODUCTION

The propagation of sound waves with viscothermal effects has been investigated extensively. Depending on the assumptions made regarding compressibility and thermal effects, the models in existence can be categorized into three major groups; full linearized Navier Stokes models, simplified Navier Stokes models and the Low Reduced Frequency (LRF) model. The different categories were put in perspective by Beltman [1]. The Low Reduced Frequency model proved to be most efficient in the majority of cases.

The motivation for simplification of the full set of equations is the fact that viscosity and heat conduction only have significant effects in thin boundary layers near boundaries. This makes it possible to split the solution into a part that describes the propagation of waves inside the medium and a part that describes the boundary layer effects. However, suitable boundary conditions that support this approach are only available for simple geometries. Therefore, these simplified models can not be used for wave propagation in complex geometries or geometries for which it is not trivial to define a coordinate system for which there is a significant difference in length scales. An example of such a 'complex' 2D geometry is given in figure 1. Assume we are interested in the frequency range for which L_b , the scale on which viscous and thermal boundary layer effects occur, is $L_h < L_b < < L_w$, V_w , V_h , where L_h , L_w , V_h and V_w are the length scales of the geometry depicted in figure 1.

In this case, the left part of the geometry can be considered as a thin air layer. This means that the viscous and thermal boundary layer will become large compared to length scale in the thickness direction. The model that describes the wave propagation in this part of the geometry must therefore include viscothermal effects.

The right part of the geometry has length scales which can be considered large compared to the viscous and thermal boundary layer. Therefore, there is no need to include viscothermal effects in the model that describes this part of the system and a standard acoustic model is sufficient.

The middle part of the geometry has both length scales that are small and large compared to the viscous and thermal boundary layers, so viscothermal effects can not be neglected. However, there is no suitable set of boundary conditions that supports a simplified Navier Stokes model. This means a model based on the full linearized Navier Stokes equations is necessary to properly describe wave propagation in this part of the geometry.



Figure 1 Geometry including different length scales

In case of complex 3D structures that include length scales of the same order as the viscous and thermal boundaries, it is in most cases not possible to formulate suitable boundary conditions that support simplified models. In this case one must also resort to a model based on the full linearized Navier Stokes equations. Examples of geometries that require a model based on this set of equations include multi resonator mufflers and transducers in for instance hearing aids or cellular phone headsets.

The present study is concerned with developing a 2D finite element formulation of the system of equations that describes acoustical wave propagation including viscous and thermal effects.

2. FULL LINEARIZED NAVIER STOKES MODEL

The basic equations governing the propagation of sound waves are the linearized Navier Stokes equations, the equation of continuity, the equation of state for an ideal gas and the energy equation. In the absence of mean flow, gravitational forces and energy sources, the equations can be written as:

$$\rho_0 \frac{\partial \overline{\mathbf{v}}}{\partial t} = -\overline{\nabla}\overline{p} + \left(\frac{4}{3}\mu + \eta\right)\overline{\nabla}\left(\overline{\nabla}\cdot\overline{\mathbf{v}}\right) - \mu\overline{\nabla}\times\left(\overline{\nabla}\times\overline{\mathbf{v}}\right)$$
(1)

$$\rho_0 \left(\overline{\nabla} \cdot \overline{\mathbf{v}} \right) + \frac{\partial \overline{\rho}}{\partial t} = 0 \tag{2}$$

$$\overline{p} = \overline{\rho} R_0 \overline{T} \tag{3}$$

$$\rho_0 C_p \frac{\partial T}{\partial t} = \lambda \overline{\Delta} \overline{T} + \frac{\partial \overline{p}}{\partial t}$$
(4)

where $\overline{\mathbf{v}}$, \overline{p} , \overline{p} , \overline{T} , μ , η , R_0 , ρ_0 , λ , C_p and *t* denote the velocity vector, pressure, density, temperature, viscosity, bulk viscosity, gas constant, mean density, thermal conductivity, specific heat at constant pressure and time, respectively. The gas constant can also be described by the relation $R_0 = C_p - C_V$, with C_v specific heat at constant volume. The following small dimensionless harmonic perturbations are introduced:

$$\overline{\mathbf{v}} = c_0 \mathbf{v} e^{i\omega t} \qquad \overline{p} = p_0 \left(1 + p e^{i\omega t} \right) \qquad \overline{\rho} = \rho_0 \left(1 + \rho e^{i\omega t} \right) \qquad \overline{T} = T_0 \left(1 + T e^{i\omega t} \right) \tag{5}$$

where c_0 , T_0 , p_0 , ω and *i* are the undisturbed speed of sound, mean temperature, mean pressure, angular frequency and the imaginary unit. The coordinates, the gradient and the Laplace operators are all made dimensionless with a length scale l so $\nabla = l\overline{\nabla}$ and $\Delta = l^2\overline{\Delta}$. Equations (1) to (4) are further linearized and the following set of equations is obtained:

$$i\mathbf{v} = -\frac{1}{k\gamma}\nabla p + \frac{1}{s^2}\left(\frac{4}{3} + \varepsilon\right)\nabla(\nabla \cdot \mathbf{v}) - \frac{1}{s^2}\nabla \times (\nabla \times \mathbf{v})$$
(7)

$$(\nabla \cdot \mathbf{v}) + ik\rho = 0 \tag{8}$$

$$p = \rho + T \tag{9}$$

$$iT = \frac{1}{s^2 \sigma^2} \Delta T + i \left(\frac{\gamma - 1}{\gamma}\right) p \tag{10}$$

where the dimensionless parameters s the shear wave number, k the reduced frequency, γ ratio of specific heats, σ square of the Prandtl number and ε the viscosity ratio, are defined as:

$$s = l \sqrt{\frac{\rho_0 \omega}{\mu}}$$
 $k = \frac{\omega l}{c_0}$ $\gamma = \frac{C_p}{C_v}$ $\sigma = l \sqrt{\frac{\mu C_p}{\lambda}}$ $\varepsilon = \frac{\eta}{\mu}$ (11)

Equation (7) is rewritten using the mathematical identity $\nabla \times (\nabla \times \mathbf{v}) = \nabla (\nabla \cdot \mathbf{v}) - \nabla \cdot \nabla \mathbf{v}$ and the relation for the bulk viscosity $\eta = \hat{\lambda} + \frac{2}{3}\mu$ with $\hat{\lambda}$ the second viscosity. Rewriting equation (8) with equation (9) yields a system in terms of unknown velocity, temperature and pressure.

$$i\mathbf{v} = -\frac{1}{k\gamma}\nabla p + \frac{1}{s^2}(\hat{\eta} + 1)\nabla(\nabla \cdot \mathbf{v}) + \frac{1}{s^2}\nabla \cdot \nabla \mathbf{v}$$
(12)

$$-ikp + ikT = (\nabla \cdot \mathbf{v}) \tag{13}$$

$$iT = \frac{1}{s^2 \sigma^2} \Delta T + i \left(\frac{\gamma - 1}{\gamma}\right) p \tag{14}$$

where $\hat{\eta} = \hat{\lambda}/\mu$ the ratio between the second and first viscosity (for air under standard conditions $\hat{\eta} = -2/3$). This set of equations can be easily rewritten into the set of equations discretized by Malinen [2] to obtain a finite element formulation of the governing equations (1) to (4). The equations posed by Malinen are given here in the dimensionless coordinates defined in equation (11) and the dimensionless variables defined by equation (5).

$$i\mathbf{v} = -\frac{1}{k\gamma}\nabla T - \nabla\phi + \frac{1}{s^2}\nabla(\nabla\cdot\mathbf{v}) + \frac{1}{s^2}\nabla\cdot\nabla\mathbf{v}$$
(15)

$$\frac{\mathcal{K}^2 s^2}{is^2 - \mathcal{K}^2 \hat{\eta}} \phi = (\nabla \cdot \mathbf{v})$$
(16)

$$\frac{k}{(\gamma-1)s^2\sigma^2}\Delta T = \frac{ik}{\gamma-1}T + \frac{\varkappa^2 s^2}{is^2 - \varkappa^2 \hat{\eta}}\phi$$
(17)

Where the auxiliary variable ϕ can be expressed as a linear combination of the temperature and the pressure:

$$\phi = \frac{is^2 - \chi^2 \hat{\eta}}{is^2 \chi} (p - T)$$
(18)

Substitution of equation (9) demonstrates that this auxiliary unknown is simply the density multiplied by a combination of the dimensionless parameters given by equation (11). In the present work, the system given by equation (12) to (14) is discretized. In this way the pressure is obtained directly by solving the system and it's more natural to apply pressure loads.

4. FINITE ELEMENT DISCRETIZATION

The mixed system given by equation (12) to (14) is discretized using the method of weighed residuals. As a first attempt a quadrilateral element with equal order approximation for all fields is implemented, since this is most convenient from an implementation point of view. However, if no precautions are taken, this approach is known to produce an unstable solution. In literature it is well documented that the equal order bilinear velocity, bilinear continues pressure element suffers from spurious modes [3]. In order to stabilize the solution, three degrees of freedom are added to each element. One degree of freedom is added to each velocity component, and one degree of freedom is added that is shared by both velocity components. In this way a so called quadrilateral 'Mini' finite element is obtained, which is proved to be LBB-stable [3]. Since, the three added velocity related degrees of freedom are 'nodeless', they can be condensed out of the system that describes the elements behaviour. In this way an LBB-stable bilinear element is obtained, with a pressure, a temperature and two velocity degrees of freedom at each node. The Petrov-Galerkin method is used to minimize the residual of the system of differential equations at hand.

4.1 Weak formulation

Equation (12) tot (14) are multiplied by weighing functions \mathbf{w} , φ , q and integrated over the domain Ω to obtain the weak form of the system.

$$i\int_{\Omega} \mathbf{w} \cdot \mathbf{v} d\Omega + \frac{1}{k\gamma} \int_{\Omega} \mathbf{w} \cdot \nabla p d\Omega + \frac{1}{s^2} (\hat{\eta} + 1) \int_{\Omega} (\nabla \mathbf{v} : \nabla \mathbf{w}) d\Omega + \frac{1}{s^2} \int_{\Omega} ((\nabla \mathbf{v})^T : \nabla \mathbf{w}) d\Omega + V_{\partial\Omega} = 0$$
(19)

$$-ik\int_{\Omega}\varphi pd\Omega + ik\int_{\Omega}\varphi Td\Omega - \int_{\Omega}\varphi(\nabla\cdot\mathbf{v})d\Omega = 0 \qquad (20)$$

$$i\int_{\Omega} qTd\Omega + \frac{1}{s^2 \sigma^2} \int_{\Omega} (\nabla q \cdot \nabla T) d\Omega - i \left(\frac{\gamma - 1}{\gamma}\right) \int_{\Omega} qpd\Omega + T_{\partial\Omega} = 0 \qquad (21)$$

The integrals over the boundary of the domain $\partial \Omega$ resulting from integration by parts are given by:

$$V_{\partial\Omega} = -\frac{1}{s^2} (\hat{\eta} + 1) \int_{\partial\Omega} (\nabla \mathbf{v} \cdot \mathbf{w}) \, d\partial\Omega - \frac{1}{s^2} \int_{\partial\Omega} (\mathbf{w} \cdot \nabla \mathbf{v}) \, d\partial\Omega$$
(22)

$$T_{\partial\Omega} = -\frac{1}{s^2 \sigma^2} \int_{\partial\Omega} (q \nabla T) \cdot d\partial\Omega$$
(23)

4.2 Boundary conditions

If the system is subject to a velocity boundary condition $\mathbf{v} = \mathbf{g}$ along the complete boundary, the integrals in equation (22) vanish since the weighing functions \mathbf{w} can be chosen equal to zero on the parts of the boundary where the Dirichlet condition is prescribed. On the same grounds, the integral in equation (23) vanishes on the parts of the boundary for which the temperature is prescribed. It is also possible to prescribe a heat flux Q on parts of the boundary by rewriting the integral as

$$T_{\partial\Omega} = -\frac{1}{s^2 \sigma^2 \lambda} \int_{\partial\Omega} qQ d\partial\Omega$$
 (24)

5. VALIDATION

To validate the results of the finite element formulation a numerical experiment is carried out for which the results can be compared with an analytical solution. A rectangular layer of air is modelled with the finite element described above (see figure 2). It is assumed that there is no pressure, velocity or temperature gradient in y-direction, making it possible to model the air layer with 2D elements. The air layer is 5mm thick (z-direction) and 200mm long (x-direction). The finite element mesh has a relatively higher element density near the walls of the air layer in order to capture the viscous and thermal boundary layer effects.

The walls surrounding the air layer are considered rigid and isothermal. The air layer has an inlet for which a velocity profile is prescribed at $x = x_2$. A rigid, adiabatic wall is modelled at $x = x_1$. The analytical solution is based on the Low Reduced Frequency model for air layers [1].



Figure 2 Finite element mesh and overview of boundary conditions and parameter values

Tests are carried out at three frequencies: 20Hz, 200Hz and 2000Hz. The shear wave numbers corresponding to these frequencies (and the geometry under consideration) are 7.2, 22.9 and 72.3, respectively. The frequencies range from a case for which viscothermal effects are dominant (s = 7.2) and a case for which viscothermal effects can be neglected (s = 72.3).

5.1 Low Reduced Frequency solution

Both the analytical solution for the problem described above and the boundary conditions for particle velocity and temperature that are applied to the finite element model are obtained with the Low Reduced Frequency model for thin air layers [1, 4];

$$p(x) = p_A e^{\tau \bar{k}x} + p_B e^{-\tau \bar{k}x}$$
(25)

$$T(x,z) = -\left\lfloor \frac{\gamma - 1}{\gamma} \right\rfloor C(s\sigma, z) p(x)$$
(26)

$$v_x(x,z) = -\frac{i}{k\gamma}A(s,z)\frac{\partial p}{\partial x}$$
(27)

$$v_{z}(x,z) = ik \left[\frac{\Gamma^{2}}{\gamma} \int_{0}^{z} A(s,z) - \left(z + \left[\frac{\gamma - 1}{\gamma} \right] C(s\sigma,z) \right] p(x)$$
(28)

with $\overline{k} = \omega/c_0$ and A(s, z) a function describing the velocity profile defined as:

$$A(s,z) = \frac{\cosh(zs\sqrt{i})}{\cosh(s\sqrt{i})} - 1$$
(29)

And $C(s\sigma, z)$ a function describing the thermal profile. The function $C(s\sigma, z)$ equals $A(s\sigma, z)$ for isothermal boundary conditions and $C(s\sigma, z) = -1$ for adiabatic boundary conditions.

The averaged particle velocity in propagation (x-) direction is set to unity at x_2 and set to zero for x_1 . The velocity profile for v_x and v_z as a function of the thickness (z-) direction are calculated at these locations with the LRF model (see figure 3) and applied as boundary conditions in the finite element model. The heat flux is set to zero along these boundaries, so the integral in equation (23) vanishes. The other boundaries found at $z = z_1$ and $z = z_2$, represent rigid isothermal walls, so the particle velocity in both directions and the temperature are set to zero.



Figure 3 Velocity profiles applied as boundary conditions at 20Hz, 200Hz and 2000Hz: (a) particle velocity in propagation (x-) direction at $x = x_2$, (b) particle velocity in thickness (z-) direction at $x = x_2$,(c) Velocity in thickness (z-) direction at $x = x_1$

5.2 Numerical results

The results obtained with the finite element model described above converge to the analytical solution if the element size is decreased. The convergence rate of the different variables was found to be dependent on the shear wave number. For higher shear wave numbers the profiles of particle velocity and temperature are less smooth and the boundary layer effects are present only close to the boundary. Therefore, more elements are needed to accurately describe the thermal and viscous influences on the wave propagation. The rate of convergence is slowest for the particle velocity in thickness (*z*-) direction especially for higher shear wave numbers. This might be due to the fact that the elements near the boundary of the layer are stretched considerably (large aspect ratios). It is pointed out by Malinen [2] that the performance of the method under consideration may degenerate if elements, so it may be advantageous to adopt another method of discretization for the set of equations under consideration.

The convergence rate of the element presented here was also compared with the element developed by Malinen (obtained by discretizing equation (15) to (17)). It was found that the convergence rate is comparable for both elements for different shear wave numbers. In figure 4, an example of the convergence rate of both elements at 20Hz at location $x = x_0$ is given.



Figure 4 Relative error at 20 Hz at location $x = x_0$: (a) *particle velocity in propagation* (x-) *direction, (b) particle velocity in thickness (z-) direction, (c) temperature, (d) pressure*



Figure 5 Real part of the particle velocity and temperature for the solution at 2000Hz. (a) particle velocity in propagation (x-) direction, (b) particle velocity in thickness (z-) direction, (c) temperature

The solution for temperature and particle velocity in both directions at 2000Hz is given in figure 5. The solution clearly shows how the velocity profile that is applied as boundary condition is present along the entire length of the air layer. At a relatively small distance from the boundary, the particle velocity in propagation (*x*-) direction is found to be only a function of the propagation (*x*-) direction, which is expected for a shear wave number of s = 72.3.

6. CONCLUSIONS

The need for a finite element that includes viscous and thermal effects for modelling wave propagation in arbitrary geometries was outlined. A system of equations with pressure, temperature and particle velocity as unknown variables was derived from the set of governing equations.

A similar set of equations with temperature, particle velocity and an auxiliary variable as unknowns was preciously discretized by Malinen. The system presented in this paper can be easily rewritten to obtain the system of equations discretized by Malinen. In the present formulation, however, the auxiliary variable introduced by Malinen is replaced by a pressure unknown. Therefore, the pressure is obtained directly by solving the finite element system. Furthermore, it is more natural to apply pressure loads to the system that is being modeled.

The Petrov-Galerkin method was used to minimize the residual of both sets of differential equations. Convergence rates were determined for both elements and it was found that both finite elements perform equally well. However, it must be noted that the method used to descretize both elements may degenerate if elements are highly stretched. Further research is needed to point out if adopting a different discretization scheme is necessary. Current research focuses on the development of a three node triangle element and a family of viscothermal 3D elements.

REFERENCES

- [1] W.M. Beltman, "Implementation and experimental validation of a new viscothermal acoustic finite element for acousto-elastic problems", *Journal of Sound and Vibration* **216**(1), 159-185 (1998).
- [2] M. Malinen, M. Lyly, P. Råback, A. Kärkkäinen and L. Kärkkäinen, "A Finite element method for the modelling of thermo-viscous effects in acoustics", *Proceedings of the European Congress on Computational Methods in Applied Sciences and engineering (ECCOMAS 2004)*, Jyväskylä, Finland, 24– 28 July 2004.
- [3] W. Bai, "The quadrilateral 'Mini' finite element for the Stokes problem", *Computer methods in applied mechanics an engineering* 141, 41-47 (1997).
- [4] M.J.J. Nijhof, Y.H. Wijnant and A. de Boer, "Optimizing circular side-resonators to reduce computer fan noise", *Proceedings of NOISE-CON 2004*, Baltimore, Maryland, USA, 12-14 July 2004.