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STOCHASTIC FINITE ELEMENT ANALYSIS OF THE FREE VIBRATION OF FUNCTIONALLY GRADED MATERIAL PLATES

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Abstract

The superior properties of Functionally Graded Materials (FGM) are usually accompanied by randomness in their properties due to difficulties in tailoring the gradients during manufacturing processes. Using the Stochastic Finite Element Method (SFEM) proved to be a powerful tool in studying the sensitivity of the static response of FGM plates to uncertainties in their material properties. This tool is yet to be used in studying free vibration of FGM plates. The aim of this work is to use a Second Order Reliability Method (SORM), combined with a nine-noded isoparametric Lagrangian element based on the third order shear deformation theory to investigate sensitivity of the fundamental frequency of FGM plates to material uncertainties. These include uncertainties in ceramic and metal Young's modulus and Poisson's ratio, their densities and the ceramic volume fraction. The developed code utilizes MATLAB capabilities to derive the derivatives of the stiffness and mass matrices symbolically with a considerable reduction in calculation time. Calculating the eigenvectors at the mean values of the variables and updating them only at the last iteration significantly increases solution speed. The results of the stochastic finite element code are compared to published results and to the results of the well-established Monte Carlo simulation technique with importance sampling. Results show that the relative importance of variations in the constituents' properties is highly dependent on the volume fraction and is virtually independent of the frequency ratio for practical values of solution reliability. SORM is proven to be an excellent rapid tool in the stochastic analysis of free vibration of FGM plates, when compared to the slower Monte Carlo simulation techniques.

1. INTRODUCTION

One way to overcome the adverse effects of abrupt changes in material properties of conventional laminated composites is to use Functionally Graded Materials (FGM). In these materials, properties vary continuously across the thickness by gradually changing the volume fraction of the constituent materials, usually in the thickness direction only. Due to difficulties in manufacturing methods, properties of FGM's are not deterministic in nature. There is a reasonable body of recent research on studying the effect of uncertainties in material properties on the accuracy of static and thermal analyses of FGM's. In [1], for example, Ferrante and Graham used simulation to study the effect of microstructural randomness on stress and temperature distributions in FGM's. Later, they included the effect of non-Gaussian porosity randomness in their reliability analysis [2]. Yang, et al. [3] investigated the stochastic bending response of moderately thick FGM plates.

Reliability analysis of the dynamic behavior, however, has not received as much attention, even for the more commonly used laminated composites. In [4] Salim, et al. used first order perturbation techniques and FEM formulation to investigate the sensitivity of the natural frequencies of single ply and double ply laminates to randomness in material properties. In [5], Senthil and Batra, on the other hand, obtained exact solutions to the free vibration of FGM rectangular plates using deterministic properties.

Since uncertainties in mechanical properties, material density, and plate dimensions greatly affects dynamic response, the sensitivities of the dynamic characteristics of FGM plates to random changes in these properties need to be investigated. In this work, our previously developed stochastic finite element SFEM analysis of the free vibration of composite laminates [6] is adopted for FGM plates. Laminate mechanical behavior is modeled using a higher order shear deformable element. The code is built using the MATLAB 7.1 compiler and all runs are made on a P4 2.8 GHz machine with 512 MB RAM.

2. FINITE ELEMENT MODEL

Figure 1 shows the geometry of a rectangular FGM plate. Without losing generality, it can be assumed that the top surface of an FGM plate is ceramic rich and the bottom is metal rich. The region between the two surfaces consists of material blended with both of them.

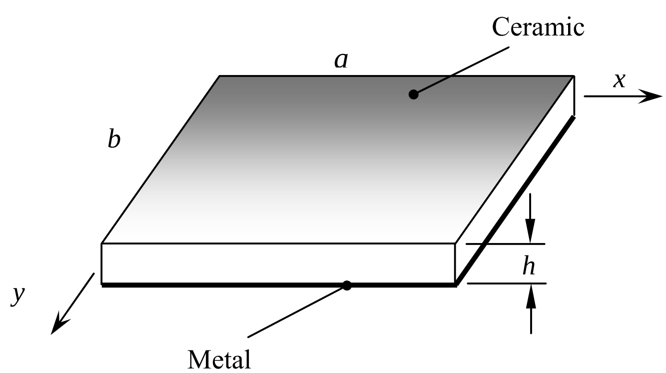

Figure 1. Geometry of the FGM plate.

To include transverse shear stresses and rotatory inertia effects into the free vibration analysis of this plate, the Higher-Order Shear Deformation Theory (HSDT) is utilized. The displacement field is described in terms of midsurface displacements u, v and w, the perpendicular to the midplane, ζ, and the rotations of the normal to the midsurface at $\zeta = 0$, ϕ_1 and ϕ_2. Considering the derivatives of the out-of-plane displacement as separate independent degrees of freedom transforms this system, into one with 7 degrees of freedom per node and C^0 continuity. The displacement field may be modified to accommodate C^0 continuity, see [7], as:

$$\bar{u}\left(x_1, x_2, \zeta, t\right) = u + f_1(\zeta)\,\phi_1 + f_2(\zeta)\,\theta_1 \tag{1.a}$$

$$\bar{v}\left(x_1, x_2, \zeta, t\right) = v + f_1(\zeta)\,\phi_2 + f_2(\zeta)\,\theta_2 \tag{1.b}$$

$$\bar{w}\left(x_1, x_2, \zeta, t\right) = w \tag{1.c}$$

where:

$$\theta_1 = \partial w / \partial x_1,\ \theta_2 = \partial w / \partial x_2,\ f_1(\zeta) = \zeta - 4\zeta^3 / 3h^2\text{, and } f_2(\zeta) = -4\zeta^3 / 3h^2\,. \tag{2}$$

The components of the effective stiffness matrix Q_e of the FGM material are calculated using the mixture law:

$$\left[Q(\zeta)\right]_e = \left[Q\right]_C V_C + \left[Q\right]_M \left(1 - V_C\right) \tag{3}$$

where the subscripts C and M stand for the isotropic ceramic and matrix constituents, respectively, and V_C is the ceramic volume fraction given by:

$$V_C = \left(0.5 + \frac{\zeta}{h}\right)^n \quad , -h/2 \le \zeta \le h/2\ ,\ \ 0 \le n < \infty \tag{4}$$

Following a classical FEM formulation, and using variational principles, the characteristic equation of the system can be derived, see [7], as:

$$Aq - \lambda q = 0\,, \tag{5}$$

where $A = M^{-1}K$, with K and M being the global stiffness and mass matrices of the element, respectively, and where q is the global displacement vector.

3. RELIABILITY MODEL OF THE FGM PLATE

The plate is assumed to be subjected to a periodic load with frequency ω_L, which can take any value up to the plate fundamental frequency ω_p. This upper limit is not a unique value, but has a certain distribution. At the design point, ω_p, is equal to a certain specified value ω_r, which may be taken as that of the periodic load. Accordingly, the performance function is defined as:

$$g(X) = (\lambda_p / \lambda_r) - 1 \tag{6}$$

where $\lambda_{p,r} = \omega^2_{p,r}$ are the eigenvalues, and X is a vector of the basic variables. Here, the components of X are Young's modulus, Pisson's ratio and density of each constituent; namely E_C, E_M, ν_C, ν_M, ρ_C and ρ_M. According to Eq. (6), a failure surface or a limit state of interest can be defined as $g(X) = 0$, with a certain probability of failure p_f. In order to calculate p_f, and following our procedure in [6], we shall use the Second-Order Reliability Method (SORM). A detailed account of SORM can be found in [8] and will only be summarized here.

The method utilizes a second order Taylor approximation of the nonlinear limit state function around a given point X^* in the standard normalized space of the random variables:

$$g(X_1) \cong g(X^*) + \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}(X_i - X_i^*) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(X_i - X_i^*\right)\left(X_j - X_j^*\right)\frac{\partial^2 g}{\partial X_i \partial X_j} \tag{7}$$

A simple closed-form solution for the probability of failure using this second-order approximation is derived using the theory of asymptotic approximations as:

$$p_f = \Phi(-\beta)\prod_{i=1}^{n-1}(1+\beta\kappa_i)^{-0.5} \tag{8}$$

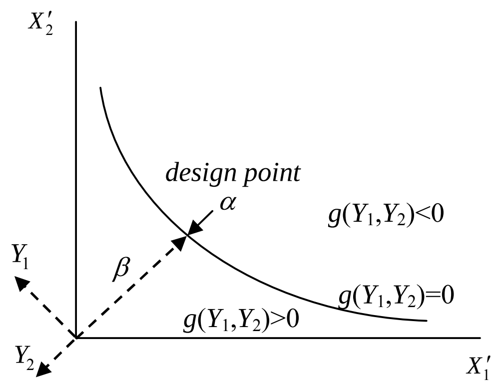

Figure 2. SORM rotation of coordinates.

where β is the reliability index obtained using a First Order Reliability Method (FORM), and κ_i are the principal curvatures of the limit state at the minimum distance point. These curvatures can be obtained as follows. First the X' standard normal variables are rotated to another set of coordinates, denoted as Y, such that the last component of the new set, Y_n, coincides with the unit gradient vector of the limit state at the design point. This transformation is shown in Figure 2 for the case two variables. The rows of the rotation matrix R of this orthogonal transformation are calculated using Gram-Schmidt orthogonalization procedure, see [8]. Accordingly Y_n is calculated from:

$$Y_n = \beta + \frac{1}{2}Y^T AY \tag{9}$$

where A is a matrix whose elements a_{ij} are computed as:

$$a_{ij} = \frac{(RDR^T)_{ij}}{\left|\nabla g(Y^*)\right|} \tag{10}$$

and D being the second-derivative matrix of the limit state surface in the standard normal space evaluated at the design point. The required curvatures κ_i are computed as the eigenvalues of the matrix A. The probability of failure can now be calculated from Eq. (8). The distance from the origin to this new design point in the X'-space is:

$$\beta = \sqrt{\sum_{i=1}^{n} X_i'^{*2}} \tag{10}$$

The procedure is terminated when reaching the Most Probable Point (MPP). MPP is assumed to be reached when values of β and g in two successive iterations are very close to each other. Finally, the partial derivatives of the performance function $g(X)$ with respect to all random variables X_i can be expressed, using the chain rule. The partial derivatives of the jth eigenvalue with respect to the random variables are:

$$\frac{\partial \lambda_j}{\partial X} = \frac{\frac{\partial}{\partial X}\left[\phi_j^T (K - \lambda_j M)\phi_j\right]}{\phi_j^T M \phi_j} \tag{11}$$

Standard finite difference routines can be used to evaluate the derivatives of Eq. (11) with respect to the random variables. This, however, becomes time consuming because the process is repeated at each iteration point. Also, using SORM in computing the probability of failure requires calculating the second derivatives as well, which deems the finite difference choice impractical. Use is made of MATLAB symbolic capabilities in evaluating the derivatives of the reduced stiffness and mass matrices. In evaluating the derivative in the numerator of Eq. (11), the eigenvectors at the mean value of X are used, and updated only at the last iteration. This is justifiable for large frequency ratios because as the frequency ratio increases, MPP tends to be closer to the mean value of X. Validity of this simplification, and confidence in the whole modeling, is established in our recent work [6] by comparisons with available published results.

4. NUMERICAL ILLUSTRATION

Our numerical illustration is that of a SSSS aluminum-zirconia FGM plate. The plate has a thickness ratio a/h = 10. To facilitate comparisons with published results, the ceramic volume fraction exponent is initially assumed deterministic, n=2. Material properties of the constituents are taken as normal random variables with the distributions shown in Table 1. The tolerances of convergence on β and g are taken as 0.001.

Table 1. Statistical distribution of the material properties of a SSSS aluminum -zirconia plate.

Property	E_C (GPa)	E_M (GPa)	ν_C	ν_M	ρ_C (Kg/m^3)	ρ_M (Kg/m^3)
Mean	151	70	0.3	0.3	3,000	2,707
COV	0.036	0.037	0.0	0.03	0.036	0.036

The calculated nondimensional natural frequency $\bar{\omega} = \omega a^2 \sqrt{\rho_M / E_M h^2}$ for a 5x5 mesh is 4.7799 which compares very well with the published result of 4.7756 reported in [9]. This latter value was obtained using a 10x10 mesh of quadratic 8-node serendipity elements.

To the best of our knowledge, full stochastic analyses, with calculated probability of failure, reliability index, and MPP, of the free vibration of FGM plates do not exist in literature. Therefore, the obtained results of the probability of failure will be compared only to those of a developed and verified code in [6] employing Monte Carlo simulation with importance sampling. The same aluminium-zirconia plate is analyzed for a frequency ratio

FR= $\omega_r / \omega_p = 0.93$. The probability of failure using Monte Carlo simulation technique depends on the number of simulations, as can be seen in Table 2. Therefore p_f can be taken as a dependent random variable, for which one can calculate a mean, a standard deviation and a skewness coefficient. These values for the p_f distribution of Table 2 are 5.362x10^{-5}, 0.01586, -0.00301 respectively. The small value of standard deviation suggests that the value of p_f does not change much around the mean. The negative skewness coefficient means that dispersion is more below the mean than above it. Therefore, taking the mean of Monte Carlo calculated p_f as a reference for comparison is reasonable.

Table 2. Variation of the probability of failure of Monte Carlo simulation for SSSS FGM square plate with *a*/*h* =10, *n*=2, and a frequency ratio of 0.93

No. of Simulations	900	1000	1100	1200	1300	1400	1500	1600	1700	1500
p_f(x10^5)	5.04	5.26	5.34	5.46	5.48	5.46	5.41	5.38	5.34	5.45

The developed code has the option of using either FORM or SORM as an optimization method. Probability of failure, calculated using FORM, is p_f=5.42x10^{-5}, while that calculated using SORM is p_f =5.3625x10^{-5}. In both methods, the system has to be solved only five times. FORM overestimates the value of the mean of probability of failure by about 1.13%, while SORM overestimates it by only 0.01%. This means that this problem is quasi linear with a small introduced error when the nonlinearity is ignored in FORM. Table 3 shows a comparison of the values of the reliability index and the probability of failure, calculated for three frequency ratios using both FORM and SORM methods.

Table 4 presents the MPP results of the plate for the three values of the frequency ratio. It is clear from this table that Poisson's ratio of both constituents does not affect the reliability of the solution.

Table 3. Comparison of the safety index and probability of failure of SSSS FGM square plate with *a*/*h* =10, *n*=2, for three values of the frequency ratio.

ω_r / ω_p	FORM		SORM	
	β	p_f	β	p_f
0.90	5.5951	1.10E-8	5.5978	1.09E-8
0.93	3.8710	5.42E-5	3.8737	5.36E-5
0.95	2.7414	3.10E-3	2.7440	3.00E-3

Table 4. MPP of SSSS FGM square plate with *a*/*h* =10, *n*=2, for three values of the frequency ratio.

ω_r / ω_p	E_C (GPa)	E_M (GPa)	ν_C	ν_M	ρ_C (Kg/m^3)	ρ_M (Kg/m^3)
0.90	137.3	61.3	0.3	0.3	3,190.8	3019.1
0.93	141.8	64.2	0.3	0.3	3,136.8	2,930.8
0.95	144.6	66.0	0.3	0.3	3,099.0	2,869.0

The sensitivities of the performance function, *g*, for changes in the random variables are plotted in Figure 3. The figure shows that, at this particular value of *n*, with more metal than ceramic, metal properties have a more pronounced effect on the solution. The natural frequency is most sensitive to changes in Young's moduli, and is least sensitive to changes in Poisson's ratio, which explains why the values of ν_C and ν_M at the MPP point are almost equal to their mean values. Finally, the figure shows that the relative importance of the variables is the same at all reliability levels of this range.

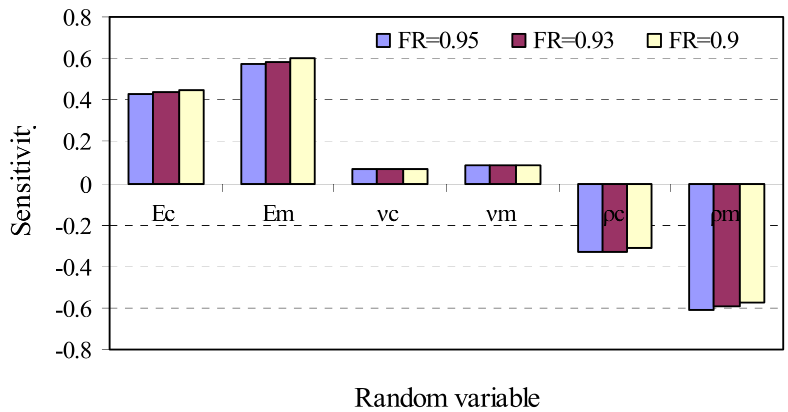

Figure 3. Sensitivity of the performance function to changes in the random variables.

Finally, the variation of the covariance of the eigenvalue, COV(λ), due to individual uncertainties in the basic random variables is investigated. Unlike results presented so far, the randomness in the ceramic volume fraction is considered now. Figure 4 shows the variation of COV(λ) when the COV of each of the random variables varies from 0 to 20%, for the case when the mean value of n is 2. From this figure it can be concluded that variations in the volume fraction exponent n and Poisson's ratios have small effects on the randomness of the eigenvalue compared to the effects of the density and Young's modulus. Table 5 shows the order of importance of the uncertainties on the calculated distribution of λ in a decreasing order of importance for three different compositions of the plate represented by three values of the mean of n. It can be seen that as n increases, signifying more metal, the plate natural frequency becomes more sensitive to the metal properties than to those of the ceramic.

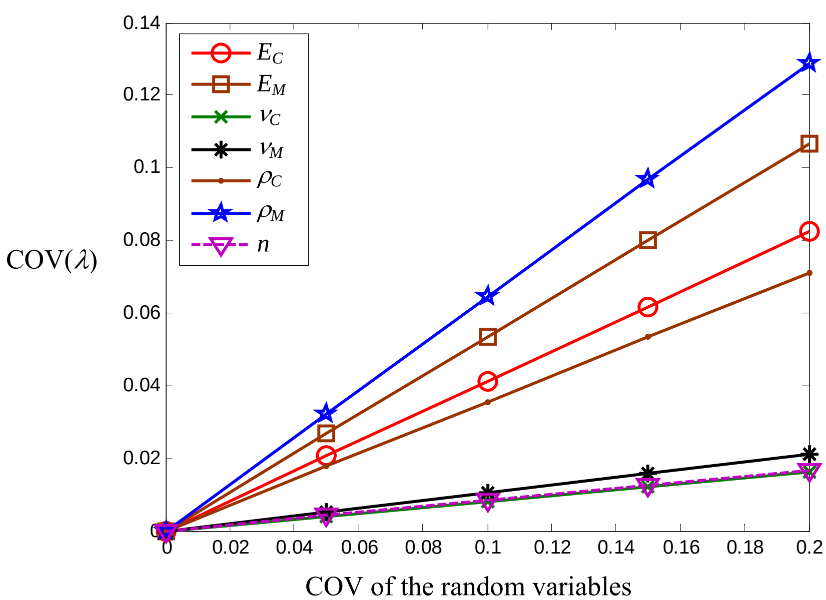

Figure 4. Effect of randomness in the basic variables on the covariance of the fundamental frequency.

Table 5. Order of importance of uncertainties in material properties on the randomness of the fundamental eigenvalue.

$\mu(n)$	E_C	E_M	ν_C	ν_M	ρ_C	ρ_M	n
0.5	1	4	6	7	2	3	5
1.0	1	4	6	7	2	3	5
2.0	3	2	7	5	4	1	6

5. CONCLUSIONS

The potential and versatility of a suggested procedure were demonstrated by applying it to reliability analysis of the free vibration of FGM plates. Using the developed code, the derivative of the performance function with respect to each of the random variables was calculated symbolically. These variables included the properties of both constituents and the ceramic volume fraction. SORM technique was used to optimize the solution and obtain MPP of the plate. Natural frequency results obtained showed excellent agreement with the limited published results and with Monte Carlo simulation results. SORM was found to be an appropriate optimization method for this problem, and converged after a small number of iterations. Randomness in Young's modulus and density of both constituents was found to have a strong effect on the randomness of the fundamental eigenvalue. The algorithm can be modified to solve other classes of problems with minor programming modifications and smart choice of the performance function. These include optimization and forced vibrations.

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