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# PARAMETRIC VIBRATIONS OF THE ROTOR 

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#### Abstract

In this paper the parametrically excited vibrations of the rotor are analyzed. The rotor is considered as a shaft-disc system. The parameters of the system are time dependent. The mathematical model of the rotor is a system of two coupled differential equations of Hill's type. The parametric excitation has the form of the Jacobi elliptic function. The vibrations of the system are obtained analytically and numerically. Two analytical methods for solving are developed: the method of harmonic balance based on the elliptic functions and the KrylovBogolubor method with variable amplitude and phase. The analytical solutions are compared with numerical ones. They are in a good agreement.


## 1. INTRODUCTION

The rotors with time variable parameters are the fundamental working elements of many machines in cable, paper, carpet, textile industry as well as in process industry. Usually the variation of parameters is periodical. In the most papers dealing with the rotor dynamics a simplification is made and the parameters are assumed to be constant. Unfortunately, this model differs significantly from the real one. The aim of this paper is to analyze the influence of parametrical excitation on the dynamics of the rotor.

The rotor is modeled as a shaft-disc system. Mass of the shaft is negligible in comparison to the mass of the disc. The disc is settled in the middle of the shaft. The shaft is supported in two rigid bearings. The parameter of the rotor is time dependent. It varies periodically in time and represents the parametrical excitation of the system.

The mathematical model of the rotor is a second order differential equation with complex function of Hill's type

$$
\begin{equation*}
\ddot{z}+\left[C+D c n^{2}\left(\omega t, k^{2}\right)\right] z=0, \tag{1}
\end{equation*}
$$

where $C$ and $D$ are constants, $z=x+i y$ is the complex deflection function, $x, y$ are coordinates of mass center, $i=\sqrt{-1}$ is the imaginary unit, $c n$ is the Jacobi elliptic function [1]. $k$ is the modulus of the Jacobi function.

Separating the real and imaginary part of the eq. (1) two second order ordinary differential equations are obtained

$$
\begin{align*}
& \ddot{x}+\left[C+\operatorname{Dcn}^{2}\left(\omega t, k^{2}\right)\right] x=0, \\
& \ddot{y}+\left[C+D c n^{2}\left(\omega t, k^{2}\right)\right] y=0 . \tag{2}
\end{align*}
$$

The equations have the same form. The initial conditions for $t=0$ are $x_{0}, y_{0}, \dot{x}_{0}, \dot{y}_{0}$.
There are many papers considering the solution of the Mathieu's equation as the special case of Hill's equation (see [2]-[4]). In all of these papers it is assumed that the excitation has the form of the circular harmonic function (sinus or cosines) and the excitation is small. For solving such equations some approximate analytical methods are developed. In the paper [5] the excitation is of Jacobi elliptic type. The system has one degree of freedom.

In this paper the extension to the two degree of system with parametric excitation of elliptic type is done. The Jacobian elliptic function is with the period $4 K$ which is the complete elliptic integral of the first kind. The methods developed for solving Mathieu's equation are adopted for this problem.

## 2. MATHEMATICAL MODEL OF THE ROTOR

Let us consider the vibrations of the rotor in the polar coordinates. For $z=\rho \exp (i \theta)$, where $\rho$ is the deflection of mass center and $\theta$ is the polar angle, the eq. (1) transforms into

$$
\begin{align*}
& \ddot{\rho}-\rho \dot{\theta}^{2}+\left[C+\operatorname{Dcn}^{2}\left(\omega t, k^{2}\right)\right] \rho=0,  \tag{3}\\
& \rho \ddot{\theta}+2 \dot{\rho} \dot{\theta}=0,
\end{align*}
$$

with the initial conditions $\rho_{0}, \theta_{0}, \dot{\rho}_{0}, \dot{\theta}_{0}$. The connections between the initial conditions for the eq. (2) and $(3,4)$ are

$$
\begin{equation*}
\rho_{0}=\sqrt{x_{0}^{2}+y_{0}^{2}}, \tan \theta_{0}=\frac{y_{0}}{x_{0}}, \dot{\rho}_{0}=\frac{x_{0} \dot{x}_{0}+y_{0} \dot{y}_{0}}{\sqrt{x_{0}^{2}+y_{0}^{2}}}, \dot{\theta}_{0}=\frac{x_{0} \dot{y}_{0}-y_{0} \dot{x}_{0}}{x_{0}^{2}+y_{0}^{2}} . \tag{5}
\end{equation*}
$$

The equation (4) has the first integral

$$
\begin{equation*}
\rho^{2} \dot{\theta}=\rho_{0}^{2} \dot{\theta}_{0}=\kappa=\text { const } . \tag{6}
\end{equation*}
$$

Introducing (6) into (3) the Ermakov-Pinney equation is obtained

$$
\begin{equation*}
\ddot{\rho}+\left[C+D c n^{2}\left(\omega t, k^{2}\right)\right] \rho=\frac{\kappa}{\rho^{3}} . \tag{7}
\end{equation*}
$$

It is a non-linear second order differential equation with periodically time variable parameter. Let us solve the eq.(7). Two groups of problems will be considered: a) $\kappa=0$ and $\kappa \neq 0$. Which of these two cases will appear depends on the initial conditions.

## 3. HARMONIC BALANCE FOR THE CASE WHEN $\kappa=0$

For the case when the initial conditions are

$$
\begin{equation*}
\rho(0)=\rho_{0}, \theta(0)=\theta_{0}, \dot{\rho}(0)=\dot{\rho}_{0}=0, \dot{\theta}(0)=\dot{\theta}_{0}=0, \tag{8}
\end{equation*}
$$

it is $\kappa=0$ and the equation of motion is

$$
\begin{equation*}
\ddot{\rho}+\left[C+\operatorname{Dcn}^{2}\left(\omega t, k^{2}\right)\right] \rho=0 . \tag{9}
\end{equation*}
$$

Let us assume the solution of the equation (9) in the form
$\rho=A-B c n^{2}\left(\omega t+\psi, k^{2}\right)$,
where $A, B$ and $\psi$ are constants. Substituting the assumed solution in (9) and separating the terms with the same order of the function cn, we obtain

$$
\begin{equation*}
A=B \frac{3 C+2 D}{3(2 C+D)}, \tag{11}
\end{equation*}
$$

and

$$
\begin{equation*}
\omega^{2}=\frac{3 C^{2}+4 C D+D^{2}}{6(2 C+D)}, k^{2}=\frac{D}{6 \omega^{2}}=\frac{D 92 C+D)}{3 C^{2}+4 C D+D^{2}} . \tag{12}
\end{equation*}
$$

It means that the solution (10) is correct only for the case when the relations between the parameters are (11) and (12). From the eq.(12) it is evident that the coefficients of Jacobi function do not depend on the initial conditions. Substituting (11) into (12) it is
$\rho=B \frac{3 C+2 D}{3(2 C+D)}-B c n^{2}\left(\omega t+\psi, k^{2}\right)$.
The coefficients $B$ and $\psi$ have to be obtained according to the initial conditions. Introducing the initial conditions it is

$$
B=-3 \rho_{0} \frac{2 C+D}{3 C+D}, \psi=0,
$$

i.e.,
$\rho=3 \rho_{0} \frac{2 C+D}{3 C+D}\left[c n^{2}\left(t \sqrt{\frac{3 C^{2}+4 C D+D^{2}}{6(2 C+D)}}, \frac{D(2 C+D)}{3 C^{2}+4 C D+D^{2}}\right)-\frac{3 C+2 D}{3(2 C+D)}\right]$.

## 4. THE METHOD OF VARIABLE PHASE AND AMPLITUDE FOR THE CASE WHEN $\kappa \neq 0$

For the case when $\kappa \neq 0$, and the nonlinearity is small the solution of the equation (7) is obtained applying the method of slow variable amplitude and phase.

Let us form the trial solution according to (10) as
$\rho=A(t)-B(t) c n^{2}\left[\theta(t), k^{2}\right]$,
where

$$
\begin{equation*}
\theta(t)=\int \omega d t+\psi(t), \quad A(t)=B(t) \frac{3 C+2 D}{3(2 C+D)} . \tag{15}
\end{equation*}
$$

The first time derivative of (15) is
$\dot{\rho}=2 B \omega s n c n d n$,
with the constraint

$$
\dot{A}-\dot{B} c n^{2}\left[\theta(t), k^{2}\right]+2 B \dot{\psi} s n c n d n=0,
$$

where $s n \equiv s n\left[\theta(t), k^{2}\right], \quad c n \equiv c n\left[\theta(t), k^{2}\right], d n \equiv d n\left[\theta(t), k^{2}\right]$. Substituting (15) and (17) into (7) it is
$2 \dot{B} \omega s n^{2} c n^{2} d n^{2}\left(A-B c n^{2}\right)^{3}+\omega \dot{B}\left(c n^{2}-p\right)\left(A-B c n^{2}\right)^{3}$
$\left(k^{2}-1+2 c n^{2}+3 k^{2} c n^{4}-4 k^{2} c n^{2}\right)=k s n c n d n$
$-4 B^{4} \omega s n^{2} c n^{2} d n^{2}\left(p-c n^{2}\right)^{2} \dot{\psi}+2 B^{4} \dot{\psi} \omega\left(p-c n^{2}\right)^{3}$
$\left(k^{2}-1+2 c n^{2}+3 k^{2} c n^{4}-4 k^{2} c n^{2}\right)=\kappa$
where $p=\frac{3 C+2 D}{3(2 C+D)}$.
Averaging the eqs.(19) and (20) in the period of $4 K$ as $\langle\ldots\rangle=\int_{0}^{4 K} \ldots d \theta$ and integrating the so obtained equations using the initial conditions it is

$$
\begin{align*}
B & =B_{0}=\text { const. }  \tag{21}\\
\psi & =\frac{\kappa}{2 B_{0}^{4} \omega a} t+\psi_{0}, \tag{22}
\end{align*}
$$

where $B_{0}$ and $\psi_{0}$ are constants obtained according to the initial conditions, and

$$
a=\int_{0}^{4 K}\left(p-c n^{2}\right)^{3}\left(k^{2}-1+2 c n^{2}+3 k^{2} c n^{4}-4 k^{2} c n^{2}\right) d \psi-2 \int_{0}^{4 K}\left(p-c n^{2}\right)^{2} s n^{2} c n^{2} d n^{2} d \psi .
$$

The solution of (7) is

$$
\begin{equation*}
\rho=B_{0} p-B_{0} c n^{2}\left(\omega t+\frac{\kappa}{2 B_{0}^{4} \omega a} t+\psi_{0}, k^{2}\right) . \tag{23}
\end{equation*}
$$

For the initial conditions $\rho(0)=\rho_{0}$ and $\dot{\rho}(0)=0$ it is $B_{0}=\frac{\rho_{0}}{p-1}$ and $\psi_{0}=0$ and the solution (23) is

$$
\begin{equation*}
\rho=\frac{\rho_{0}}{p-1}\left[p-c n^{2}\left(\omega t+\frac{\kappa(p-1)^{4}}{2 \rho_{0}^{4} \omega a} t, k^{2}\right)\right] \tag{24}
\end{equation*}
$$

The main disadvantage of the method is that the so obtained solution is evident only for a special type of periodically time variable function when the relations between parameters are given as (12) and (21).

## 5. VIBRATIONS OF THE ROTOR WITH SMALL EXCITATION

Let us consider the case when the excitation is small and $D \ll C$. According to Stegun [6] the series expansion of the Jacobi function $c n$ to circular function cos is

$$
\begin{equation*}
c n\left(\omega t, k^{2}\right)=\frac{2 \pi}{k K} \sum_{n=1}^{\infty} \frac{q^{n-\frac{1}{2}}}{1+q^{2 n-1}} \cos (2 n-1) \frac{\pi \omega t}{2 K}, \tag{25}
\end{equation*}
$$

where $q=\exp \left(-\pi K^{\prime} / K\right), \quad K \equiv K(k)$ and $K^{\prime} \equiv K\left(\sqrt{1-k^{2}}\right)$. Substituting (25) into (1) and assuming only the first two terms of the series expansion the differential equation of motion simplifies to

$$
\begin{equation*}
\ddot{z}+C_{1} z=\varepsilon z a \cos \Omega t, \tag{26}
\end{equation*}
$$

where $\varepsilon=\frac{4 \pi^{2} q}{k^{2} K^{2}}, \Omega=\frac{\pi \omega}{K}, \quad a=-\frac{D}{2\left(1+q^{2}\right)}, \quad C_{1}=C+\frac{2 \pi^{2} D q}{k^{2} K^{2}\left(1+q^{2}\right)}$.
Let us solve the equation applying the method of variable amplitude and phase. For $\varepsilon=0$ the generating solution is

$$
\begin{equation*}
z=A \cos \left(\omega_{1} t+\alpha\right)+i B \cos \left(\omega_{2} t+\beta\right), \tag{27}
\end{equation*}
$$

where $\omega_{1}=\omega_{2}=\sqrt{C_{1}}$. According to (27) let us assume the trial solution for (26) in the form

$$
\begin{equation*}
z=A(t) \cos \psi_{1}(t)+i B(t) \cos \psi_{2}(t), \tag{28}
\end{equation*}
$$

where $\psi_{1}(t)=\omega_{1} t+\alpha(t)$ and $\psi_{2}(t)=\omega_{1} t+\beta(t)$. The first time derivative is
$\dot{z}=-A \omega_{1} \sin \psi_{1}-i B \omega_{1} \sin \psi_{2}$,
with the constraint
$\dot{A} \cos \psi_{1}-A \dot{\psi}_{1} \sin \psi_{1}+\dot{B} i \cos \psi_{2}-i B \dot{\psi}_{2} \sin \psi_{2}=0$.
Substituting (28) and the derivative of (20) into (26) and separating the real and the imaginary terms two differential equations are obtained
$-\dot{A} \omega_{1} \sin \psi_{1}-A \omega_{1} \dot{\alpha} \cos \psi_{1}=\varepsilon a A \cos \psi_{1} \cos \frac{\Omega\left(\psi_{1}-\alpha\right)}{\omega_{1}}$
$-\dot{B} \omega_{1} \sin \psi_{2}-B \omega_{1} \dot{\beta} \cos \psi_{2}=\varepsilon a B \cos \psi_{2} \cos \frac{\left(\psi_{2}-\beta\right) \Omega}{\omega_{2}}$

Using the relations (30)-(32) it is
$\dot{A} \omega_{1}=-\frac{\varepsilon a A}{2} \sin 2 \psi_{1} \cos \left(\frac{\Omega \psi_{1}}{\omega_{1}}-\frac{\Omega \alpha}{\omega_{1}}\right)$,
$\dot{\alpha}=-\frac{\varepsilon a}{\omega_{1}} \cos ^{2} \psi_{1} \cos \left(\frac{\psi_{1}-\alpha}{\omega_{1}} \Omega\right)$,
$\dot{B} \omega_{1}=-\frac{\varepsilon a B}{2} \sin 2 \psi_{2} \cos \left(\frac{\Omega \psi_{2}}{\omega_{1}}-\frac{\Omega \beta}{\omega_{1}}\right)$,
$\dot{\beta}=-\frac{\varepsilon a}{\omega_{1}} \cos ^{2} \psi_{2} \cos \left(\frac{\psi_{2}-\beta}{\omega_{1}} \Omega\right)$.
Averaging the angles $\psi_{1}$ and $\psi_{2}$ in the period of $2 \pi$ the eqs.(33) - (36) are
$\dot{A} \omega_{1}=-\frac{\varepsilon a A}{2}\left(r_{1} \cos \frac{\Omega \alpha}{\omega_{1}}+r_{2} \sin \frac{\Omega \alpha}{\omega_{1}}\right)$,
$\dot{\alpha}=-\frac{\varepsilon a}{\omega_{1}}\left(p_{1} \cos \frac{\Omega \alpha}{\omega_{1}}+p_{2} \sin \frac{\Omega \alpha}{\omega_{1}}\right)$,
$\dot{B} \omega_{1}=-\frac{\varepsilon a B}{2}\left(r_{3} \cos \frac{\Omega \beta}{\omega_{1}}+r_{4} \sin \frac{\Omega \beta}{\omega_{1}}\right)$,
$\dot{\beta}=-\frac{\varepsilon a}{\omega_{1}}\left(p_{3} \cos \frac{\Omega \beta}{\omega_{1}}+p_{4} \sin \frac{\Omega \beta}{\omega_{1}}\right)$,
where

$$
\begin{aligned}
& p_{1}=\int_{0}^{2 \pi} \cos ^{2} \psi_{1} \cos \frac{\Omega \psi_{1}}{\omega_{1}} d \psi_{1}= \\
& \frac{\omega_{1}}{4\left(2 \omega_{1}-\Omega\right)} \sin 2 \pi \frac{2 \omega_{1}-\Omega}{\omega_{1}}+\frac{\omega_{1}}{4\left(2 \omega_{1}+\Omega\right)} \sin 2 \pi \frac{2 \omega_{1}+\Omega}{\omega_{1}}+\frac{\omega_{1}}{2 \Omega} \sin \frac{2 \Omega \pi}{\omega_{1}}, \\
& p_{2}=\int_{0}^{2 \pi} \cos ^{2} \psi_{1} \sin \frac{\Omega \psi_{1}}{\omega_{1}} d \psi_{1}=-\frac{\omega_{1}}{4\left(2 \omega_{1}+\Omega\right)} \cos 2 \pi \frac{2 \omega_{1}+\Omega}{\omega_{1}} \\
& \quad+\frac{\omega_{1}}{4\left(2 \omega_{1}-\Omega\right)} \cos 2 \pi \frac{2 \omega_{1}-\Omega}{\omega_{1}}-\frac{\omega_{1}}{2 \Omega} \cos \frac{2 \Omega \pi}{\omega_{1}}-\frac{\omega_{1}\left(\Omega^{2}-2 \omega_{1}^{2}\right)}{\Omega\left(4 \omega_{1}^{2}-\Omega^{2}\right)}, \\
& p_{3}=\int_{0}^{2 \pi} \cos ^{2} \psi_{2} \cos \frac{\Omega \psi_{2}}{\omega_{1}} d \psi_{2}=p_{1}, \quad p_{4}=\int_{0}^{2 \pi} \cos ^{2} \psi_{2} \sin \frac{\Omega \psi_{2}}{\omega_{1}} d \psi_{2}=p_{2}, \\
& \quad r_{1}=\int_{0}^{2 \pi} \sin 2 \psi_{1} \cos \frac{\Omega \psi_{1}}{\omega_{1}} d \psi_{1}=\frac{2 \omega_{1}^{2}}{4 \omega_{1}^{2}-\Omega^{2}} \\
& \quad-\frac{\omega_{1}}{2\left(2 \omega_{1}+\Omega\right)} \cos 2 \pi \frac{2 \omega_{1}+\Omega}{\omega_{1}}-\frac{\omega_{1}}{2\left(2 \omega_{1}-\Omega\right)} \cos 2 \pi \frac{2 \omega_{1}-\Omega}{\omega_{1}},
\end{aligned}
$$

$$
\begin{gathered}
r_{2}=\int_{0}^{2 \pi} \sin 2 \psi_{1} \sin \frac{\Omega \psi_{1}}{\omega_{1}} d \psi_{1}= \\
\frac{\omega_{1}}{2\left(2 \omega_{1}-\Omega\right)} \sin 2 \pi \frac{2 \omega_{1}-\Omega}{\omega_{1}}-\frac{\omega_{1}}{2\left(2 \omega_{1}+\Omega\right)} \sin 2 \pi \frac{2 \omega_{1}+\Omega}{\omega_{1}}, \\
r_{3}=\int_{0}^{2 \pi} \sin 2 \psi_{2} \cos \frac{\Omega \psi_{2}}{\omega_{1}} d \psi_{2}=r_{1}, \quad r_{4}=\int_{0}^{2 \pi} \sin 2 \psi_{2} \sin \frac{\Omega \psi_{2}}{\omega_{1}} d \psi_{2}=r_{2} .
\end{gathered}
$$

Solving the eq.(38) and eq.(40) for the initial conditions $\alpha(0)=\alpha_{0}$ and $\beta(0)=\beta_{0}$ we obtain

$$
\begin{array}{r}
\alpha=2 \frac{\omega_{1}}{\Omega} \tan ^{-1}\left(\frac{p_{2}}{p_{1}}-\frac{\sqrt{p_{1}^{2}+p_{2}^{2}}}{p_{1}} \tanh \left(Q t+\alpha_{0}\right)\right), \\
\beta=2 \frac{\omega_{1}}{\Omega} \tan ^{-1}\left(\frac{p_{4}}{p_{3}}-\frac{\sqrt{p_{3}^{2}+p_{4}^{2}}}{p_{3}} \tanh \left(Q_{1} t+\beta_{0}\right)\right), \tag{42}
\end{array}
$$

where $Q=\frac{\varepsilon a \Omega \sqrt{p_{1}^{2}+p_{2}^{3}}}{2 \omega_{1}^{2}}$ and $Q_{1}=\frac{\varepsilon a \Omega \sqrt{p_{3}^{2}+p_{4}^{2}}}{2 \omega_{1}^{2}}$.
Substituting (41) and (42) into (37) and (39) and integrating for the initial conditions $A(0)=A_{0}$ and $B(0)=B_{0}$ it gives

$$
\begin{align*}
& A=A_{0} \exp \left(-\frac{\varepsilon a}{2 \omega_{1}}\left(r_{1} \cos \frac{\Omega \alpha}{\omega_{1}}+r_{2} \sin \frac{\Omega \alpha}{\omega_{1}}\right)\right),  \tag{43}\\
& B=B_{0} \exp \left(-\frac{\varepsilon a}{2 \omega_{1}}\left(r_{3} \cos \frac{\Omega \beta}{\omega_{1}}+r_{4} \sin \frac{\Omega \beta}{\omega_{1}}\right) .\right. \tag{44}
\end{align*}
$$

The values $A_{0}, B_{0}, \alpha_{0}, \beta_{0}$ are obtained from the following relations

$$
A_{0}=\sqrt{x_{0}^{2}+\dot{x}_{0}^{2}}, \quad B_{0}=\sqrt{y_{0}^{2}+\dot{y}_{0}^{2}}, \alpha_{0}=\tan ^{-1}\left(-\frac{\dot{x}_{0}}{x_{0}}\right), \beta_{0}=\tan ^{-1}\left(-\frac{\dot{y}_{0}}{y_{0}}\right) .
$$

The deflection of the rotor center is

$$
\begin{equation*}
\rho_{A}=\sqrt{A^{2} \cos ^{2}(\omega t+\alpha)+B^{2} \cos ^{2}(\omega t+\beta)}, \tag{45}
\end{equation*}
$$

according to relations (41) - (44).

## 6. EXAMPLE

Let us compare the analytically obtained solutions with numerically obtained one. The RungeKutta numerical method is applied for solving the system of differential equations (2). The parameters of the system are: $C=\sqrt{3}$ and $D=3$. According to (12) it is $\omega=1$ and $k^{2}=0.5$. The initial conditions are $\rho_{0}=0.5, \dot{\rho}_{0}=0, \theta_{0}=0, \dot{\theta}_{0}=1$. The constant coefficient is $\kappa=0$.
In Fig.1. the analytically obtained solution (45) is compared with numerically obtained one (7).


Figure 1. The time-history diagrams obtained analytically-averaged and exact numerical.

## 7. CONCLUSIONS

In this paper the parametrically excited vibrations of the rotor are analyzed. It is concluded:
For a special group of parameters and initial conditions using the method of harmonic balance the exact solution of the differential equation (1) is obtained.

Comparing the analytical and numerical solutions (see Fig.1.) it can be concluded that the simplification of the Jacobi elliptic function to a simple harmonic function is possible only for the case when the excitation is small. The analytical solution is the averaged value of the exact solution. The difference between the solutions is significant for the longer time period. The vibrations depend on the initial conditions: the simplification of the excitation function is limited with initial conditions.

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