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A MODIFIED GREEN'S FUNCTION FOR A DUCT AND ITS APPLICATION TO THE CALCULATION OF THE SOUND FIELD GENERATED BY A PISTON

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Abstract

A two-dimensional acoustic duct with rigid parallel walls is considered. For such a duct, a novel formula for the Green's function is derived by means of adding and subtracting the Green's function for Laplace equation with the boundary conditions of the duct. The Green's function is used to obtain a new formula for the pressure field generated by a vibrating piston mounted to a duct wall. The formula is a sum of a quickly converging series and a singular integral. It is shown that a singularity in the integral is logarithmic and, therefore, the integral is converging and can be evaluated. The formula derived here as well as a formula derived on the basis of a well-known expression for the duct Green's function are utilised to calculate the near field of the piston in the area of the duct directly above the piston. The dependency of the pressure on the vertical coordinate above the middle of the piston is calculated. During these calculations, the numbers of terms required for the convergence of the series in both formulae are obtained. It is shown that, with the same criterion of convergence, the formula obtained here requires the number of terms up to one and a half orders of magnitude smaller than the traditional formula. Amplitude and phase discrepancies for both formulae are also calculated. It is shown that the formula obtained here results in the discrepancies up to three orders of magnitude smaller than the traditional formula with the same convergence criterion.

1 INTRODUCTION

Methods of control of sound propagation in ducts are currently under active theoretical and experimental investigation, as the issue of noise control in ducts is important for the design of muffling devices and controlling the air conditioning duct noise. Most, if not all, existing strategies of active noise control in ducts are based on a formula, derived by Doak [1]. This formula determines the pressure field of a vibrating piston in a waveguide as an infinite sum of normal waveguide modes.

In real circumstances, usually several lower order modes are propagating, whereas all higher order modes are evanescent and exponentially decay with increasing distance from the source. As the evanescent modes do not give a significant contribution to the far field, the practice has been to neglect them when considering the cancellation of an incident plane sound wave in an active noise control system.

It is known, however, that evanescent modes are important for determining the amplitude and phase of the vibrations of the controlling sound source in an active noise control system, as they may lead to significant additional fluid-loading of the source. The contribution of the evanescent modes is acknowledged, for example, by Huang [2], who used the Doak's formula for a detailed analysis of control of sound propagation in a two-dimensional duct by means of a rigid piston mounted in a duct wall.

To solve scattering problems in acoustic waveguides, the author and his former co-authors have previously suggested a method, which is based on a waveguide Green's function transformed into a quickly converging form [3]. This method has been further developed by the author in a series of publications in application to two-dimensional [4] and three-dimensional [5,6] fluid layers.

In the present article, the method developed by the author is applied to the problem of sound generation by a rigid piston, vibrating with given amplitude in a two-dimensional duct. The objective of the article is to derive a formula allowing an efficient calculation of the piston pressure field and to compare its convergence with the convergence of the formula obtained by Huang [2] on the basis of Doak's formula [1].

2 THE LAYOUT OF THE DUCT AND THE REPRESENTATION OF THE RADIATED ACOUSTIC FIELD

Consider a two-dimensional plane infinite duct of width, D, filled with a compressible perfect fluid of density, ρ, and sound speed, c (Figure 1).

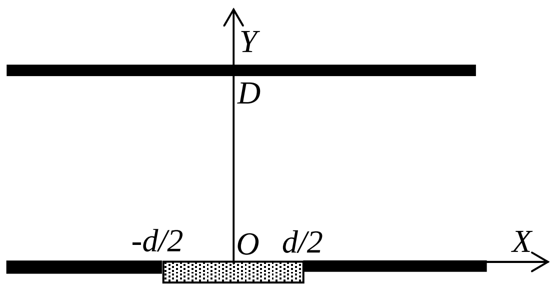

Figure 1. Layout of the duct.

A rigid piston of width, d, is mounted to one of the duct walls and vibrates in the normal direction with the velocity amplitude, V, and the angular frequency, ω. The x-axis is parallel to the duct walls, the y-axis is normal to the walls, and the origin coincides with the medium point of the piston. The harmonic temporal dependence, $\mathrm{e}^{-\mathrm{i}\omega t}$, is assumed, and all variables having the dimension of length are normalised on D/π.

In the analysis below, (x, y) is the observation point, (x_0, y_0) is the source point, $k = 2D/\lambda$ is the non-dimensional wavenumber, λ is the acoustic wavelength.

Let the pressure field, $P(x, y)$, generated by the piston in the duct, be expressed as a field of a single layer with the strength, $\mu(x_0)$:

$$P(x,y)=\int_{-d/2}^{d/2}\mu(x_0)G(x,y;x_0,0)dx_0\,, \tag{1}$$

where $G(x,y;x_0,y_0)$ is the duct Green's function, and by the conditions of zero velocity on the top and bottom boundaries of the duct outside the limits of the piston:

$$\frac{\partial}{\partial y}P(|x|>d/2,y)\Big|_{y=0}=\frac{\partial}{\partial y}P(|x|>d/2,y)\Big|_{y=D}=0. \tag{2}$$

Based on Euler's equation, the following boundary condition on the surface of the piston can be obtained:

$$\frac{\partial}{\partial y}P(|x|<d/2,y)\Big|_{y=0}=\mathrm{i}V\rho kc. \tag{3}$$

The substitution of Eq. (3) into Eq. (1) determines the following formula for the non-dimensional pressure field $\overline{P}(x,y)=P(x,y)/\rho cV$ in the duct:

$$\overline{P}(x,y)=-\mathrm{i}k\int_{-d/2}^{d/2}G(x,y;x_0,0)\,dx_0\,. \tag{4}$$

3 THE DUCT GREEN'S FUNCTION

3.1 The Green's function in the traditional form

The Green's function of the duct in its traditional form is an infinite series of normal duct modes:

$$G(x,y;x_0,y_0)=\frac{\mathrm{i}}{2\pi}\sum_{n=0}^{\infty}\frac{\varepsilon_n}{g_n}\cos ny_0\cos ny\,\mathrm{e}^{\mathrm{i}g_n|x-x_0|}\,, \tag{5}$$

where $g_n=\sqrt{k^2-n^2}$ are longitudinal wavenumbers, and Neumann factor, $\varepsilon_n=2$ for $n>0$ and $\varepsilon_0=1$.

The convergence of Eq. (5) depends strongly on the horizontal distance, $|x-x_0|$, between the source and observation points. On the one hand, at large $|x-x_0|$ the series becomes finite, as its terms of orders $n>k$ decay exponentially with increasing *n*. On the other hand, at $x\to x_0$ the convergence of the series significantly worsens, and the use of the Green's function in the form of Eq. (5) becomes impractical.

Moreover, at $|x-x_0|=|y-y_0|=0$ Eq. (5) is divergent. This means that the Green's function is singular at observation points coinciding with source points. At the same time, analytical or numerical integration of a function with a singularity in the form of divergent series is difficult. Thus, transformation of the duct Green's function into a more convenient form is required for the efficient use of Eq. (4) for calculating the pressure field in the duct.

3.2 Transformation of the Green's function of the duct into a more efficient form

In the work described here, the duct Green's function in the form of Eq. (5) will be transformed by a method based on Kummer's transformation (Formula 3.6.26 of the reference [7]) of the series of the duct modes. This method has been successfully developed by the author and his co-authors in application to scattering problems in two-dimensional [3,4] and three-dimensional [5,6] plane acoustic waveguides with both boundaries acoustically soft as well as with one rigid and one soft boundary. Kummer's transformation has been also used by Linton [8] to derive the Green's function in a plane two-dimensional duct with acoustically soft boundaries.

Here this method will be applied to the two-dimensional duct with acoustically rigid boundaries. The duct Green's function, transformed by the method, is a sum of a quickly converging infinite series and an asymptotic term, which accommodates the Green's function singularity at $|x-x_0|=|y-y_0|=0$.

To derive a modified expression for the duct Green's function, consider the Green's function, $G_L(x,y;x_0,y_0)$, for the Laplace equation:

$$G_L(x,y;x_0,y_0)=\frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{\varepsilon_n}{n}\cos ny_0\cos ny\,\mathrm{e}^{-n|x-x_0|}. \tag{6}$$

$G_L(x,y;x_0,y_0)$ can be obtained from $G(x,y;x_0,y_0)$, determined by Eq. (5), by substituting $k=0$.

With the use of known formulae [9], the infinite series in the above equation can be summed as follows:

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{n}\cos ny_0\cos ny\,\mathrm{e}^{-n|x-x_0|}=\\&\quad-\frac{1}{4}\ln\left(4\,\mathrm{e}^{-2|x-x_0|}\left(\cosh|x-x_0|-\cos(y+y_0)\right)\left(\cosh|x-x_0|-\cos(y-y_0)\right)\right)\end{aligned} \tag{7}$$

Subtracting Eq. (6) from Eq. (5) and adding it back in the form of Eq. (7), one can obtain the following expression for the duct Green's function:

$$\begin{aligned}G(x,y;x_0,y_0)=&\frac{\mathrm{i}}{2\pi k}\mathrm{e}^{\mathrm{i}k|x-x_0|}+\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{\mathrm{i}}{g_n}\mathrm{e}^{\mathrm{i}g_n|x-x_0|}-\frac{1}{n}\mathrm{e}^{-n|x-x_0|}\right)\cos ny\cos ny_0-\\&\frac{1}{4\pi}\ln\left[4\,\mathrm{e}^{-2|x-x_0|}\left(\cosh|x-x_0|-\cos(y+y_0)\right)\left(\cosh|x-x_0|-\cos(y-y_0)\right)\right].\end{aligned} \tag{8}$$

Eq. (8) represents one of the results of the current work. It can be shown easily that, at $n>>k$, the terms of the infinite series are of the order $O\left((k/n)^2\right)$, and the series converges quickly at any $|x-x_0|$. In addition, the Green's function singularity at $|x-x_0|=|y-y_0|=0$ is incorporated into the third term, which can be shown to have the logarithmic singularity. Therefore, the Green's function in the form of Eq. (8) can be integrated without difficulty over the domain, containing acoustic sources, if the observation point also belongs to this domain. These advantages of the Green's function so obtained are utilised below in calculating the pressure field of the piston, mounted to a duct wall.

4 PRESSURE FIELD OF THE PISTON

On the basis of the general three-dimensional formula, derived by Doak [1], Huang [2] obtained formulae for the pressure field of a plane rigid piston, shown in Figure 1 of this paper. If the frequency is below the first cut-on, so that only zero order mode propagates through the duct, the pressure field in the area directly above the piston is determined as follows [2]:

$$\frac{P\left(|x|<d/2,y\right)}{\rho cV}=-\frac{1}{\mathrm{i}\,k\pi}\left[1-\mathrm{e}^{\mathrm{i}kd/2}\cos kx\right]-\frac{2\,\mathrm{i}\,k}{\pi}\sum_{n=1}^{\infty}\frac{1}{\left|g_n\right|^2}\cos ny\left[1-\mathrm{e}^{-\left|g_n\right|d/2}\cosh\left(\left|g_n\right|x\right)\right], \quad (9)$$

On the other hand, substitution of the Green's function of the duct in the form of Eq. (8) to Eq. (4) gives the following alternative equation determining the pressure field, generated by the piston:

$$\begin{aligned}\frac{P\left(|x|<d/2,y\right)}{\rho cV}=\frac{\mathrm{i}}{k\pi}\left[1-\mathrm{e}^{\mathrm{i}kd/2}\cos kx\right]+\frac{\mathrm{i}\,k}{4\pi}\int_{-d/2}^{d/2}\ln\left[4\,\mathrm{e}^{-2|x-x_0|}\left(\cosh\left|x-x_0\right|-\cos y\right)^2\right]dx_0-\\ \frac{2\,\mathrm{i}\,k}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{\left|g_n\right|^2}\left[1-\mathrm{e}^{-\left|g_n\right|d/2}\cosh\left(\left|g_n\right|x\right)\right]-\frac{1}{n^2}\left[1-\mathrm{e}^{-nd/2}\cosh nx\right]\right\}\cos ny.\end{aligned} \quad (10)$$

In addition to Eq. (8), Eq. (10) also represents a result of the current work. The advantages of Eq. (10) are demonstrated below in numerical experiments.

5 NUMERICAL EXPERIMENTS

5.1 Parameters

The pressure field of the piston, shown in Figure 1, has been calculated numerically for the following parameters. The piston length, d, is considered to be equal to the duct width, D. The non-dimensional wavenumber, k=0.5, is chosen to be below its value, k=1, for the first cut-on frequency. The region of interest in this paper is the area of the duct above the piston.

5.2 Convergence of the series

An important issue in such calculations is the criterion for achieving the convergence of the series. In the present work, the convergence criterion is formulated as follows. A series, determined by

$$S=\sum_{n=1}^{\infty}S_n, \quad (11)$$

is considered to have converged at some $n=N_{\max}$, if any of the following two conditions are satisfied for $N=N_{\max}-2$, $N=N_{\max}-1$, and $N=N_{\max}$:

$$\left|S_N\right| \le \varepsilon \left|\sum_{n=1}^{N} S_n\right|, \quad \left|S_N\right| \le \delta. \tag{12}$$

Here the convergence errors ε and δ are small positive numbers satisfying the condition $0<\delta<\varepsilon \ll 1$.

5.3 Accuracy of the calculations of the pressure field in the duct

The calculations are carried out at 200 points across the duct at $x=0$ for two values of the convergence error $\varepsilon=10^{-3}$ and $\varepsilon=10^{-6}$ and for $\delta=10^{-12}$.

Figure 2 represents the dependence of parameters characterising the convergence of the series in Eqs. (9) and (10) on the normalised vertical coordinate, *y/D*, at *x*=0. The number of terms, $N_{\max}$ required to achieve the convergence of the series in Eqs. (9) and (10) is shown in Figures 2a and 2b. It can be seen clearly, that the convergence of the series in Eq. (10) obtained here requires only several terms for both values of the convergence error, *ε*. On the other hand, for the traditional Eq. (9), the number of terms depends strongly on ε and reaches nearly 10^3 for $\varepsilon=10^{-6}$.

Let the accuracy of the calculations of the pressure field be described by the amplitude and phase errors, ΔA and $\Delta\varphi$. Let the errors be defined with respect to a reference pressure, $\overline{P}_{ref}$, as follows:

$$\Delta A = \left|\frac{\left|\overline{P}\right|}{\left|\overline{P}_{ref}\right|}-1\right|, \quad \Delta\varphi = \left|\arg\left(\overline{P}\right)-\arg\left(\overline{P}_{ref}\right)\right|. \tag{13}$$

Figures 2c and 2d represent ΔA, whereas $\Delta\varphi$ is shown in Figures 2e and 2f. Values of $\overline{P}$, calculated by Eq. (9) for the convergence error $\varepsilon=10^{-9}$, are taken as the reference pressure, as both Eq. (9) and Eq. (10) give results tending to these values with decreasing ε.

Analysis of these Figures shows that the accuracy of determining $\overline{P}$ by means of Eq. (10) is significantly better than that by Eq. (9) with the same criterion of convergence. Indeed, at $\varepsilon=10^{-3}$ both errors for Eq. (9) for most values of y/D are between 1 and 2 orders of magnitude larger than those for Eq. (10). At $\varepsilon=10^{-6}$, the errors for Eq. (9) are, on average, only marginally smaller than the errors for Eq. (10), but this result is achieved by taking into account significantly larger number of terms in the series in comparison with Eq. (10) (Figures 2a and 2b).

The pressure on the surface of the piston is of special interest as its knowledge is necessary for the calculation of the piston impedance that determines the radiated sound power. It is clear from Figures 2c, 2d, 2e, and 2f that the accuracy of calculating the non-dimensional pressure $\overline{P}$ at *y*=0 is between 2 and 3 orders of magnitude worse for the traditional Eq. (9) than for Eq. (10) derived here. Therefore, Eq. (10) appears to be more suitable for use in solving problems of sound generation and noise control in ducts.

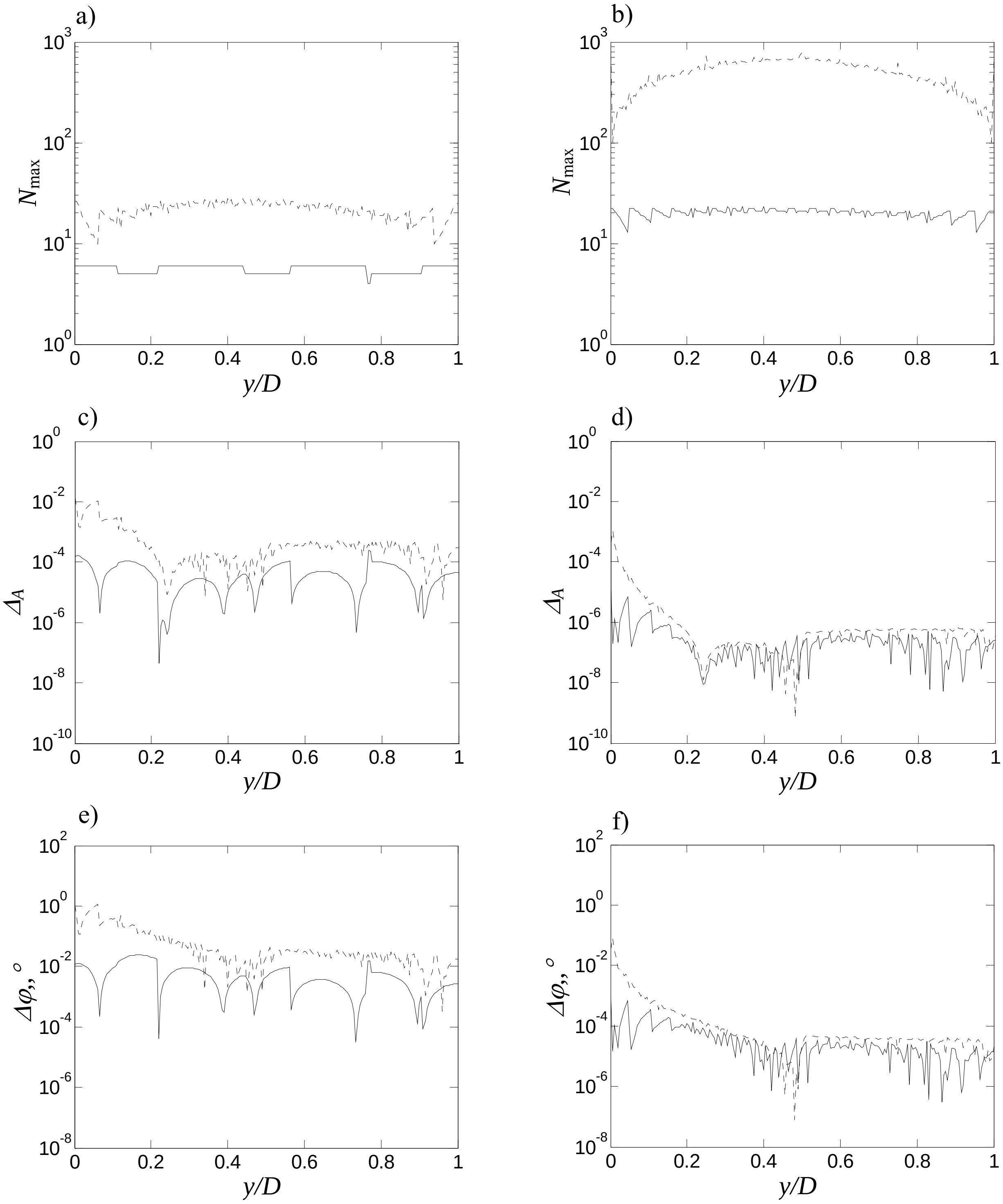

Figure 2. Results of the numerical experiments versus the normalised y at x=0. a,b) The number of terms required to achieve the convergence of the series; c,d) amplitude error; e,f) phase error; left: $\varepsilon = 10^{-3}$; right: $\varepsilon = 10^{-6}$; dotted line: Eq. (9); solid line: Eq. (9).

6 CONCLUSIONS

In the present work, the calculation of the pressure field generated by a vibrating piston mounted to a wall of a two-dimensional duct has been considered. To calculate the pressure field, a novel expression for the Green's function of the duct has been derived. The expression so obtained differs from the known expression for the duct Green's function by the presence of an asymptotic term, which takes account of all higher-order evanescent duct modes up to infinite order. The other part of the novel expression for the Green's function is an infinite series, which converges quickly.

The Green's function derived in this work has been used to obtain a formula for the pressure field generated by the piston. The numerical experiments show that the formula obtained here has a significant advantage as compared with the known formula. Whereas the latter can produce reliable results only if hundreds of the duct modes are taken into consideration, the convergence of the former can be achieved with only several terms of the series. The formula derived here can be considered to be more accurate than the traditional one as it leads to smaller amplitude and phase errors if the same convergence criterion is used.

The numerical experiments also show that the convergence errors for the new formula on the surface of the piston are two to three orders of magnitude smaller than the errors for the traditional one. As the pressure on the piston surface is used to calculate the piston impedance to predict the radiated sound, the new formula can be recommended for use in solving sound radiation and noise control problems in ducts.

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