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INFLUENCE OF A RECTANGULAR PLATE INHOMOGEINITY ON NATURAL FREQUENCIES OF A CLAMPED RECTANGULAR PLATE

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Abstract

Natural frequencies and mode shapes of rectangular and circular plates are well studied as they are common for many engineering applications. It is possible to utilize different approximate methods. However implementation of such methods in case of a distributed structural inhomogeneity frequently leads to very complex procedure.

The goal of the present work is to find comparatively simple solution for clamped rectangular plates with rectangular segment placed on its surface. Such sort of design is very important for installation of equipment on deck and floor. Influence of the inhomogeneity on the plate fundamental frequency is explored in the paper.

An analytical solution is developed for calculation of fundamental frequency of a clamped rectangular plate. The solution is gathered in a dimensionless form as a function of the plate dimensions and material properties.

Influence of an added plinth- like structure on the plate fundamental frequency is considered utilizing two approaches. First one is based on the modal stiffness and mass concept. Another one employs the perturbation method technique. Simulation runs were performed to verify accuracy of the developed methods. Deviation between fundamental frequencies determined by the derived formulas and FEM calculations is within reasonable limits. However formula based on the perturbation approach gives more accurate result. Nonetheless any of the proposed methods can be employed in engineering practice for approximate calculations to determine influence of plinth inhomogeinity on the fundamental frequency of the clamped rectangular plate.

1. INTRODUCTION

Fundamental natural frequency of a rectangular or circular plate is of particular interest for many structural dynamic and sound insulation tasks.

Rayleigh-Ritz, Galerkin and many other methods can be employed to get solution for the natural frequencies of regular plates at different boundary conditions [1-3]. It is also possible to utilize approximate methods to calculating the plate natural properties with lumped attached mass or stiffness reinforcement. However implementation of such methods in the case of distributed structural inhomogeneity leads to very complex solution procedure. Moreover accuracy of such solution could be non satisfactory.

Goal of the present work is to find out comparatively simple solution for clamped rectangular plates (Fig.1) with rectangular segment placed on its surface. Such sort of design is very important for installation of equipment on deck and floors. Influence of the inhomogeneity on the plate fundamental frequency is explored in the paper.

2. NATURAL FREQUENCIES OF A CLAMPED RECTANGULAR PLATE

Approaches to find out natural frequencies of clamped rectangular plates were presented in several papers [3-6]. Results of work [3] are most frequently utilized in engineering calculations to get natural frequencies of a rectangular plate with different side length ratio [7].

The method is based on the assumed mode shape and implementation of Galerkin method. Galerkin method is applied to gather approximate solution of equation describing the plate vibration by one of the natural mode shapes [5]:

$$\begin{gathered} D\nabla^4 W_i(X,Y) - \rho\omega^2 W_i(X,Y) = 0, \\ D = \frac{EH^3}{12(1-\nu^2)}, \end{gathered} \tag{1}$$

where $\boldsymbol{D}$ is the cylindrical stiffness of the plate, $\boldsymbol{E}$ is Young modulus of the plate material, $\boldsymbol{H}$- plate thickness, ν- Poisson ratio, ρ- material density, $W_i(x,y)$- presumed mode shape of ***i-th*** natural frequency, X,Y- Cartesian coordinates (see Fig.1).

Paper [5] contains tables for calculation of natural frequencies depending on the rectangular length/width ratio. However it would be useful to get analytical expressions for the fundamental natural frequency calculation at arbitrary lengths ratio.

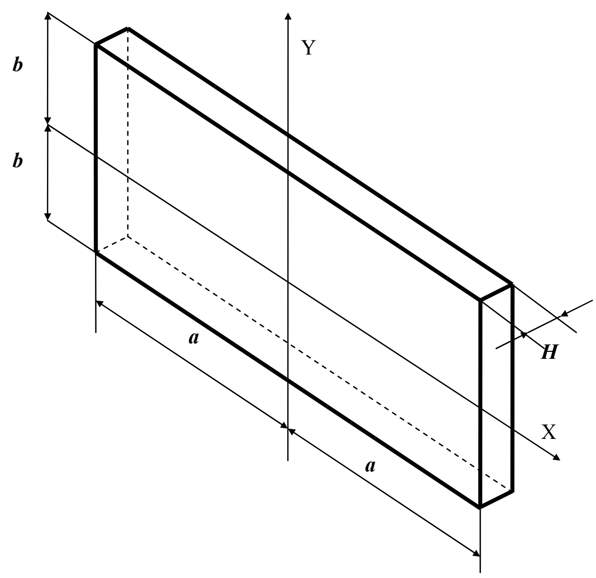

Figure 1. Dimensions of the clamped plate

3. ANALYTICAL SOLUTION FOR THE FUNDAMENTAL FREQUENCY OF A CLAMPED RECTANGULAR PLATE

Presumed mode shape for the plate fundamental frequency, which corresponds to clamped boundary conditions, can be expressed as follows [3]:

$$W_1(X,Y) = (ab)^{-7/2}(X^2 - a^2)^2(Y^2 - b^2)^2, \tag{2}$$

where a and b are linear dimensions of the rectangular plate(Fig.1). For the purpose of convenience let us introduce dimensionless parameters:

$$\begin{aligned} &X = x\sqrt{ab}, Y = y\sqrt{ab}, r = \sqrt{\frac{a}{b}}, W = w\sqrt{ab}, \\ &H = h\sqrt{ab}, r = \sqrt{\frac{a}{b}}, \omega^2 = \theta^2 \frac{E}{\rho ab}. \end{aligned} \tag{3}$$

Then equation (1) can be represented in dimensionless form:

$$\begin{aligned} &\overline{D}\nabla^4 w_1(x,y) - h\theta^2 w_1(x,y) = 0, \\ &\overline{D} = \frac{h^3}{12(1-\nu^2)}. \end{aligned} \tag{4}$$

Solution of equation (4) with respect to the fundamental frequency can be found by Rayleigh formula. It can also be interpreted as ratio of the modal (generalized) stiffness and modal mass operators which can be expressed as follows [7, 8]:

$$\begin{aligned} &\theta_1^{\,2} = c_1 / m_1, \\ &c_1 = \int_{-r}^{r}\int_{-1/r}^{1/r} \overline{D}[(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2})^2 - 2(1-\nu)(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - (\frac{\partial^2 w}{\partial x \partial y})^2)]dxdy, \\ &m_1 = \int_{-r}^{r}\int_{-1/r}^{1/r} hw^2(x,y)dxdy. \end{aligned} \tag{5}$$

Expression for c_1 is derived from the strain energy of plates at bending deformation. It can be simplified for clamped plates [7, 9]:

$$c_1 = \int_{-r}^{r}\int_{-1/r}^{1/r} \overline{D}(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2})^2 dxdy. \tag{6}$$

Taking into account that the dimensionless mode shape is:

$$w_1(x,y) = (x^2 - r^2)^2(y^2 - r^{-2})^2, \tag{7}$$

it is possible to derive an expression for the dimensionless fundamental frequency:

$$
\begin{aligned}
c_1 &= \overline{D}\,\frac{32768}{11025}\,\frac{(7+4r^4+7r^8)}{r^4},\\
m_1 &= h\,\frac{65536}{99225},\\
\theta_1^2 &= \frac{9\overline{D}}{2h}\left(4+7(r^4+r^{-4})\right).
\end{aligned}
\tag{8}
$$

Thus the dimensionless fundamental frequency can be represented as a function of 2 geometrical parameters *i.e.* dimensionless plate thickness $\boldsymbol{h}$ and ratio of the rectangular dimensions $\boldsymbol{r}$. The angular fundamental frequency can be obtained by equation (3). Comparison of the fundamental frequencies of clamped rectangular plates with results of FEM modeling indicates high coincidence of the magnitudes.

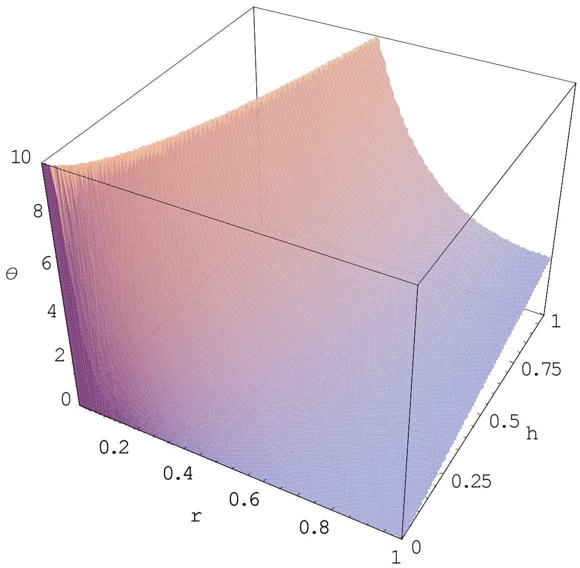

Figure 2. Dimensionless fundamental frequency for clamped rectangular plates with Poisson ratio 0.3

4. INFLUENCE OF RECTANGULAR INHOMOGENEITY ON THE FUNDAMENTAL FREQUENCY

4.1 Calculation of the Fundamental Frequency Based on Change of the Modal Mass and Stiffness

Assuming that dimensions and mass of the attached plinth (Fig.3) are comparable with those of the basic structure, it is impossible to employ a lumped approach for estimation of the natural properties. Replacement of the plinth by the attached point mass and stiffness can bring severe inaccuracy into the fundamental mode calculation. Therefore it is necessary to employ methods for continuous systems [7-9].

Let us consider plinth with coordinates $\boldsymbol{L_1}$, $\boldsymbol{L_2}$, $\boldsymbol{S_1}$, $\boldsymbol{S_2}$ and thickness $\boldsymbol{H_p}$ (Fig.3). Material of the plinth is the same as for the base structure. Dimensionless parameters of the structure are $\boldsymbol{l_1}$, $\boldsymbol{l_2}$, $\boldsymbol{s_1}$, $\boldsymbol{s_2}$ and $\boldsymbol{h_p}$ respectively. It is assumed that fundamental mode shape does not noticeably change after adding the plinth structure. Thus formula (5) can be employed for estimation of modal stiffness and mass of the modified structure with the same assumed fundamental mode function (7).

To evaluate integrals (5) for the modified structure it is more convenient to present them as follows:

$$c_{1m} = \int_{-r}^{r}\int_{-1/r}^{1/r} (\overline{D} + \overline{D}_p)[(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2})^2 - 2(1-\nu)(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - (\frac{\partial^2 w}{\partial x \partial y})^2)]dxdy,$$

$$m_{1m} = \int_{-r}^{r}\int_{-1/r}^{1/r} (h + h_p) w^2(x,y) dxdy, \tag{9}$$

$$\overline{D}_p = \frac{h_p(3h^2 + 3hh_p + h_p^2)}{12(1-\nu^2)}, h_p = \begin{cases} h, -l \le x \le l, -s \le y \le s \\ 0, -l > x > l, -s > y > s \end{cases},$$

where c_{1m} is the modal stiffness and m_{1m} is the modal mass of the modified structure.
Representing integrals in equation (9) as the sum of already known integrals and additional members it is possible to write:

$$\theta_{1m}{}^2 = (c_1 + c_{1p})/(m_1 + m_{1p}),$$

$$c_{1p} = \overline{D}_p \int_{l_1}^{l_2}\int_{s_1}^{s_2} [(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2})^2 - 2(1-\nu)(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - (\frac{\partial^2 w}{\partial x \partial y})^2)]dxdy, \tag{10}$$

$$m_{1p} = h_p \int_{l_1}^{l_2}\int_{s_1}^{s_2} w^2(x,y) dxdy,$$

where θ_{1m} is the dimensionless fundamental frequency of the modified structure, c_{1p} is the added modal stiffness due to the plinth and m_{1p} is the added modal mass. Analytical solution of equations (10) can be found. It is not represented here because the expression is too cumbersome.

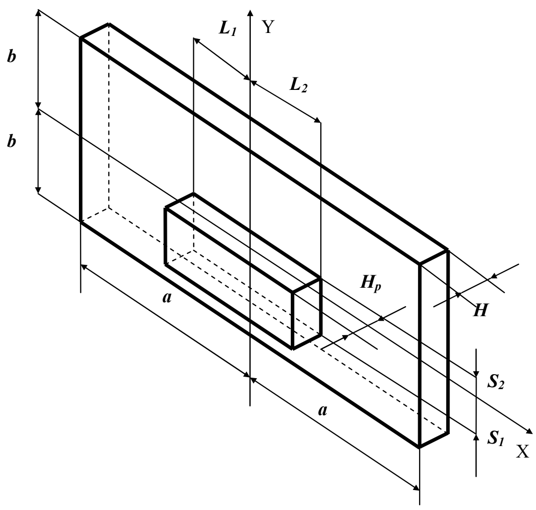

Figure 3. Dimensions of the plate with rectangular inhomogeinity

Equations (10) represent a straightforward way to evaluate influence of plinth- like structure on the fundamental frequency of the clamped rectangular plate. It can be also employed for inhomogeneity of other shape. However comparison of fundamental frequencies defined by this approach with FEM modeling indicates some discrepancy between results. Therefore it is needed to find out another approach that can bring more accurate solution.

4.2 Implementation of Perturbation Method for Evaluation of the Fundamental Frequency for the Rectangular Plate with Plinth Structure.

As previously, it is assumed that the original modal stiffness and mass of the clamped rectangular plate is changed as a result of the modification by the plinth structure which does not introduce significant distortion to the mode shape. Let us assume that original modal stiffness and mass is modified by the modal parameters of the attached plinth scaled by factor μ:

$$\begin{aligned} \mu c_{1p} &= c_{1m} - c_1, \\ \mu m_{1p} &= m_{1m} - m_1. \end{aligned} \tag{11}$$

Then the fundamental frequency of the resultant system can be expanded as the polynomial series:

$$\theta_{1m}^2 = \theta_1^2 + \mu^2\theta_{1p}^2 + \mu^4\theta_{1p}^4 + \ldots, \tag{12}$$

where the disturbance term θ_{1p} depends on change of the original fundamental frequency due to stiffness/mass of the attached plinth structure:

$$\theta_{1p}^2 = \frac{c_{1p} - \theta_1^2 m_{1p}}{m_1}. \tag{13}$$

It is possible to gather formula for the fundamental frequency of the modified structure keeping only two members in the series given by equation (12) and taking into account formulas (5), (11) and (13):

$$\begin{aligned} &\theta_{1m}^2 = \alpha\theta_{1a}^2, \\ &\theta_{1a}^2 = \frac{c_{1p}}{m_{1p}}, \alpha = 1 + 0.5\left(\sqrt{4\frac{m_{1p}}{m_1} + 1} - 1\right)\left(\frac{c_1}{c_{1p}} - \frac{m_1}{m_{1p}}\right). \end{aligned} \tag{14}$$

It should be noted that the system of equations gives 3 non-zero solutions with respect to unknowns θ_{1m} ,μ,θ_{1p} ,c_{1m} , m_{1m} . Equation (14) corresponds to non-zero solution for all of the variables and increase of the modal mass.

As it can be seen from equation (14), 'virtual' plinth fundamental frequency θ_{1a}^2 defined by ratio c_{1p} /m_{1p} is scaled by factor α which depends on ratio of the added and original modal mass and ratio of the added and original modal stiffness. Fig.4 represents behaviour of the multiplication factor α depending on the ratios of the modal parameters. Condition $\frac{c_1}{c_{1p}} = \frac{m_1}{m_{1p}}$ corresponds to special case when the plinth structure does not change the fundamental frequency of the original structure. It is equivalent to condition $\theta_{1m} = \theta_{1a} = \theta_1$.

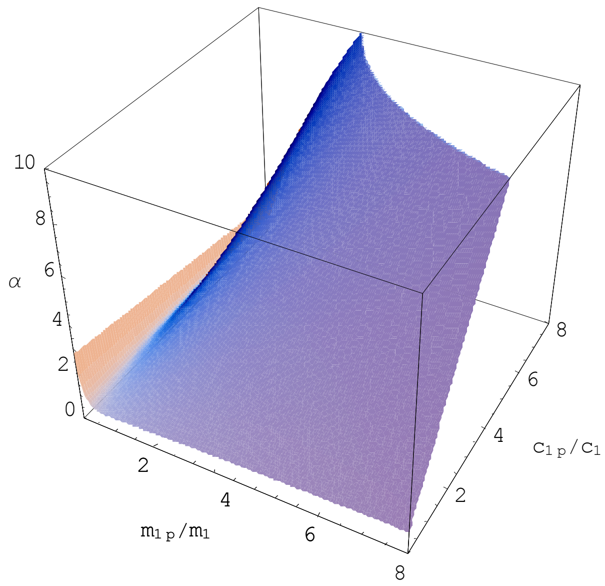

Figure 4. Dependence of the frequency multiplier α on added mass and stiffness for clamped rectangular plates

5. EXAMPLES OF THE FUNDAMENTAL FREQUENCY CALCULATION

The derived formulas can be used for engineering calculations to identify change in the fundamental frequency due to a plinth-like structures. Such inhomogeneities are common in construction practice and shipbuilding.

Table 1 contains natural frequencies of steel clamped plate with dimensions ***1x2x0.01m***. Material properties are: $\boldsymbol{E=2\cdot 10^{11}}$ ***Pa,*** $\boldsymbol{\nu=0.29}$, $\boldsymbol{\rho=7850kg/m^3}$**.** Coordinates of the rectangular inhomogenity of 0.008m thickness can be found in Table 1. The summarized results are compared with FEM calculations (ANSYS software). It should be noted that both FEM simulation and formula (8) give the same result of 59.7Hz for the fundamental frequency of the plate without plinth.

Calculations are also performed for concrete plates with dimensions **5x3x0.1m**. Quasi elastic behavior is assumed in the latest case with equivalent material properties $\boldsymbol{E=2.3\cdot 10^{10}}$ ***Pa,*** $\boldsymbol{\nu=0.15}$, $\boldsymbol{\rho=2400kg/m^3}$**.** The plinth has thickness 0.08m, coordinates are represented in Table 2. Fundamental frequency of the homogeneous plate is 41.2Hz as found by the FEM simulation and 41.5Hz if calculated by expression (8).

Table 1. Natural frequencies of the steel plate with attached plinth, Hz

Calculation	**Plinth coordinates, m**				
	L_1=-0.4 L_2=0.4 S_1=-0.25 S_2=0.25	L_1=0.1 L_2=0.9 S_1=-0.1 S_2=0.4	L_1=0.1 L_2=0.9 S_1=-0.25 S_2=0.25	L_1=-0.4 L_2=0.4 S_1=-0.1 S_2=0.4	L_1=-1 L_2=1 S_1=-0.25 S_2=0.25
FEM	68.8	63.3	64.9	63.3	73.8
Formula (10)	75.6	70.7	70.6	75.6	80.2
Formula (14)	72.6	69.3	69.1	72.8	76.2

Table 2 shows good coincidence of FEM results and calculations by formulas (10) and (17). Evaluation by the second expression is more accurate. Discrepancy between analytical calculations by formula (14) and finite element modeling does not exceed 10% for majority of considered cases.

Table 2. Natural frequencies of the concrete plate with attached plinth

Calculation	**Plinth coordinates, m**				
	L_1=-1 L_2=1 S_1=-0.75 S_2=0.75	L_1=0.25 L_2=2.25 S_1=-0.3 S_2=1.2	L_1=0.25 L_2=2.25 S_1=-0.75 S_2=0.75	L_1=-1 L_2=1 S_1=-0.3 S_2=1.2	L_1=-2.5 L_2=2.5 S_1=-0.75 S_2=0.75
FEM	68.8	63.3	64.9	63.3	73.8
Formula (10)	75.6	70.7	70.6	75.6	80.2
Formula (14)	72.6	69.3	69.1	72.8	76.2

6. SUMMARY

Analytical solution for calculation of the fundamental frequency of a clamped rectangular plate has been developed. The solution is gathered in a dimensionless form as a function of the plate dimensions and material properties.

Influence of an added plinth- like structure on the plate fundamental frequency is considered utilizing two approaches. First one is based on the modal stiffness and mass concept. The second method employs the perturbation technique. It is possible to get analytical solution for the modified structure in both of the cases.

Set of simulation runs was performed to verify accuracy of the developed methods. Deviation between fundamental frequencies determined by the derived formulas and FEM calculations is within reasonable limits. However formula based on the perturbation approach gives more accurate result if FEM calculation is taken as reference. Any of the proposed methods can be employed in engineering practice for approximate calculations to determine influence of plinth inhomogeinity on the fundamental frequency of the clamped rectangular plate.

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