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WAVE-BASED DISCONTINUOUS GALERKIN METHODS IN THE FREQUENCY DOMAIN WITH APPLICATION TO FLOW ACOUSTICS

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Abstract

Prediction of aircraft noise relies on the simulation of noise propagation through air flows. Computational techniques for flow acoustics (or aeroacoustics) have been mainly based on finite element methods for the linear full potential theory, or on finite difference schemes and Discontinuous Galerkin methods for the linearized Euler equations in the time domain. However, currently available numerical methods are difficult to apply to full-scale realistic configurations. For instance predictions of noise radiation from aircraft engines including the scattering by the pylon and the wing are still too expensive to be carried out systematically for design purposes. This paper presents a novel discontinuous Galerkin method in the frequency domain which uses local plane wave solutions of the problem at hand to approximate the solutions. This departs significantly from previous computational schemes based on polynomial interpolation techniques. The method is formulated for the linearized Euler equations and is therefore able to deal with very general mean flow configurations. The dispersion relation of the linearized Euler equations is used to discretize the solution and the trial function, it also forms the basis for the numerical flux splitting. Simple validation results of the wave-based discontinuous Galerkin method will be presented in order to illustrate the accuracy of the method. Examples of realistic applications will also be presented. In particular the problem of noise propagating through jets will be considered.

1. INTRODUCTION

Standard numerical methods used for wave propagation problems include finite element methods and finite difference schemes which are well-proven computational methods for relatively low frequencies. As frequency increases, the pollution error requires the use of very fine meshes to compensate for the accumulation of dispersion error over several wavelengths. For high frequency problems, one can use approximate techniques such as geometrical acoustics or statistical energy analysis. But there is an intermediate region, the mid-frequency regime, where neither approaches is adequate for large scale realistic applications.

Several approaches have been developed to alleviate the pollution error. One of these is the use of physics-based methods where the numerical model is devised so as to include some a priori information on the problem at hand (such as the discontinuity and tip singularity of a crack propagating in a solid, or the dispersion properties of waves in acoustics or electromagnetism). Generally this a priori information is embodied in a series of local solutions of the equations at hand which are used to build local approximations of the global solution. For wave propagation problems, these local solutions can either be plane waves or Green's functions.

The partition of unity method developed by Melenk and Babuška represents a simple way of including local solutions in the approximation basis [1]. At each node a set of local solutions is combined with standard finite element shape functions to yield a conforming approximation of the problem. With the discontinuous enrichment method, the standard finite elements shape functions are supplemented with a set of local solutions in each elements. The continuity of the solutions is then recovered by means of Lagrange multipliers at the interface between elements [2]. A variant of this approach is the discontinuous Galerkin method with Lagrange multipliers devised by Farhat *et al.* [3]. Other examples of discontinuous approximations include the ultra-weak variational formulation where continuity conditions are weakly imposed at the interfaces between elements [4], and the least square method where the discontinuity between elements is minimized [5].

This paper presents a discontinuous Galerkin method using a plane-wave basis for the approximation in the elements and numerical fluxes for the continuity of the solution. This method is presented for a general hyperbolic system of linear conservation equations. It is then applied to flow acoustic by solving the linearized Euler equations. Only the main features of the method are presented here, a more thorough description can be found in reference [6] together with details of validation and applications.

2. DESCRIPTION OF THE NUMERICAL METHOD

Consider an hyperbolic system of linear conservation equations in two dimensions:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}(\mathbf{A}\mathbf{u}) + \frac{\partial}{\partial y}(\mathbf{B}\mathbf{u}) = \mathbf{0} \,, \tag{1}$$

where $\mathbf{u}$ denotes the conserved quantities and the coefficient matrices $\mathbf{A}$ and $\mathbf{B}$ are square and independent of $\mathbf{u}$. We consider time-harmonic problems with a time dependence given by $e^{-i\omega t}$, therefore in equation (1) the time derivative $\partial\mathbf{u}/\partial t$ can be substituted by $-i\omega\mathbf{u}$.

The computational domain is decomposed into a set of elements $\{\Omega_e\}_{e=1,\ldots,N_e}$. The numerical approximation is discontinuous across element edges. Therefore, the system of conservation equations (1) is rewritten as the following variational statement:

$$\sum_e \int_{\Omega_e} -i\omega\mathbf{v}^T\mathbf{u} - \frac{\partial\mathbf{v}^T}{\partial y}\mathbf{A}\mathbf{u} - \frac{\partial\mathbf{v}^T}{\partial y}\mathbf{B}\mathbf{u}\,\mathrm{d}\Omega + \int_{\partial\Omega} \mathbf{v}^T\mathbf{F}\mathbf{u}\,\mathrm{d}\Gamma \\ + \sum_e \sum_{e'<e} \int_{\Gamma_{e,e'}} (\mathbf{v}^T\mathbf{F}\mathbf{u})_e + (\mathbf{v}^T\mathbf{F}\mathbf{u})_{e'}\,\mathrm{d}\Gamma = 0 \,, \quad \forall\mathbf{v} \,, \tag{2}$$

where $\mathbf{v}$ denotes the trial function and $\Gamma_{e,e'}$ is the edge between the elements e and e' with a unit normal $\mathbf{n}$ pointing outside element e. The flux matrix is defined as $\mathbf{F} = \mathbf{A}n_x + \mathbf{B}n_y$ and the product $\mathbf{Fu}$ represents the flux of the conserved quantities across a boundary with normal $\mathbf{n}$. The superscript ^{T} denotes the Hermitian transpose. The quantities $(\mathbf{v}^T\mathbf{Fu})_e$ and $(\mathbf{v}^T\mathbf{Fu})_{e'}$ are evaluated in the elements e and e' respectively.

2.1. Plane wave basis

In each element, the coefficient matrices are assumed to be constant (i.e. a piecewise constant approximation is used for $\mathbf{A}$ and $\mathbf{B}$) and a set of plane waves is used to approximate the solution. One seeks plane wave solutions $\mathbf{u} = \mathbf{r}\exp(ikx\cos\theta + iky\sin\theta)$ with amplitude $\mathbf{r}$, direction θ and wavenumber k. For a given direction θ, one is left with an eigenvalue problem

$$\mathbf{Er} = \lambda\mathbf{r} \ , \quad \text{with } \mathbf{E} = \mathbf{A}\cos\theta + \mathbf{B}\sin\theta \quad \text{and } \lambda = \frac{\omega}{k} \ , \tag{3}$$

where the phase speed λ is the eigenvalue and $\mathbf{r}$ is the eigenvector. This eigenvalue problem is in fact the dispersion relation of the system of equations at hand (i.e. it describe exactly the wave properties in terms of frequency, wavenumber and coefficients $\mathbf{A}$ and $\mathbf{B}$). For a given problem specified by $\mathbf{A}$ and $\mathbf{B}$, one can solved explicitly equation (3) and obtain a set of plane wave solutions, each wave corresponding to a certain type of solutions (such as S-wave and P-wave for linear elasticity, or acoustic and vortical waves for flow acoustics).

To build an approximation of the solution in each element using these plane waves, it is necessary to choose a set of wave directions $\{\theta_n\}_{n=1,\ldots,N_w}$ where N_w is the number of plane waves. Then the numerical solution is given by

$$\mathbf{u}(\mathbf{x}) = \sum_{n=1}^{N_w} a_n\mathbf{U}_n \exp(ik_n\boldsymbol{\theta}_n \cdot \mathbf{x}) \ , \quad \text{on } \Omega_e \ , \tag{4}$$

where $\mathbf{U}_n$ and k_n are obtained from the eigenvalue problem and $\boldsymbol{\theta}_n$ denotes the unit vector with direction θ_n. The degrees of freedom of this numerical model are the wave amplitudes a_n. The discontinuous Galerkin formulation is particularly flexible in terms of the interpolation used in each element. For instance, the number of plane waves and their directions can be chosen independently in each element.

2.2. The trial functions

The trial functions are also discretized using plane waves. But instead of using solutions of equation (1), it is useful to consider the adjoint equation to the problem. With uniform coefficients $\mathbf{A}$ and $\mathbf{B}$, the adjoint problem is written as follows

$$i\omega\mathbf{v} - \mathbf{A}^T\frac{\partial\mathbf{v}}{\partial x} - \mathbf{B}^T\frac{\partial\mathbf{v}}{\partial y} = \mathbf{0} \ . \tag{5}$$

Plane wave solutions of the form $\mathbf{v} = \mathbf{l}\exp(ikx\cos\theta + iky\sin\theta)$ yields the following adjoint eigenvalue problem:

$$\mathbf{E}^T\mathbf{l} = \lambda\mathbf{l} \ , \quad \text{with } \mathbf{E} = \mathbf{A}\cos\theta + \mathbf{B}\sin\theta \ , \quad \text{and } \lambda = \frac{\omega}{k} \ . \tag{6}$$

It is worth noting that the eigenvalues λ are the same as for the direct eigenvalue problem. In fact $\mathbf{r}$ and $\mathbf{l}^T$ are the right and left eigenvectors of the matrix $\mathbf{E}$. The plane-wave basis for $\mathbf{v}$ is built by choosing a set of wave direction $\{\theta_n\}_{n=1,\ldots,N_w}$. Following equation (4), the trial function $\mathbf{v}$ is then written as a sum of plane waves:

$$\mathbf{v}(\mathbf{x}) = \sum_{m=1}^{N_w} b_m \mathbf{V}_m \exp(ik_m \boldsymbol{\theta}_m \cdot \mathbf{x}) \ , \quad \text{on } \Omega_e \ . \tag{7}$$

2.3. Flux vector splitting

For the unknown $\mathbf{u}$ to be conserved, the flux $\mathbf{Fu}$ across the element edges should be conserved as well. This implies that we can write $(\mathbf{Fu})_e = -(\mathbf{Fu})_{e'} = \mathbf{f}_{e,e'}(\mathbf{u}_e, \mathbf{u}_{e'})$ where $\mathbf{f}$ is a numerical flux. In the present paper the standard upwind flux vector splitting is used. The flux matrix can be written as $\mathbf{F} = \mathbf{W}\boldsymbol{\Lambda}\mathbf{W}^{-1}$ where $\mathbf{W}$ is the matrix of eigenvectors of $\mathbf{F}$ and $\boldsymbol{\Lambda}$ is the diagonal matrix of eigenvalues. With non-uniform coefficients, the flux matrices calculated on both sides of an edge $\Gamma_{e,e'}$ are different. The numerical flux is built by combining information from both sides of the edge:

$$\mathbf{f}_{e,e'}(\mathbf{u}_e, \mathbf{u}_{e'}) = \mathbf{F}^+_{e,e'}\mathbf{u}_e + \mathbf{F}^-_{e,e'}\mathbf{u}_{e'} \ ,$$

with $\mathbf{F}^\pm_{e,e'} = \mathbf{W}_{e,e'}\boldsymbol{\Lambda}^\pm_{e,e'}\mathbf{W}^{-1}_{e,e'}$. The diagonal matrices $\boldsymbol{\Lambda}^+_{e,e'}$ and $\boldsymbol{\Lambda}^-_{e,e'}$ contains only the positive eigenvalues of $\mathbf{F}_e$ or the negative eigenvalues of $\mathbf{F}_{e'}$, respectively. The lines of the matrix $\mathbf{W}^{-1}_{e,e'}$ are taken from $\mathbf{W}^{-1}_e$ for positive eigenvalues and from $\mathbf{W}^{-1}_{e'}$ for negative eigenvalues.

It is particularly interesting to note that the eigenvalues and eigenvectors of $\mathbf{F}$ are solution of the following equation

$$\mathbf{Fw} = \lambda\mathbf{w} \ , \quad \text{with } \mathbf{F} = \mathbf{A}n_x + \mathbf{B}n_y \ .$$

By comparing this expression with the dispersion relation (3), it appears that the upwind flux-vector splitting technique is based on the dispersion relation with a direction θ corresponding to the normal of the edge. In other words it is based on the one-dimensional characteristics of the hyperbolic system of equations along the normal of the edge.

2.4. Application to the linearized Euler equations

The linearized Euler equations represent the propagation of linear disturbances on a steady base flow. For two-dimensional problems with constant and uniform entropy, these equations correspond to the following definitions:

$$\mathbf{u} = \begin{bmatrix} \rho' \\ (\rho u)' \\ (\rho v)' \end{bmatrix} \ , \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ c_0^2 - u_0^2 & 2u_0 & 0 \\ -u_0 v_0 & v_0 & u_0 \end{bmatrix} \ , \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ -u_0 v_0 & v_0 & u_0 \\ c_0^2 - v_0^2 & 0 & 2v_0 \end{bmatrix} \ , \tag{8}$$

where ρ_0 denotes the mean density, $\mathbf{v}_0 = (u_0, v_0)^T$ the velocity, and c_0 the sound speed. The components of $\mathbf{u}$ represent respectively the linear perturbations of density ρ' and momentum $(\rho u)'$, $(\rho v)'$. When the eigenvalue problems (3) and (6) are solved, two different families of plane waves are found, corresponding to acoustic waves and vortical waves. N_a acoustic waves are used and defined by

$$k_n = \frac{\omega}{\mathbf{v}_0 \cdot \boldsymbol{\theta}_n + c_0} \;, \quad \mathbf{U}_n = \begin{bmatrix} 1 \\ u_0 + c_0 \cos\theta_n \\ v_0 + c_0 \sin\theta_n \end{bmatrix} \;, \quad \text{with } 1 \leqslant n \leqslant N_a \;. \tag{9}$$

And N_h vorticity waves are used

$$k_n = \frac{\omega}{\mathbf{v}_0 \cdot \boldsymbol{\theta}_n} \;, \quad \mathbf{U}_n = \begin{bmatrix} 0 \\ -c_0 \sin\theta_n \\ c_0 \cos\theta_n \end{bmatrix} \;, \quad \text{with } N_a + 1 \leqslant n \leqslant N_w \;. \tag{10}$$

3. COMPARISON WITH HIGH-ORDER FINITE DIFFERENCE SCHEMES

To illustrate the performance of the wave-based discontinuous Galerkin method, it can be tested against a standard finite difference scheme used in computational aeroacoustics. The dispersion-relation-preserving schemes are high-order stencils which are particularly well-suited for wave propagation problems since they are optimized to minimize the dispersion error [7].

The present test case is the propagation of a single plane wave on a square computational domain of width 1. An unstructured mesh is used for the wave-based discontinuous Galerkin method (228 elements, 135 nodes, element size 0.1) with 12 acoustic waves and 7 vortical waves per element. A 38×38 uniform cartesian grid is used for the DRP schemes. The number of degrees of freedom in both methods is exactly the same. The base flow is uniform and parallel to the x axis with a Mach number $M = 0.5$. The density and sound speed are $\rho_0 = 1$ and $c_0 = 1$. The numerical error is defined as $\|\mathbf{u} - \mathbf{u}_{ex}\|_{L_2(\Omega)} / \|\mathbf{u}_{ex}\|_{L_2(\Omega)}$ where $\mathbf{u}_{ex}$ is the exact solution.

Figure 1 shows the anisotropy of the numerical models for an acoustic wave and a vorticity wave by plotting the numerical error as a function of the wave direction θ. The wave-based discontinuous Galerkin method yield results that are at least one order of magnitude more accurate with the same number of degrees of freedom.

Figure 2 shows the convergence of the two methods for a fixed mesh when the frequency is increased. Again, both for acoustic and vortical waves, the wave-based discontinuous Galerkin method is significantly more accurate than the DRP scheme. And it should be noted that the discontinuous Galerkin method can readily use unstructured meshes whereas DRP schemes rely on structured grids which cannot accommodate complex geometries easily.

It is well known that the conditioning of wave-based numerical methods can be an issue. The conditioning of the wave-based DGM is shown in figure 3 where the condition number is plotted against the number of degrees of freedom per wavelength for different numbers of plane waves. Both for acoustic and vorticity waves the rate of increase of the condition number is controlled by the number of plane waves. With large numbers of plane waves the condition number can be above 10^{15}. For Helmholtz problem, it was found that the conditioning of the wave-based DGM is similar to that of the ultra weak variational formulation (in fact for the Helmholtz equation with uniform coefficients the two methods are equivalent, see [6]). Also shown on figure 3 is the conditioning of the algebraic system after applying the pre-conditioner used in references [4, 8]. This pre-conditioner reduces significantly the condition number of the method and represents an efficient way to reduce the conditioning of wave-based methods.

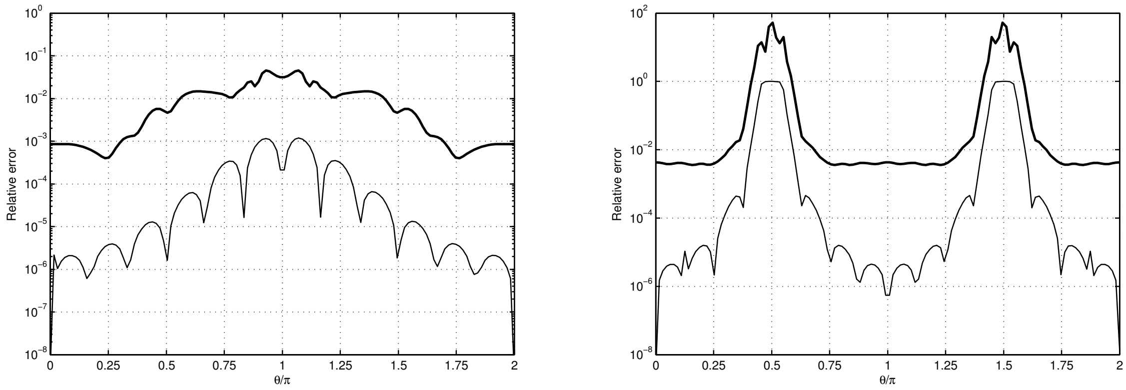

Figure 1. Anisotropy of the wave-based DGM (thin lines) and the DRP finite difference scheme (thick lines). Left: acoustic wave with $\omega = 20$; Right: hydrodynamic wave with $\omega = 10$.

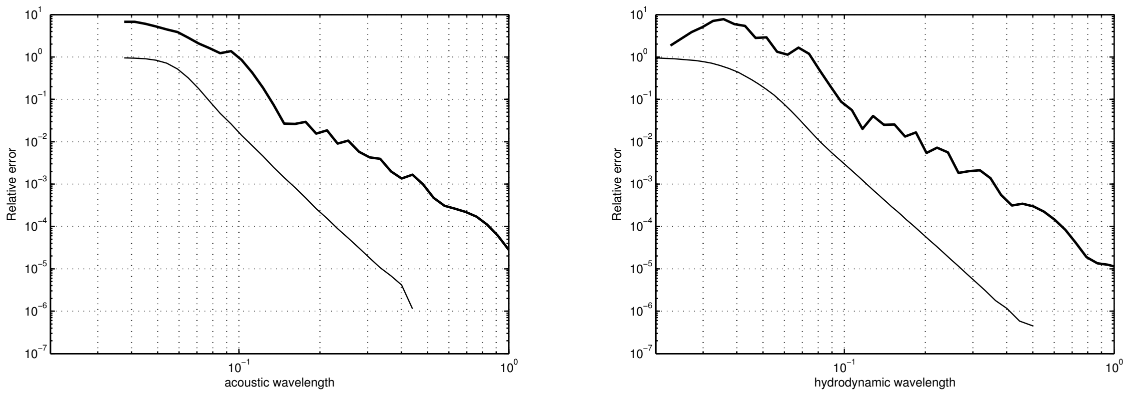

Figure 2. Convergence of the wave-based DGM (thin lines) and the DRP finite difference scheme (thick lines). Left: acoustic wave; Right: hydrodynamic wave.

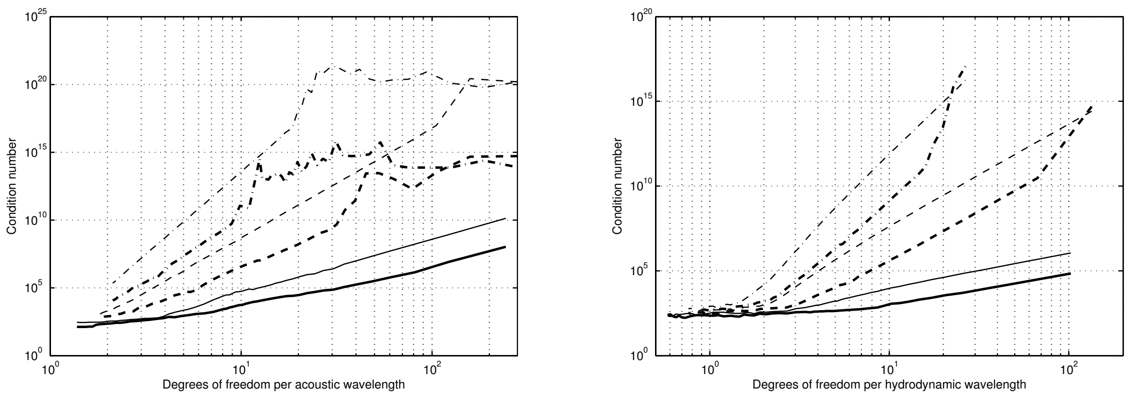

Figure 3. Influence of the pre-conditioner for acoustic waves (left) and hydrodynamic waves (right). Thin lines: no pre-conditioner; Thick lines: with pre-conditioner. Left: $N_h = 3$ (solid line), $N_h = 5$ (dashed line), $N_h = 7$ (dot-dashed line). Right: $N_a = 6$ (solid line), $N_a = 10$ (dashed line), $N_a = 14$ (dot-dashed line).

4. APPLICATION TO JET NOISE

When sound propagates through non-uniform flows, such as the mixing layer of a jet, sound is refracted by the mean flow gradient. To obtain accurate prediction of aeroacoustic noise propagation, it is crucial to describe this refraction effect accurately. As an example, the radiation of sound from a point source embedded in a two-dimensional parallel jet is now presented. In this problem the flow is parallel to, and constant in, the x direction with a velocity profile given by

$$u_0(y) = u_\infty + (u_{jet} - u_\infty)e^{-\log(2)(y/b)^2} \ ,$$

where u_∞ denotes the free stream velocity, u_{jet} is the velocity on the centerline of the jet and b denotes the width of the jet. Here we choose the parameters $u_\infty = 0.2$, $u_{jet} = 0.5$ and $b = 0.14$. The sound speed and mean density are uniform with $c_0 = 1$ and $\rho_0 = 1$. This test case is similar to the benchmark problem for computational aeroacoustics devised by Agarwal *et al.* [9].

For this simulation 14 acoustic waves and 5 vorticity waves are used in each element. The mesh is shown in figure 4 together with the numerical solution for the real part of pressure. Two important effects are well captured by the wave-based discontinuous Galerkin method. First the convective effect of the base flow reduces or increases the acoustic wavelength when the wave is propagating against or with the flow, respectively. As a consequence, in the region upstream of the point source the wavelength is particularly small and there is less than two elements per wavelength in this region of the computational domain. Secondly, the refraction effect introduced by the mean velocity gradient in the jet is also well captured with the presence of a 'cone of silence' in the region downstream of the source where the pressure amplitude is significantly reduced.

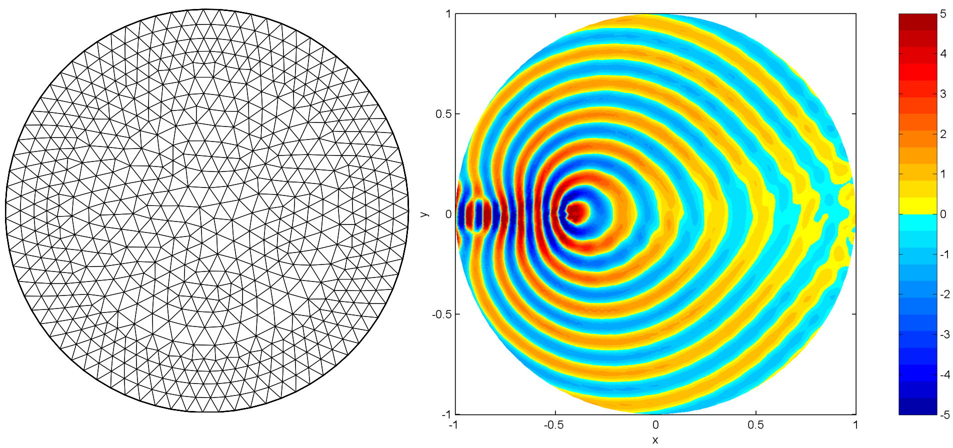

Figure 4. Left: mesh used for the jet simulation. Right: real part of pressure for the noise radiated by a point source in a jet.

5. CONCLUSIONS

The wave-based DGM owes its high-level of accuracy to the systematic use of the dispersion relation of the exact conservation equations. This dispersion relation is built into the plane wave basis (4) for the solution, it is used also for the trial function and for the flux vector splitting method. As for other wave-based numerical methods the conditioning of the algebraic system is high but comparable to that of the ultra weak variational formulation for instance. The conditioning can also be improved by means of a pre-conditioner. A comparison with DRP schemes shows that the wave-based DGM is significantly more accurate than high-order finite difference schemes.

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