

# ON THE USE OF A HIGH-ORDER FINITE ELEMENT - WAVE BASED METHOD FOR INTERIOR ACOUSTIC CAVITY ANALYSIS

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## Abstract

This paper discusses the use of wave based prediction methods for the analysis of steady-state interior acoustic problems. Conventional element based prediction methods, such as the finite element method (FEM), are commonly used, but are restricted to low-frequency applications. The wave based method (WBM) is an alternative deterministic technique which is based on the indirect Trefftz approach. The WBM is computationally very efficient, allowing the analysis of problems at higher frequencies. The efficiency of the WBM is most pronounced for problems of moderate geometrical complexity. For the analysis of problems with a more complex geometry, a hybrid finite element-wave based method is developed. This hybrid approach combines the strengths of the two methods, namely, the high computational efficiency of the WBM and the ability of the FEM to model problems of arbitrary geometrical complexity. Up till now, only low-order FE models have been coupled with WB models. This paper discusses the application of more accurate high-order FE schemes in the hybrid FE-WB approach. The performance of the resulting high-order hybrid method will be illustrated by means of a numerical validation example.

# 1. INTRODUCTION

Deterministic element-based methods, like the finite element method (FEM), are generally accepted for steady-state interior cavity analysis. Due to the approximating nature of the applied shape functions, element sizes need to decrease with raising frequency in order to maintain a reasonable prediction accuracy [1]. As a result, numerical model sizes grow with frequency and become prohibitively large at high frequencies [2]. However, due to the discretization into small elements, the FEM is able to model problems of arbitrary geometry.

In recent years, a Wave Based Method (WBM) [3] has been developed for steady-state acoustic cavity analysis. This method is based on an indirect Trefftz approach [4], in that the dynamic pressure response is expanded in terms of wave functions, which are exact solutions

of the governing dynamic equation. In this way, the unknown wave function contribution factors in the pressure response expansion are merely determined by the evaluation of boundary and continuity conditions. This results in small numerical models, which exhibit an enhanced computational efficiency as compared to the element-based methods, allowing the method to tackle problems also in the mid-, and sometimes even high-, frequency range [5]. However, in order to fully benefit from the enhanced computational efficiency of the WBM, the geometrical complexity of the considered problem should be moderate.

A hybrid approach combines the strong points of both the FEM and the WBM by using the WBM for modelling large, homogeneous problem subdomains, while the FEM is applied to model the geometrically more complex regions [6]. This hybrid FE-WB method exhibits similar high performance characteristics as the WBM, but overcomes the limitation of the moderate geometrical complexity. Up till now, only first-order FE models have been coupled with WB models [7]. This paper discusses the application of more accurate high-order FE schemes in the hybrid FE-WB approach.

# 2. PROBLEM DEFINITION

Consider a steady-state interior acoustic problem, as shown in figure 1. A closed boundary surrounds a bounded fluid domain V, which is characterized by its speed of sound c and its ambient fluid density  $\rho_0$ . The fluid domain is excited by an acoustic volume velocity point source q at circular frequency  $\omega$ . The time-harmonic pressure response is given by  $p(\mathbf{r},t) = p(\mathbf{r},\omega)e^{j\omega t}$  with  $\mathbf{r} = [x y z]^T$  the position vector, T the transpose operator, j the imaginary unit  $\sqrt{-1}$  and with t denoting the time. From here onwards, the steady-state solution  $p(\mathbf{r},\omega)$  is abbreviated as  $p(\mathbf{r})$ .



Figure 1. An interior acoustic problem

Figure 2. Acoustic domain partitioning

Assuming that the system is linear, the fluid is inviscid and the process is adiabatic, p(r) is governed by the Helmholtz equation

$$\nabla^2 p(\mathbf{r}) + k^2 p(\mathbf{r}) = -j\rho_0 \omega \delta(\mathbf{r}, \mathbf{r}_q) q , \qquad (1)$$

with  $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$  the Laplacian operator,  $k = \omega/c$  the acoustic wave number and with  $\delta$  a Dirac-delta function.

The boundary of the considered acoustic problem domain V is denoted as  $\partial V = \Omega$  and consists of three parts:  $\Omega = \Omega_v \cup \Omega_Z \cup \Omega_p$  imposing, respectively, predefined normal velocity values  $\bar{v}_n(\mathbf{r})$ , predefined normal impedance values  $\bar{Z}_n(\mathbf{r})$  and predefined pressure values  $\bar{p}(\mathbf{r})$ :

$$\boldsymbol{r} \in \Omega_{v} : \mathcal{L}_{v}(p(\boldsymbol{r})) = \bar{v}_{n}(\boldsymbol{r}) ,$$
  

$$\boldsymbol{r} \in \Omega_{Z} : \mathcal{L}_{v}(p(\boldsymbol{r})) = p(\boldsymbol{r})/\bar{Z}_{n}(\boldsymbol{r}) ,$$
  

$$\boldsymbol{r} \in \Omega_{p} : p(\boldsymbol{r}) = \bar{p}(\boldsymbol{r}) .$$
(2)

 $\mathcal{L}_v$  represents the normal velocity operator  $\mathcal{L}_v = \frac{j}{\rho_0 \omega} \frac{\partial}{\partial n}$  with  $\frac{\partial}{\partial n} = \mathbf{n}^T \nabla$  the derivative in the normal direction.  $\mathbf{n} = [n_x n_y n_z]^T$  is the vector normal to the fluid domain V and  $\nabla = \left[\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z}\right]^T$  is the gradient vector.

Together with the associated boundary conditions (2), the Helmholtz equation (1) defines a unique pressure field  $p(\mathbf{r})$ .

### 3. THE WAVE BASED METHOD

The wave based method (WBM) [3] is a deterministic prediction technique for the analysis of steady-state interior acoustic problems. In general, two major steps are distinguished in the WB modelling procedure.

#### 3.1. Wave function selection

Due to the non-convexity of a general bounded acoustic problem domain V, see figure 2, an initial partitioning into  $N_V$  non-overlapping, convex subdomains is required  $V = \bigcup_{\alpha=1}^{N_V} V^{(\alpha)}$ . Subsequently, continuity conditions must be applied at the resulting interfaces  $\Omega_I^{(\alpha,\beta)} = \Omega_I^{(\beta,\alpha)} = \partial V^{(\alpha)} \cap \partial V^{(\beta)}$  in order to ensure continuity of the solution at the interfaces [7].

In contrast with the FEM, the WBM describes the field variables as an expansion of wave functions which exactly satisfy the governing differential equations. In this way, only an approximation error is introduced at the boundaries and the interfaces. The steady-state acoustic pressure field  $p^{(\alpha)}(\mathbf{r})$  in acoustic subdomain  $V^{(\alpha)}$  is approximated as solution expansion  $\hat{p}^{(\alpha)}(\mathbf{r})$ 

$$p^{(\alpha)}(\boldsymbol{r}) \simeq \hat{p}^{(\alpha)}(\boldsymbol{r}) = \sum_{w=1}^{n_w^{(\alpha)}} p_w^{(\alpha)} \Phi_w^{(\alpha)}(\boldsymbol{r}) + \hat{p}_q^{(\alpha)}(\boldsymbol{r}) = \boldsymbol{\Phi}^{(\alpha)}(\boldsymbol{r}) \ \boldsymbol{p}_w^{(\alpha)} + \hat{p}_q^{(\alpha)}(\boldsymbol{r}) \ . \tag{3}$$

The wave function contributions  $p_w^{(\alpha)}$  are the weighting factors for each of the selected wave functions  $\Phi_w^{(\alpha)}(\mathbf{r})$ . Together they form the  $(n_w^{(\alpha)} \times 1)$  vector of degrees of freedom  $\mathbf{p}_w^{(\alpha)}$ . The corresponding known wave functions are collected in the  $(1 \times n_w^{(\alpha)})$  row vector  $\boldsymbol{\Phi}^{(\alpha)}$ . Each acoustic wave function exactly satisfies the homogeneous Helmholtz equation

$$\Phi_{w}^{(\alpha)}(\boldsymbol{r}(x,y,z)) = \begin{cases} \Phi_{w_{r}}^{(\alpha)}(x,y,z) = \cos(k_{xw_{r}}^{(\alpha)}x) \cos(k_{yw_{r}}^{(\alpha)}y) e^{-jk_{zw_{r}}^{(\alpha)}z} \\ \Phi_{w_{s}}^{(\alpha)}(x,y,z) = \cos(k_{xw_{s}}^{(\alpha)}x) e^{-jk_{yw_{s}}^{(\alpha)}y} \cos(k_{zw_{s}}^{(\alpha)}z) & . \end{cases}$$

$$\Phi_{w_{t}}^{(\alpha)}(x,y,z) = e^{-jk_{xw_{t}}^{(\alpha)}x} \cos(k_{yw_{t}}^{(\alpha)}y) \cos(k_{zw_{t}}^{(\alpha)}z)$$

$$(4)$$

with wave number components defined as

$$\begin{pmatrix} k_{xw_r}^{(\alpha)}, k_{yw_r}^{(\alpha)}, k_{zw_r}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} \frac{w_1^{(\alpha)}\pi}{L_x^{(\alpha)}}, \frac{w_2^{(\alpha)}\pi}{L_y^{(\alpha)}}, \pm \sqrt{k^2 - \left(k_{xw_r}^{(\alpha)}\right)^2 - \left(k_{yw_r}^{(\alpha)}\right)^2} \end{pmatrix}, \\ \begin{pmatrix} k_{xw_s}^{(\alpha)}, k_{yw_s}^{(\alpha)}, k_{zw_s}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} \frac{w_3^{(\alpha)}\pi}{L_x^{(\alpha)}}, \pm \sqrt{k^2 - \left(k_{xw_s}^{(\alpha)}\right)^2 - \left(k_{zw_s}^{(\alpha)}\right)^2}, \frac{w_4^{(\alpha)}\pi}{L_z^{(\alpha)}} \end{pmatrix}, \\ \begin{pmatrix} k_{xw_t}^{(\alpha)}, k_{yw_t}^{(\alpha)}, k_{zw_t}^{(\alpha)} \end{pmatrix} = \begin{pmatrix} \pm \sqrt{k^2 - \left(k_{yw_t}^{(\alpha)}\right)^2 - \left(k_{zw_t}^{(\alpha)}\right)^2}, \frac{w_{5(\alpha)}\pi}{L_{y(\alpha)}}, \frac{w_{6(\alpha)}\pi}{L_{z(\alpha)}} \end{pmatrix}, \end{cases}$$
(5)

with  $w_1^{(\alpha)}$ ,  $w_2^{(\alpha)}$ ,  $w_3^{(\alpha)}$ ,  $w_4^{(\alpha)}$ ,  $w_5^{(\alpha)}$  and  $w_6^{(\alpha)} = 0, 1, 2, \ldots$  The dimensions  $L_x^{(\alpha)}$ ,  $L_y^{(\alpha)}$  and  $L_z^{(\alpha)}$  represent the dimensions of the (smallest) bounding box, circumscribing the considered subdomain.

In (3),  $\hat{p}_q^{(\alpha)}$  represents a particular solution resulting from the acoustic source term  $q^{(\alpha)}$  in the inhomogeneous Helmholtz equation (1). The free-field solution of a point source is used [3].

#### 3.2. Wave based model construction and solution

With the use of the proposed pressure expansion (3), the Helmholtz equation (1) is always exactly satisfied, irrespective of the values of the  $n_W = \sum_{\alpha=1}^{N_V} n_w^{(\alpha)}$  unknown wave function contribution factors  $p_w$ . Due to the partitioning of the acoustic problem domain V into a number of  $N_V$  acoustic subdomains  $V^{(\alpha)}$ , with  $\alpha = 1 \dots N_V$ , continuity conditions along the subdomain interfaces  $\Omega_I^{(\alpha,\beta)}$  must be taken into account, in addition to the problem boundary conditions (2). The unknown wave function contribution factors  $p_w$  are merely determined by these boundary and continuity conditions.

Since both the boundary conditions and the continuity conditions are defined at an infinite number of boundary positions, while only finite sized prediction models are amenable to numerical implementation, the boundary and the continuity conditions are, for each subdomain, transformed into a weighted residual formulation. Combination of all formulations yields a square WB matrix equation, which is denoted in a condensed form as

$$A p_w = b . (6)$$

Solution of (6) and backsubstitution of the  $n_W$  wave function contribution factors  $p_w$  in the pressure expansions (3) yields an analytical description of the dynamic pressure field  $\hat{p}$  in all subdomains  $V^{(\alpha)}$ .

## 4. THE HYBRID FINITE ELEMENT - WAVE BASED METHOD

The hybrid finite element - wave based (FE-WB) method [6, 7] brings together the enhanced convergence properties of the WBM and the ability of the FEM to model any geometry, without restrictions on geometrical complexity. Applied to a general acoustic problem, large homogeneous acoustic domains are modelled with the WBM, while the geometrically more complex regions are tackled with the FEM. The saved computational resources, due to the enhanced convergence characteristics of the FE-WB method, as compared to the conventional FEM, can be used in a model refinement of the involved FE part. As a result, the refined hybrid models can be applied for predictions at higher frequencies, without loss of geometrical flexibility. Figure 3 illustrates the proposed modelling strategy for the case of an interior car cavity.

Consider an acoustic FE model consisting of  $n_{fe}$  nodal degrees of freedom (dofs) and an acoustic WB model consisting of  $n_W$  wave function contribution dofs, see figure 4. At the resulting interfaces between the FE and the WB model, continuity conditions are imposed along the common interface  $\Omega_H$ 

$$\begin{aligned} v_{fe}(\boldsymbol{r}) &= -v_w(\boldsymbol{r}) \\ p_w(\boldsymbol{r}) &= p_{fe}(\boldsymbol{r}) \end{aligned} . \tag{7}$$



Figure 3. Hybrid FE-WB modelling strategy

Figure 4. The fundamental notations for a hybrid FE-WB coupling

These continuity conditions result in additional terms in the weighted residual formulations of the FE and the WB model and link the two models together, yielding a matrix equation of the following form

$$\begin{bmatrix} S_{fe} & Q_{fw} \\ Q_{wf} & S_w \end{bmatrix} \begin{cases} p_{fe} \\ p_w \end{cases} = \begin{cases} s_{fe} \\ s_w \end{cases},$$
(8)

collecting  $n_{tot} = n_{fe} + n_W$  algebraic equations in the  $n_{fe}$  nodal FE dofs and the  $n_W$  wave function contribution dofs.

Up till now, only first-order FE parts have been considered in assembling hybrid models. However, the hybrid formulations (7) are not restricted to low-order models, but may also be applied for FE parts of higher order. The following section will discuss the application of highorder FE parts and discuss their performance.

## 5. NUMERICAL VALIDATION EXAMPLE

#### 5.1. Problem definition

Consider the non-convex three-dimensional cavity ( $V = 2.033m^3$ , outer dimensions  $L_x \simeq 3.5m$ ,  $L_x \simeq 2m$  and  $L_z \simeq 1m$ ), shown in figure 5. The cavity is an assembly of three trapezoidal volumes [7]. The system is excited by an acoustic point source located at point 17. The source is characterised by its volume velocity  $Q = \frac{4\pi}{j\rho_0 ck}$ . The cavity is filled with air c = 340 m/s,  $\rho_0 = 1.225 kg/m^3$  and all walls are acoustically rigid  $\bar{v}_n = 0 m/s$ . Inside the cavity, 1323 response points are defined, which are distributed equally throughout the whole cavity.





Figure 5. Non-convex simply shaped 3D cavity

Figure 6. Hybrid FE-WB model definition (two WB subdomains and one FE subdomain)

All the calculations are performed on a Centrino Intel Pentium M computer system (2.13GHz, 2Gb RAM) running a Windows XP-Professional operating system. The FE models are solved with MSC/Nastran2005. The WB and the hybrid models are solved with a C++ implementation of the involved routines.

# 5.2. Model descriptions

- **FEM:** Several linear 8-noded and quadratic 20-noded hexahedral FE meshes of the full acoustic cavity have been constructed. Model sizes vary from 1223 dofs up to 501809 dofs.
- WBM: Due to its non-convex shape, an initial partitioning into  $N_V = 3$  convex subdomains precedes the selection of the wave functions. Several WB models, applying from 72 up to 8406 wave functions, are constructed.
- **FE-WB method:** In order to apply the hybrid FE-WB method, two acoustic subdomains are modelled with the WBM, while the third subdomain is modelled with the FEM, see figure 6. 83% of the total cavity volume is occupied by the two WB subdomains. The remaining 17% is modelled with linear 8-noded and quadratic 20-noded hexahedral finite elements.

# 5.3. Numerical results

Figure 7 shows the pressure contour plot of a plane through the simply shaped 3D cavity at 400Hz, predicted with the hybrid FE-WB method. The applied model consists of 1956 wave functions and 10881 FE dofs resulting from 2304 20-noded quadratic hexahedral elements. The figure clearly illustrates that the continuity conditions between the different subdomains are accurately represented, since the pressure field is continuous over the subdomain interfaces. Also the rigid boundary conditions are taken correctly into account, since the pressure contour lines are perpendicular to the rigid walls.



Figure 7. Pressure contour plot (real part) predicted with the hybrid FE-WB method at 400Hz [Pa]

In order to characterize the computational efficiency of the hybrid FE-WBM as compared to the FEM and the WBM, a convergence analysis is performed. Figures 8, 9 and 10 show the relative prediction accuracy  $\epsilon$  with respect to the frequency dependent CPU time, at 200Hz, 400Hz and 700Hz, respectively.  $\epsilon$  is defined as the average of the relative prediction accuracies in the  $n_{\epsilon} = 1323$  response points inside the cavity. The most detailed quadratic FE model is used as reference model.

$$\epsilon = \sum_{j=1}^{n_{\epsilon}} \frac{\epsilon_j}{n_{\epsilon}} \quad \text{with} \quad \epsilon_j = \left| \frac{p_{prediction}[Pa] - p_{reference}[Pa]}{p_{reference}[Pa]} \right| \tag{9}$$

The hybrid FE-WBM results are represented by the solid lines with the *o* markers when the FE part applies linear 8-noded hexahedral elements and  $\triangle$  markers when the FE part applies quadratic 20-noded elements. The hybrid predictions are obtained using hybrid models with a fixed number of FE dofs (2873 for the linear models and 1589 for the quadratic models) and an increasing number of wave functions. The solid lines with the  $\Diamond$  markers represent pure WBM predictions with increasing number of wave functions. The solid lines with  $\nabla$  markers represent pure FEM predictions using quadratic elements, while the dashed lines with *o* markers are pure FEM predictions applying linear elements. Both curves are obtained by increasing the associated mesh density.

These figures illustrate that

- the prediction accuracy of the FEM (both linear and quadratic) deteriorates with increasing frequency.
- the prediction accuracy of the WBM is more or less frequency independent.
- at low frequencies, the quadratic FEM is most efficient, while, with increasing frequency, the WBM becomes more efficient.
- at 200Hz, both the linear and quadratic hybrid FE-WB models yield accurate results. Both convergence curves fall back onto the WBM convergence curve, illustrating that at



Figure 8. Pressure convergence curves at 200Hz



Figure 10. Pressure convergence curves at 700Hz



Figure 9. Pressure convergence curves at 400Hz



Figure 11. Pressure convergence curves at 700Hz, refined hybrid models

this low frequency, the accuracy of the WB part in the hybrid models governs the overall prediction accuracy.

- at 400Hz and 700Hz, the hybrid convergence curves stagnate due to the limited accuracy of the involved FE parts. This is illustrated by figure 11, which shows hybrid predictions resulting from more dense FE discretizations in the hybrid models, i.e. 9025 linear dofs and 10881 quadratic dofs. Comparison between figures 10 and 11 shows that the stagnation level decreases towards higher prediction accuracy.
- at 700Hz, the accuracy of the models involving linear FE fail to converge due to a too coarse element discretization.

# 6. CONCLUSIONS

This paper discusses the application of the hybrid FE-WB models, applying high-order FE submodels, for the analysis of steady-state interior acoustic problems. The hybrid approach combines the high computational efficiency of the WBM and the ability of the FEM to model problems of arbitrary geometrical complexity. Applied to an interior acoustic problem, large homogeneous acoustic domains are modelled with the WBM, while the geometrically more complex regions are tackled with the FEM.

A direct coupling approach between high-order FE submodels and WB submodels is discussed. A numerical validation example shows that, especially for higher frequencies, the hybrid FE-WB method becomes an efficient alternative for the conventional FEM.

Future research will study the applicability of the high-order FE-WB method for more complex problems and will investigate the use of indirect coupling approaches.

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