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# SOME NEW GENERAL RESULTS ON THE FLUID-STRUCTURE 

# INTERACTION PROBLEM 

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#### Abstract

An impedance-based approach is applied to the general fluid-structure interaction problem. For a passive finite-size lineally elastic body or structure of arbitrary configuration in fluid medium, the following new results are obtained. First, the range of allowable values of the sound power absorbed and scattered by the body is found, and a simple equation for the curve that bounds the range is derived. Second, explicit boundary conditions of the impedance type on the body surface are obtained for the limiting cases - for the most efficient absorber, the best scatterer, and for other bodies that possess extreme absorption and scattering properties. It is shown that the efficiency of the best absorber is several orders of magnitude higher than that of commonly used absorbers. Possible ways of achieving the extreme acoustic properties are discussed. Illustrative examples are presented.


## 1. INTRODUCTION

The problem of interaction of a vibrating elastic body or structure with surrounding fluid medium or, in other words, the problem of radiation, scattering, and absorption of sound by elastic bodies in fluid has been a subject of interest for many years. Despite the vast literature on the topic (it numbers, together with the corresponding works in electromagnetic theory, thousands of papers and dozens of monographs - see, e.g., [1-3]), there are still practically important problems that remain unsolved. These are, for example, the problems of the black body and the best absorber (that, among all bodies of the same configuration, absorbs maximum incident field energy), of the best scatterer and of other bodies that possess the limiting acoustic properties. Even these limiting properties themselves are unknown in most cases to say nothing about such questions as how to construct the most efficient absorbers and non-reflecting surfaces, do black bodies exist in Nature and so on.

The commonly used approach to treating the fluid-structure interaction problem consists in solving a set of differential equations governing the fluid and structure vibrations together with the conditions at all boundaries including the structure-fluid interface.

In this paper, a new impedance approach is used [4, 5]. It is based on describing the
subsystems (the medium and structure) by the special impedance characteristics (matrices) defined with respect to their interface. Three impedance matrices are needed here - one matrix describes vibrations of the structure and two matrices characterize the fluid vibrations. The full solution of the fluid-structure interaction problem is then written in terms of these three matrices. The solution thus obtained is compact, physically transparent and much simpler than those obtained by the commonly used methods. This allows one to further use it in treating more difficult problems mentioned above. Several such problems are already solved in $[4,5]$ - the variational problem of the best absorber, the problem of how to make an arbitrary body to be acoustically transparent, and others.

In the present paper, a new problem (of the constrained best absorber) is posed and solved. This variational problem is formulated as follows: among all bodies (structures) of a given configuration, find such that absorbs maximum incident sound power under condition that the scattered sound power is fixed. Analytical solution is obtained for this problem. It follows from the solution that the values of the sound power absorbed and scattered by a passive linearly elastic body cannot go beyond a certain bounded range. This range is graphically presented below and a simple equation for its boundary is given. Several particular cases of the constrained best absorber, including the most efficient absorber, are studied. For them, the surface impedances are explicitly derived and a way of their practical implementation is discussed. Illustrative examples are also presented.

## 2. BASIC EQUATIONS OF THE IMPEDANCE THEORY

The main assumption and basic equations of the impedance theory are the following.
Consider a finite-size elastic body (structure) of arbitrary geometry immersed in an acoustic fluid medium. The body occupies a volume $V$ and has an outer surface $A$ which is the interface with the fluid. The medium is not necessarily homogeneous and unbounded. It is deemed inviscid thus exerting only normal loads on the body surface. Both the body and fluid are assumed to obey linear differential equations. Time dependence of all field variables is assumed to be harmonic and exponential coefficient $\exp (-i \omega t)$ is abbreviated throughout the paper, $\omega$ being circular frequency.

In the fluid medium outside the body, some acoustic sources are present. In the absence of the body, they produce a pressure field with complex amplitude $p_{i}(x)$, which is called the incident field. In the presence of the body, the pressure field component $p_{s}(x)$ scattered by the body also appears. The total pressure field at a point with coordinates $x$ in the exterior of the body is, thus, represented by the sum of these two components (radiation is excluded),

$$
\begin{equation*}
p(x)=p_{i}(x)+p_{s}(x) \tag{1}
\end{equation*}
$$

The problem is stated as to determine the scattered field component $p_{s}(x)$ if given is the incident field component $p_{i}(x)$.

To introduce the needed impedance characteristics, it is convenient to represent the surface $A$ of the body as a set of $N$ small-size surface elements $\Delta A_{n}, n=1,2, \ldots, N$. The number $N$ is not fixed but the dimensions of the elements are assumed to be less than half of the fluid wavelength. The pressure, normal velocity and other field characteristics may then be taken as uniform within each element and the quantities continuously distributed over surface $A$ may be represented on $A$ as $N$-vectors. For example, the total pressure field, equation (1), and the corresponding fluid particle normal velocity $v(y)$ are represented on $A$
as the following $N$-vectors

$$
\begin{equation*}
\boldsymbol{p}=\left[p\left(y_{1}\right) \Delta A_{1}, \ldots, p\left(y_{N}\right) \Delta A_{N}\right]^{T}, \boldsymbol{v}=\left[v\left(y_{1}\right), \ldots, v\left(y_{N}\right)\right]^{T} . \tag{2}
\end{equation*}
$$

The components of vector $\boldsymbol{p}$ are the complex amplitudes of the forces acting on areas $\Delta A_{n}$ and vector $\boldsymbol{v}$ consist of $N$ complex amplitudes of the outward normal velocity, $y_{n}$ being the coordinates of a point of the surface element $\Delta A_{n}$. Similarly, $N$-vectors $\boldsymbol{p}_{\boldsymbol{i}}, \boldsymbol{v}_{\boldsymbol{i}}$ and $\boldsymbol{p}_{s}, \boldsymbol{v}_{s}$ are introduced to describe the incident and scattered fields on $A$. Relations between the pressure vectors and velocity vectors can be written via impedance Nx N -matrices.

Three such impedance matrices are needed for solving the scattering and absorption problem. One matrix, the body impedance matrix $\mathbf{Z}$, characterizes the scatterer. It is a matrix of the input surface impedances of the body in vacuo. Two impedance matrices, $\boldsymbol{Z}_{\boldsymbol{i}}$ and $\boldsymbol{Z}_{\boldsymbol{r}}$, are necessary to characterize the fluid. Matrix $\boldsymbol{Z}_{\boldsymbol{i}}$ of the internal impedances of the fluid is defined similarly to the body matrix $\boldsymbol{Z}$ and represents the matrix of the input surface impedances of isolated volume $V$ filled with fluid. Matrix $\boldsymbol{Z}_{\boldsymbol{r}}$ of the radiation impedances is defined as a matrix of the input surface impedances of the fluid in the exterior of surface $A$ with all the acoustic sources in it switched off. It can be shown that the scattered and incident field components satisfy the following equations

$$
\begin{equation*}
p_{i}+Z_{i} v_{i}=0, \quad p_{i}+p_{s}+Z\left(v_{i}+v_{s}\right)=0, p_{s}-Z_{r} v_{s}=0 . \tag{3}
\end{equation*}
$$

The three impedance matrices are assumed to be complex-valued and symmetric with respect to the main diagonal. Physically, the symmetry means that the reciprocity theorem is valid in the elastic body and fluid. For bodies of simple geometries in homogeneous unbounded fluid the matrices may be obtained analytically. In general case, they can be computed using one of the available numerical methods.

Let us introduce two square scattering matrices of order $N, \boldsymbol{S}$ and $\boldsymbol{Q}$, that relate the pressure and normal velocity of the scattered field on the interface $A$ correspondingly to the pressure and normal velocity of the incident field on $A$ in the form of linear equations

$$
\begin{equation*}
\boldsymbol{v}_{s}=\boldsymbol{Q} \boldsymbol{v}_{\boldsymbol{i}}, \quad \boldsymbol{p}_{s}=\boldsymbol{S} \boldsymbol{p}_{\boldsymbol{i}} \tag{4}
\end{equation*}
$$

Using relations (3) one can obtain the following equations for the scattering matrices

$$
\begin{equation*}
\boldsymbol{Q}=\left(\mathbf{Z}_{r}+\mathbf{Z}\right)^{-1}\left(\mathbf{Z}_{\boldsymbol{i}}-\mathbf{Z}\right), \quad \boldsymbol{S}=\left(\mathbf{Y}_{\boldsymbol{r}}+\boldsymbol{Y}\right)^{-1}\left(\mathbf{Y}_{\boldsymbol{i}}-\boldsymbol{Y}\right), \tag{5}
\end{equation*}
$$

where $\boldsymbol{I}$ is the identity matrix of order $N$ and the mobility matrices $\boldsymbol{Y}_{\boldsymbol{k}}$ are the inverse impedance matrices $\boldsymbol{Z}_{\boldsymbol{k}}, k=\varnothing, i, r$. Equations (5) express the scattering matrices through the impedance and mobility characteristics of the medium and scatterer. They give the solution to the scattering problem that is very similar, by the form and essence, to the well-known solution for the reflection coefficients obtained by C. Fresnel in the simplest case of plane waves reflecting from a plane interface between two fluid media. The proposed theory can, thus, be regarded as a straightforward extension of the Fresnel's theory to the general case of the scattering problem. However, in general case, three impedance (mobility) matrices are needed instead of two in the Fresnel's case. The reason is that, in the particular case of two connected fluid half-spaces, the internal and radiation impedances, $\boldsymbol{Z}_{\boldsymbol{i}}$ and $\boldsymbol{Z}_{r}$, coincide.

## 3. PARTICULAR CASES

In this section, particular cases of the impedance theory that are necessary for further consideration are briefly documented. More details can be found elsewhere [4, 5]. The first case is related to acoustically transparent (non-scattering) bodies.
It is apparent from equations (5) that a body of volume $V$ does not scatter sound and its scattering matrices are zero matrices, $\boldsymbol{Q}=\boldsymbol{S}=0$, if its matrix of the in vacuum surface impedances coincides with the matrix of the fluid internal impedances

$$
\begin{equation*}
Z=Z_{i} . \tag{6}
\end{equation*}
$$

It follows from equation (6) that it is impossible to construct a non-scattering passive coating that is made of a locally reacting material. It is because the fluid-filled volume $V$ is a vibratory system with low damping and high quality resonances, so that its matrix of the surface impedances, $\boldsymbol{Z}_{\boldsymbol{i}}$, is far from being diagonal (Note that the impedance matrix of a locally reacting system is always diagonal).

One way of implementing equality (6) in practically interesting cases is to use active structures, e.g. smart skins. Such an active structure should be of the global type in that the active force applied to a certain part of the body surface should be controlled by the field quantities measured at other parts of the surface. A global feedback control system of this type is described in paper [4].

The second important particular case is an impedance solution of the best absorber problem. The sound power entering the absorber is, by definition, equal to

$$
\begin{equation*}
\Phi=-\frac{1}{2} \operatorname{Re}\left(\boldsymbol{v}^{*} \boldsymbol{p}\right)=\frac{1}{2} \operatorname{Re}\left(\boldsymbol{v}^{*} \boldsymbol{Z} \boldsymbol{v}\right)=\frac{1}{2} \boldsymbol{v}^{*} \boldsymbol{R} \boldsymbol{v}=\frac{1}{2} \boldsymbol{v}_{\boldsymbol{i}}^{*}\left(\boldsymbol{I}+\boldsymbol{Q}^{*}\right) \boldsymbol{R}(\boldsymbol{I}+\boldsymbol{Q}) \boldsymbol{v}_{\boldsymbol{i}}, \tag{7}
\end{equation*}
$$

where $\boldsymbol{v}$ and $\boldsymbol{p}$ are the normal velocity and pressure vectors as defined in Eq.(2), $\boldsymbol{Z}=\boldsymbol{R}+\boldsymbol{i} \boldsymbol{X}$ is an in vacuo impedance symmetric matrix of the absorber, $\boldsymbol{R}$ and $\boldsymbol{X}$ being, correspondingly, the resistance and reactance matrices. The problem of the best absorber can be formulated as follows: find such an impedance matrix $\mathbf{Z}$ that renders maximum to $\Phi$ or, in other words, for which the Hermitian form (7) is stationary. Giving matrix $\mathbf{Z}$ a variation $\Delta \boldsymbol{Z}=\boldsymbol{\Delta}+\boldsymbol{i} \Delta \boldsymbol{X}, \boldsymbol{\Delta} \boldsymbol{R}$ and $\boldsymbol{\Delta X}$ being arbitrary symmetric real-valued matrices of small matrix norm, considering the matrices $\boldsymbol{\Delta} \boldsymbol{R}$ and $\boldsymbol{\Delta} \boldsymbol{X}$ as independent variations of $\boldsymbol{Z}$ and going through the matrix algebra, one can find that the absorber impedance matrix $\boldsymbol{Z}$ is equal to the Hermitian conjugate of the radiation impedance matrix:

$$
\begin{equation*}
\boldsymbol{Z}=\boldsymbol{Z}_{r}^{*} \quad \text { or } \quad \boldsymbol{R}=\boldsymbol{R}_{r}, \quad \boldsymbol{X}=-\boldsymbol{X}_{r} . \tag{8}
\end{equation*}
$$

By analyzing variations of the absorbed power in the vicinity of (8), it can be shown that this stationary value is the maximum value. A body with the surface impedance matrix (8) is, thus, the best absorber and, among variety of bodies of the same configuration, it absorbs maximum of the incident field energy. This maximum value is equal to

$$
\begin{equation*}
\Phi_{\max }=\frac{1}{8} \boldsymbol{p}_{i}^{*}\left(\boldsymbol{I}+\boldsymbol{Y}_{i}^{*} \mathbf{Z}_{r}^{*}\right) \boldsymbol{R}_{r}^{-1}\left(\boldsymbol{I}+\mathbf{Z}_{r} \boldsymbol{Y}_{i}\right) \boldsymbol{p}_{\boldsymbol{i}} . \tag{9}
\end{equation*}
$$

Some general conclusions can be drawn directly from equation (8). First, the properties of
the best absorber do not depend on the incident field and are fully determined by the acoustic environment, i.e. by the radiation impedances. Second, a body with a locally reacting surface cannot be the best absorber. It follows from the fact that the radiation impedance matrix is not diagonal in most real situations. The condition (8) also means that the best absorber should be of the resonant type: its reactances must compensate the reactances of the surrounding medium, $\boldsymbol{X}+\boldsymbol{X}_{r}=0$. As a result, the best absorber is at the same time a good scatterer: the absorbed power is equal to the scattered power. Besides, the resistances of the best absorber are equal to the radiation resistances. This is similar to the optimal load condition in electric circuit theory. Equation (8) can, thus, be interpreted as a matching condition of the absorber and acoustic environment. Note that the link between the amount of the absorbed power and the radiation impedance has been reported earlier, though is simplified 1-D form, in the acoustic literature [2, 6].

One of the rare examples a natural best absorber is a gas bubble in liquid,. The bubble is assumed to be a pulsating sphere of a small radius $a$ having one vibratory degree of freedom.


Fig.1. Sound power absorbed by the best spherical absorber of radius $a$ (solid line) and by a pulsating air-bubble in water (dashed lines). The curve 1 corresponds to the optimal internal damping of the bubble that is equal to the resonant radiation damping; the curves 2 and 3 correspond to internal damping, which is equal to 0.01 and 50 of the optimal value. The absorbed power is normalized with the incident power $\pi a^{2}\left|p_{i}\right|^{2} / 2 \rho c$

In Figure 1, the solid line corresponds to the relative absorption cross-section, i.e. the absorbed sound power of the best spherical absorber normalized with the incident power $\Phi_{0}=\pi a^{2}\left|p_{i}{ }^{2}\right| / 2 \rho c$. Other curves in Fig. 1 correspond to absorption of a gas bubble (more exactly, an air-bubble in water). In the frequency range of our concern, $x=k a$ is small and the reactance of the air-bubble is spring-controlled giving, together with mass of the entrained water, the natural frequency $x_{0}=0.014$. Dashed line $l$ in Fig. 1 corresponds to the particular bubble of radius $a=3.3 \mathrm{~mm}$ whose internal damping (due to heat transfer and viscosity) is equal to the damping due to radiation. In this case, both conditions (8) are satisfied and the air-bubble in water behaves like the true best absorber. If the bubble radius is smaller or greater than 3.3 mm the internal damping of the bubble becomes greater or smaller than the radiation damping at $x=x_{0}$, and the absorbed power decreases (dashed lines 2 and 3 in

Fig.1). Note that the relative cross-section of the corresponding matched sphere that has the specific impedance $\rho c$ is equal to four, i.e. three orders of magnitude smaller than that of the best absorber.

Similarly, the problem of the best scatterer can be solved. The impedance matrix of the best scatterer should be pure imaginary (the body is lossless) and close to minus radiation reactance matrix. The maximum possible scattered power of a passive body is four times the maximum absorbed power (9) - see details in [5].

## 4. CONSTRAINED BEST ABSORBER

Consider now the following variational problem: find such an impedance matrix $\mathbf{Z}$ that renders maximum to the absorbed power (7) if the scattered power

$$
\begin{equation*}
F=\frac{1}{2} \operatorname{Re}\left(\boldsymbol{v}_{s}^{*} \boldsymbol{p}_{s}\right)=\frac{1}{2} \boldsymbol{v}_{\boldsymbol{i}}^{*} \boldsymbol{Q}^{*} \boldsymbol{R}_{\boldsymbol{r}} \boldsymbol{Q} \boldsymbol{v}_{\boldsymbol{i}}, \tag{10}
\end{equation*}
$$

where $\boldsymbol{R}_{r}=\operatorname{Re}\left(\boldsymbol{Z}_{r}\right)$ is a matrix of the radiation resistances and $\boldsymbol{Q}$ is the scattering matrix given in equations (4), (5), is fixed and equals to $F_{0}$. The solution to this problem is obtained by the method of the Lagrange multipliers. Here are some results.

A solution to the problem exists not for all fixed values of the scattered power, but when it exists it is unique. Introducing the following notation for the normalized scattering and absorption powers

$$
f=\frac{F_{0}}{\Phi_{\max }}, \quad \varphi=\frac{\Phi}{\Phi_{\max }},
$$

where $\Phi_{\max }$ is the absolute maximum value of the absorbed power (9), it can be shown that the constrained best absorber should have the scattering matrix $\boldsymbol{Q}$ that is proportional to the scattering matrix $\boldsymbol{Q}_{0}$ of the best absorber

$$
\begin{equation*}
\boldsymbol{Q}=\sqrt{\boldsymbol{f}} \boldsymbol{Q}_{0} \tag{11}
\end{equation*}
$$

and the surface impedance matrix that is equal to

$$
\begin{equation*}
\mathbf{Z}=\left(\mathbf{Z}_{r}+\mathbf{Z}_{i}\right)\left(\boldsymbol{I}+\sqrt{\boldsymbol{f}} \boldsymbol{Q}_{0}\right)^{-1}-\mathbf{Z}_{\boldsymbol{r}} . \tag{12}
\end{equation*}
$$

The maximum value of the absorbed power relates to the fixed value of the scattered power according to the following simple equation

$$
\begin{equation*}
\varphi=2 \sqrt{f-f} . \tag{13}
\end{equation*}
$$

Let us consider more closely Eq.(13) and properties of the constrained best absorber.
One of the consequences of solution (11)-(13) is the existence of a finite range of allowable values for the absorption and scattering powers of passive bodies. In Fig. 2 this range is represented on the $(f, \varphi)$ plane. From below this range is bounded by the $f$-axis and from above - by the curve (13). Each point of the range corresponds to a body with a certain impedance matrix and the scattering and absorption powers represented by the coordinates of this point.

The plot shown in Fig. 2 is universal: it holds for passive bodies of any geometry at any frequency, while all the individual features of scatterers and absorbers, as well as those of the incident field, are enclosed in the value $\Phi_{\max }$ (Eq.(9)), which serves as a normalizing factor of the power value and is assumed to be finite.


Fig. 2. Allowable values of the scattered and absorbed power of a passive body normalized with the absolute maximum value of the absorbed power (9)

The plot in Fig. 2 reflects many general properties of scatterers and absorbers. In particular, it suggests that a finite-size body that absorbs sound energy but scatters nothing is impossible; i.e. if the scattered power is equal to zero, the absorbed power should also be zero. By contrast, there is an infinitely large number of bodies that can scatter without absorption; i.e. if $\varphi=0, f$ can take any value from 0 to 4 , which corresponds to a multitude of bodies with all of the possible reactive impedances. If the absorption power takes some fixed nonzero value within the interval $[0,1]$, the scattering power can take an infinite number of values.

The plot shown in Fig. 2 is useful for developing absorbers and scatterers with given physical properties. For example, if it is necessary to develop an efficient absorber, it is natural to construct it as the best absorber or a body close to it with impedances (8) and absorption power (9), which corresponds to the vicinity of the absolute maximum of absorption, i.e. the point with coordinates ( 1,1 ) in Fig.2. If an efficient scatterer is required, it can be found near the point (4, 0). If it is necessary to design non-scattering (non-reflecting) coating, it should be sought in the vicinity of the origin of coordinates, where the inequality $f \ll 1$ is valid. A characteristic feature of this region of Fig. 2 is that, here, the ratio of the absorbed power to the scattered power may take any values from zero to very large ones. Indeed, for bodies that correspond to the points on the $f$-axis in the Fig.2, this ratio is equal to zero, while for bodies corresponding to points of the boundary curve (13) it is expressed as

$$
\begin{equation*}
\frac{\Phi}{F}=\frac{\varphi}{f}=\frac{2}{\sqrt{f}}-1 \tag{14}
\end{equation*}
$$

and, depending on $f$, may be arbitrarily large. Thus, this region contains coatings with low scattering and virtually any ratio between the absorption and scattering powers. In particular, the coating with the greatest value of ratio (14) should have surface impedances (12) and scattering matrix (11).

## 5. IMPLEMENTATION OF THE RESULTS

Analysis shows that the efficiency of existing absorption and non-scattering means [6] is very far from the theoretical limiting values of the best sound absorber and other bodies with the extreme acoustic properties considered above. Hence, there is a real perspective to improve the absorption efficiency by implementing the above obtained surface impedances of the bodies using, for example, appropriate coatings.

The main feature of the bodies with extreme properties is a surface of global reaction. The response of such a surface to a point excitation is noticeable not only at the driving point (as in locally reacting surfaces) but at all other points. As a result, the surface impedance matrices (6), (8), (12) are fully populated. Implementation of such globally reacting surfaces is very difficult using either smart skins or passive coatings, since each two points of the surface should be coupled. However it is possible to approximate them with the help of passive surfaces of extended reaction. Such surface is thought as a set of discrete identical elements covering the body surface with appropriate coupling between neighbouring elements. The present author verified on simple examples (plane absorbers) that the absorption coefficient of a surface with extended reaction is not so high as that of the surface with global reaction (of the best absorber), but is considerably higher than the coefficients of locally reacting absorptive surfaces usually employed today. Besides, surfaces with extended reaction are rather simple and can be implemented using commercially available materials.

## 6. SUMMARY

A new variational problem of a constrained best absorber is formulated and analytically solved. Several new general results concerning vibrational fluid-structure interaction follow from the obtained solution. First, for any passive elastic structure, there exists a finite range of allowable values of its absorption and scattering powers. Second, explicit equations for surface impedances for the structures that possess limiting absorption and scattering properties are derived. A way of achieving such properties in practice is discussed.

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