

# COMPARISON OF HIGHER-ORDER NUMERICAL SCHEMES AND SEVERAL FILTERING METHODS APPLIED TO NAVIER-STOKES EQUATIONS WITH APPLICATIONS TO COMPUTATIONAL ACOUSTICS

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#### Abstract

In computational acoustics, fluid-acoustic coupling methods for the computation of sound have been widely used by researchers for the last five decades. In the first part of the coupling procedure, the fully unsteady incompressible or compressible flow equations for the near-field of the unsteady flow are solved by using a Computational Fluid Dynamics (CFD) technique, i.e., Direct Numerical Simulation (DNS), Large Eddy Simulation (LES) or unsteady Reynolds averaged Navier-Stokes equations (RANS); results of these simulations are then used to calculate sound sources using the acoustic analogy or by solving a set of acoustic perturbation equations (APE) leading to the solution of the acoustic field. It is possible to use a 2-D reduced problem to provide preliminary understanding of many acoustic problems. Unfortunately 2-D CFD simulations using a fine-mesh-small-time-step-LES-alike numerical method cannot be considered as LES which applies to 3-D simulations only. Therefore it is necessary to understand the similarities and the effect between filters applied to unsteady compressible Navier-Stokes equations and the combined effect of high-order schemes and mesh sizes. The aim of this study is to provide suitable LES-alike methods for 2-D simulations. An efficient software implementation of high order schemes is also proposed. Numerical examples are provided to illustrate these empirical similarities.

## **1. INTRODUCTION**

From a computational viewpoint, there are two solution strategies, i.e., the direct sound computation and coupling computation of sound. In the former prediction strategy, the unsteady flow and the sound generated by the unsteady flow can be computed together using the unsteady compressible Navier-Stokes equation, i.e., the unsteady flow and its sound are regarded as correlated parts of the same flow field. There are mainly three different techniques which are normally used by researchers in Computational Aeroacoustics (CAA). By placing them in the decreasing order in terms of computational accuracy as well as computational cost, they are direct numerical simulation (DNS), Large Eddy Simulation

(LES) and Reynolds-averaged-Navier-Stokes equations (RANS). However, it can be easily shown that it will be impossible to apply DNS for practical flow and aeroacoustic problems (high Reynolds number) in the foreseeable future; direct sound computation based on LES for application to engineering flows still remains expensive due to accurate computation in time and space, fine mesh (or high-order schemes) and small time-steps are required in calculation of the motion of the large scales; direct simulations of acoustic field based on RANS cannot usually obtain reasonable acoustic results due to their excessive turbulent dissipation [1], [2]. Under the circumstances, researchers in computational aeroacoustics field have to seek for more practical solution strategy. The development of coupling methods for aeroacoustic problems has been an active area of research in CAA.

In CAA, computational domain (domain of interest) is often divided into two parts; one is the 'near field' where main acoustic sources (sound generation) are located, where detailed flow structures can be resolved may be simulated by a Computational Fluid Dynamics (CFD) technique (DNS, LES or RANS); the other part is the 'far field' in which concerns are the propagation/radiation of the resulting acoustic waves, which is then calculated via an acoustic analogy or by solving a set of acoustic perturbation equations. A coupling method was developed to couple CFD calculations and the acoustic propagation inside the car compartment using Helmholtz equation [7]. The CFD calculations were based on a fine-mesh-small-time-step-LES-alike numerical method in two-dimension. Such simulation cannot be considered as an LES simulation. The aim of this study is to understand the similarities and the effect between filters applied to unsteady compressible Navier-Stokes equations and the combined effect of high-order schemes and mesh sizes. It is envisaged to supplement suitable LES-alike methods for 2-D simulations. In conjunction to LES-alike methods an efficient software implementation of high order schemes is proposed.

## 2. AN AEROACOUSTIC NOISE ANALYSIS METHOD

The rapid advance of computational power in recent years allows LES being used on many applications with reasonably high Reynolds number. The main advantage of LES over those computationally less expensive methods such as Reynolds-averaged Navier-Stokes equations (RANS) is the increased level of detail it can deliver. While RANS methods provide "averaged" results and turbulence models over-damping the high frequency fluctuations, LES is able to predict instantaneous flow characteristics and resolve turbulent flow structures of large scales (i.e., the energy-containing eddies), which are know to contribute most to the sound generation in many problems.

The difficulty in achieving predictive simulations is perhaps best illustrated by the wide range of approaches that have been developed and are still being used by the turbulence modelling community; Implicit Large Eddy Simulation (ILES) is one of them. ILES is a relatively new approach that combines generality and computational efficiency with documented success in many areas of complex fluid flow. Instead of using a subgrid-scale model for a classic LES to model the motion of those non-energy-contained eddies, ILES uses a higher-order discretisation method with a limiter. The limiter is originally meant to avoid numerical oscillations in the solution, but it also works as a subgrid model for small eddies [3]. The concept of using a higher-order discretisation method as a subgrid scale model in ILES, with a fine mesh, small time-steps numerical approach to resolve the unsteady flow field is implemented in this paper.

A hypothetical car configuration with an open sunroof with part of the compartment forming the resulting cavity is used as an example to illustrate the noise analysis method. The car is travelling at a cruising speed with induced flow fluctuation due to the open sunroof. The pressure perturbation along the sunroof is computed by solving the two dimensional unsteady compressible Navier-Stokes equations using a typical commercial Finite Volume CFD package, PHOENICS [4], and the pressure fluctuation due to the sunroof is extracted and analysed. For the second part of the coupling procedure, the acoustic response inside the car compartment is calculated by solving the Helmholtz equation.

## 2.1 Resolve Unsteady Navier-Stokes Equations

A hypothetical car with an open sunroof as depicted in Figure 6. In order to excite the flow to get stronger pressure fluctuation response on top of the sunroof, an artificial sinusoidal vertical-velocity disturbance is used to represent a single vortex generated by vehicle travelling at upstream of this car. Numerical schemes applied to resolve the unsteady Navier-Stokes equations are high-order, very fine spatial mesh, and small time-steps resemble many features of a DNS simulation. Previous experience of a similar problem by solving RANS can be found in [5].



Figure 1. A hypothetical car with open sunroof.

To satisfy both the mass and momentum conservation laws, the velocity and pressure field are solved iteratively by using the SIMPLE pressure-correction algorithm proposed by Patankar and Spalding [6]. QUICK scheme is used on all variables as the numerical differencing scheme.

In a snapshot of vertical velocity disturbance at t = 0.5s (Figure 2), it shows the amplitude of aerodynamic disturbances is gradually becoming weaker and weaker. This is due to the numerical diffusion from the differencing scheme. However, a clear vortex shearing on top the sunroof can still be observed.



Figure 2. A snap shot of vertical velocity component disturbance at t = 0.5s; top: zoom-in image around the sunroof opening.

# 2.2 Analysis of Acoustic Response

Frequency components of the pressure fluctuations are then examined by producing an acoustic power spectrum of the time history at all seven points on the sunroof via sampling a

512-point Fast Fourier Transform (FFT), which returns the dominant frequency at all observation points on the sunroof, occurs roughly at 13Hz.

The validity of the results of the dominant frequency is checked against a Helmholtz resonator with similar shaped and sized cavity. The resonant frequency for a typical may approximately Helmholtz resonator be calculated by the formula,  $f = (c/2\pi)\sqrt{A/(l_{eff}V)}$ , where  $l_{eff} = l + l_{cor} = l + \eta r$  denotes the effective length of the air at the opening, l is the geometric neck length (i.e., 0.05m, in Figure 1).  $l_{cor}$  is the end correction on the neck length, which can be expressed by a product of r, the radius of the neck, and  $\eta$ , an empirical coefficient which significantly depends on geometrical configuration and sizes. A is cross sectional area of the neck, V represents the volume of the inside cavity. An approximate value of the dominant resonant frequency with  $\eta = 16.9$  is around 10.5Hz. This is not a strict comparison due to the coefficient unavailable currently for the cavity of the car compartment considered. However this crude comparison shows that the dominant frequency value obtained through the unsteady computation is a physically acceptable approximation.

#### 2.3 Sound Propagation inside the Car Compartment

To implement the acoustic propagation by Helmholtz equation in this case, it is assumed that the flow inside the car compartment is negligible. For the present study the analysis of sound distribution for the dominant frequency of 13Hz due to an incoming disturbance of 50 Hz is examined. The power spectrum density along the sunroof is used as Dirichlet boundary conditions for the Helmholtz equation, which calculates the acoustic pressure distribution inside the car compartment.

It is shown that the highest acoustic pressure is experienced at x at 7.1m (end of sunroof opening). On the other hand, along the horizontal line just below the sunroof, the acoustic pressure shows an oscillatory behaviour resulting from the pressure fluctuation above the sunroof. This oscillatory behaviour gradually becomes weaken as one moves deeper into the car compartment. This shows that the solution obtained is reliable. The acoustic pressure tends to be more stable at the bottom of the car compartment.

More details of this method for aeroacoustic analysis can be found in the paper published in ICSV13 proceedings.

# 3. COMPARISON BETWEEN HIGH ORDER SCHEMES AND FILTERING EFFECTS

As presented in the test case in last section, the buffeting noise along the sunroof is computed by solving an Implicit-LES-like method with high-order-scheme-filter-effect, instead of using the classical LES supplemented with a sub-grid scale turbulent model, but in two-dimension. Fine time steps and spatial mesh are used. Spatial discretisation in higher order provides better numerical approximation than using 2<sup>nd</sup> order CFD schemes. The method described in previous section is in essence a LES method, however, not strictly in its sense, since LES applies to three dimensional problems.

In order to understand high order schemes and the equivalent/similar filter effect, this paper uses a convection and diffusion problem as a testbed. The results are then compared with two types of commonly used filters, Box filter and Gaussian filter. The aim of this study is to establish the relationship between high order schemes and filters. It is hoped that such relationships may be extended to Navier-Stokes equations.

#### **3.1 Higher Order Schemes**

An example of a linear elliptic two point boundary value convection and diffusion problem in 1-D is given as

$$-\phi''(x) + a(x)\phi'(x) + b(x)\phi(x) = f(x, k, \alpha_k), x \in [0, 1],$$
(1)

where  $\phi(x)$  is the resolved variable, a(x) and b(x) are two given functions of x, and  $\alpha_k$  is a series of random numbers. The problem is subject to boundary conditions  $\phi(0) = \phi(1) = -3.568E - 1$ . This problem is supplemented with the analytic solution (see Figure 3) is

$$\phi(x) = \sum_{k=0}^{k=N} A \frac{2\pi k}{L} \sin\left(\frac{2\pi k x}{L} + \pi \alpha_k\right), \tag{2}$$

where A is the amplitude, N normally takes half of the number of grid points.



Figure 3. Analytic solution for  $\phi(x), x \in [0,1]$ .

To obtain different high order finite difference discretisation for eqn (1) is a tedious task affecting the software development. A systematic algorithm has been developed for easy implementation of high order schemes based on the concept of the *defect correction method*. The given differential equation (1) can be rewritten as

$$\left(-\frac{1}{h^2} - \frac{a(x)}{2h}\right)\phi_{i-1} + \left(\frac{2}{h^2} + b(x)\right)\phi_i + \left(-\frac{1}{h^2} + \frac{a(x)}{2h}\right)\phi_{i+1} + \tau_i = f_i,$$
(3)

where *h* is the step size,  $\tau_i$  is the high order truncation term from the Taylor's Series Expansion. Hence for 2<sup>nd</sup>, 4<sup>th</sup> and 6<sup>th</sup> order accuracy, the corresponding  $\tau_i$  s are expressed as

$$\tau_i^{(2)} = 0,$$
 (4)

$$\tau_i^{(4)} = \frac{h^2}{12} \phi_i^{(4)} - a(x) \frac{h^2}{6} \phi_i'', \tag{5}$$

$$\tau_i^{(6)} = \frac{h^2}{12}\phi_i^{(4)} + \frac{h^4}{360}\phi_i^{(6)} + a(x)\left(-\frac{h^2}{6}\phi_i''' - \frac{h^4}{120}\phi_i^{(5)}\right),\tag{6}$$

respectively.

By introducing the defect correction method, it eases the complexity and obtains

flexibility of the calculation in order to help us to create a systematic way to calculate a given problem to different levels of high order accuracies. The concept is by writing Equation (1) in the form

$$L\phi_i^* = f_i, \tag{7}$$

where  $\phi_i^*$  is the resolved solution, *L* is the matrix structure from the given equation. Taylor's Series Expansion is used to reconstruct *L* into a unified matrix structure  $L_h$  with the high order truncation  $\tau_i$  term, i.e.,

$$L_h \phi_i^* + \tau_i = f_i. \tag{8}$$

Since  $\tau_i$  is considered to be fairly small, we can calculate an approximated solution  $\phi_i$  by neglecting  $\tau_i$  term, i.e., as we are solve

$$L_h \phi_i = f_i. \tag{9}$$

Subtract Equation (9) from (8) leads to

$$L_h \varphi_i = -\tau_i \tag{10}$$

where  $\varphi_i$  is referred as the corrector,  $\varphi_i = \phi_i^* - \phi_i$ . Such an algorithm avoids reformulating the finite difference matrix structure *L* every time a different order of accuracy method is used. The same unified matrix structure  $L_h$  to solve for both the approximated solution  $\phi_i$  and the correction  $\varphi_i$  which is then used to obtain the final solution  $\phi_i^* = \phi_i + \varphi_i$ . Only the truncation term  $\tau_i$  needs to be replaced in different computation.

The result for the calculation based on 2<sup>nd</sup> order and 4<sup>th</sup> order schemes are shown in Table 1. As expected, higher order (4<sup>th</sup> order in this case) calculation improves the accuracy of the numerical calculation. Note that the algorithm resembles similar procedures as how the filtering system employed in Large Eddie Simulations works. If we assume  $\phi_i$  being the large scale component which is resolved by solving unsteady Navier-Stokes equations,  $\varphi_i$  is the small scale (filtered-out) component which is modelled by Subgrid Scale model. It certainly exhibits certain relationships between the higher order schemes and filtering systems.

#### **3.2 Filter Effects**

In order to compare the type of filters which have equivalent/similar filtering effect as those of high order schemes, two types of common used filters in image processing, Box and Gaussian filters, have been applied on analytic solution to compare the effects.

The Box filter, also known as the mean filter, is a simple, intuitive and easy to implement method of reducing the amount of intensity variation between one pixel of a picture and the next to smooth the image. The idea of box filtering is simply to replace each pixel value in an image with the mean ("average") value of its neighbours, including itself. This has the effect of eliminating pixel values which are unrepresentative of their surroundings. Box filtering is usually thought of as a convolution filter, which is based around a kernel that represents the shape and size of the neighbourhood to be sampled when calculating the mean.

The Gaussian filtering is a different type of convolution filter, which is used in image processing to "blur" images and remove detail and noise, and in fluid dynamic it is used to damp-out fluctuations (small scales) in CFD simulation. In this sense it is similar to the box filter, but it uses a different kernel that represents the shape of a Gaussian ("bell-shaped") hump.

The Gaussian distribution in one dimensional has the form

$$G(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}},\tag{11}$$

where  $\sigma$  is the standard deviation of the distribution.

After applying each filter onto the analytic solution, results are shown in Table 2.

$2^{nd}$ order accuracy with step-size	$4^{\text{th}}$ order accuracy with step-size
dx = 1.25E - 2	dx = 1.25E - 2
20 15 10 5 -5 -10 -15 -10 -15 -10 -15 -10 -5 -10 -15 -10 -10 -5 -10 -5 -10 -5 -5 -10 -5 -5 -5 -5 -5 -5 -5 -5 -5 -5	20 15 10 5 0 0 5 - 0 - - - - - - - - - - - - -

Table 1. Numerical approximation vs. Analytic solution.

Table 2. Filtering solution vs. Analytic solution.

Box Filtering solution with filter-size	Gaussian Filtering solution with filter-size
$\Delta x = 1.25E - 2$	$\Delta x = 1.25E - 2$
20 15 10 5 0 0 0 0 0 0 0 0 0 0 0 0 0	20 15 10 5 - 0 - 5 - 0 - 5 - 0 - 5 - 0 - - - - - - - - - - - - -

Comparing the two results in Table 2, it is found that the result obtained by Gaussian filter has much less error than the one obtained by Box filter. The Gaussian filter outputs a "weighted average" of each grid point's neighbourhood, with the average weighted more towards the value of the central grid points. This is in contrast to the Box filter which uses uniformly weighted average. Hence Gaussian filter provides gentler smoothing and preserves fluctuations and peaks up to certain frequency better than a similarly sized Box filter.

Based on these investigations, there seems to be a certain relationship between a given high order scheme with a mesh size and the effect of a filter. As shown in previous comparison, it is not difficult to notice that both  $4^{th}$  order accuracy numerical approximation and Gaussian filtered solution show good agreement with the analytic solution roughly on the same level. Same observation applies to  $2^{nd}$  order accuracy numerical approximation and Box filtered solution, however, they show agreement with the analytic solution respect to a larger scale. From the comparison of their total errors from analytic solution, it is very interesting to find out that there are similarities, observed while step-sizes applied are twice as large as filter-sizes (i.e.,  $dx = 2\Delta x$ ), between  $2^{nd}$  order numerical solution and Box filtered solution, as well as between  $4^{th}$  order numerical solution and Gaussian filtered solution. Further investigation and more test cases are needed to explore the relationship between higher order numerical schemes and filtering effects.

#### **4. CONCLUSION**

Large Eddy Simulation still remains expensive for CAA calculations. To develop an economical method, an acoustic analysis method inside a car compartment is reviewed. The main aim is to point out that the numerical method used is a fine-mesh-small-time-step-LES-alike numerical method, which uses high order schemes rather than a subgrid scale model to resolve the small scale motions.

A one dimensional numerical example has been used to demonstrate the similarity between higher order schemes and filtering effects of CFD calculation. In order to improve computer implementation efficiency, a systematic algorithm has been developed based on the concept of the defect correction method of solving a given problem up to different levels of higher order accuracy. 2<sup>nd</sup> and 4<sup>th</sup> order schemes are compared with two different types of filtering effects: Box and Gaussian filters; and similarity between them has been discussed.

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