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EFFICIENT NUMERICAL MODELS FOR INVESTIGATING THE STATISTICS OF RANDOM DYNAMIC SYSTEMS

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Abstract

The study of random dynamic systems usually requires the definition of an ensemble of structures and the solution of the eigenproblem for each member of the ensemble. If the process is carried out using a conventional numerical approach, the computational cost becomes prohibitive for complex systems. In this work, an alternative numerical method is proposed. The results for the response statistics are compared with values obtained from a detailed stochastic FE analysis of plates. The proposed method seems to capture the statistical behaviour of the response with a reduced computational cost.

1. INTRODUCTION

The presence of uncertainties arising from the manufacturing process is unavoidable. These uncertainties can be observed in noise and vibration measurements [1] and some effort has been made to develop methods for predicting the response statistics of uncertain structures. One approach would be to use a conventional deterministic method (e.g. The Finite Element method) together with a probabilistic (e.g. the Monte Carlo method) or possibilistic (e.g. interval analysis) approach to include the uncertainties in the analysis. This procedure has two drawbacks: (i) it becomes computationally unfeasible as the frequency increases and (ii) the amount of information required on the statistics of the input parameters is usually not available. More recently, a method for assessing the response statistics of random dynamic systems was proposed by Langley and Brown [2]. The method assumes that the natural frequencies and mode shapes are in accordance with the Gaussian Orthogonal Ensemble (GOE) statistical model. The agreement of the natural frequencies and mode shape statistics with the GOE statistics has already been reported in the literature [3,4]. However, this agreement usually requires the system to be "sufficiently random" and some research is still underway to verify the GOE assumption and its applicability.

In order to study the statistics of random dynamic systems and the applicability of the GOE model it is necessary to generate ensembles of dynamic systems, obtain their eigenvalues and then calculate their statistics. The choice for a conventional numerical

approach would require a great computational power to solve the eigenproblem for each member of the ensemble covering the frequency range of interest. The method adopted should also be capable of applying different probabilistic models and allow the calculation of the energy density statistics.

In view of the limitations of other methods, an artificial approach is proposed here to study the statistics of the response of a random dynamic system. In this new approach, the stiffness matrix is assumed to be random and the only source of uncertainty. The new approach allows the calculation of the energy density for each member of the ensemble in a reasonable time and the modification of the input statistics of the ensemble in a practical way. The statistics of the eigenvalues and the energy density obtained using the method are compared with the results for random plates modelled using the FE method. The method seems to capture the behaviour of the dynamic systems when the level of randomness is reduced or symmetries are introduced in the system. The results suggested that the method can be used with confidence to investigate the statistics of random system in an efficient way.

2. NATURAL FREQUENCIES STATISTICS AND ENERGY DENSITY VARIANCE

In order to verify the applicability of the new approach, results for the energy density variance and the statistics of the natural frequencies as generated by both the new approach and the stochastic FE method are compared with analytical results in what follows. In the case of the energy density variance, the numerical results are also compared with values from the Variance Theory given in [2]. Considering that the statistics of the eigenvalues of a dynamic system follow the GOE, Langley and Brown derived the following equation for the energy density relative variance

$$r_T^2 = \frac{{\sigma_T}^2}{{\mu_T}^2} = \frac{1}{\pi M}\left\{\alpha - 1 + \frac{1}{2\pi M}\left[1 - e^{-2\pi M}\right] + \mathrm{E}_1(\pi M)\left[\cosh(\pi M) - \frac{1}{\pi M}\sinh(\pi M)\right]\right\}, \qquad (1)$$

where $M = \omega\eta\nu$ is the modal overlap factor, ν is the modal density, α is the spatial factor and E_1 is the exponential integral.

Prior to the comparison of the results for the energy density variance, it is necessary to verify the agreement between the natural frequencies statistics obtained numerically and those predicted by the GOE model. In what follow, use will be made of the same statistics used by Weaver [5] to check the validity of the GOE for aluminium blocks: the natural frequency spacing pdf, the number variance Σ^2 and the Δ_3 function.

2. RANDOM DYNAMIC SYSTEMS

2.1 Response of random dynamic systems

The aim of the following derivation is to establish a link between the probabilistic model used to define an ensemble of random structures, its eigenvalue statistics and the statistics of the energy density. The equations of motion of a general linear dynamic system can be written in the form

$$\mathbf{M}\ddot{\mathbf{x}} + \overline{\mathbf{K}}\mathbf{x} = \mathbf{f}\,, \tag{2}$$

where $\overline{\mathbf{K}} = \mathbf{K}(1+i\eta)$ assuming proportional damping with loss factor η, **K** is the stiffness matrix, **M** is the mass matrix, **f** is a vector containing the external forces and **x** is a vector of the displacements in generalized coordinates. A coordinate transformation from the generalized coordinate system to the modal coordinate system can be performed giving

$$\ddot{\mathbf{q}} + \overline{\mathbf{\Lambda}}\mathbf{q} = \mathbf{U}^T\mathbf{f}\,, \tag{3}$$

where $\mathbf{x} = \mathbf{U}\mathbf{q}$, **U** is the matrix whose columns are the eigenvectors $\mathbf{u}_j$, **q** is the vector with the displacements in modal coordinates, $\overline{\mathbf{\Lambda}} = \mathbf{\Lambda}(1+i\eta)$ and $\mathbf{\Lambda}$ is a diagonal matrix containing the square of the natural frequencies ω_j^2. Consider now that the system is a member of an ensemble of random systems, each one with its matrices **U** and **Λ**. Instead of adopting the modal coordinates of each system, we can write the equations of motion of all the members of the ensemble based on the natural coordinates of the original or nominal system. In this case, we can write

$$\ddot{\mathbf{q}} + \overline{\mathbf{A}}\mathbf{q} = \mathbf{U}^T\mathbf{f}\,, \tag{4}$$

where $\overline{\mathbf{A}} = \mathbf{A}(1+i\eta)$, with **A** being a random symmetric matrix. **A** is diagonal only in the case of the original system. Once more, Eq. (4) may be written in a new coordinate system, here called "modal random coordinates", yielding

$$\ddot{\mathbf{q}}_R + \overline{\mathbf{\Lambda}}_R\mathbf{q}_R = \mathbf{U}_R^T\mathbf{U}^T\mathbf{f}\,, \tag{5}$$

where $\overline{\mathbf{\Lambda}}_R = \mathbf{\Lambda}_R(1+i\eta)$, $\mathbf{U}_R$ is a matrix containing random eigenvectors, $\mathbf{\Lambda}_R$ is a diagonal matrix containing random eigenvalues and $\mathbf{q_R}$ is a vector with the displacements in modal random coordinates. Eq. (5) can be solved to obtained $\mathbf{q_R}$, which can then be transformer to generalized coordinates to give

$$\mathbf{x} = \mathbf{U}\mathbf{U}_R\left[-\omega^2\mathbf{I} + \overline{\mathbf{\Lambda}}_R\right]^{-1}\mathbf{U}_R^T\mathbf{g}\,, \tag{6}$$

where it has been noted that the mass matrix is equal to the identity matrix, and $\mathbf{g} = \mathbf{U}^T\mathbf{f}$.

The aim of this analysis is to find the energy density associated with a random stiffness matrix **A**. The kinetic energy density can be written as

$$T(\omega) = \frac{\omega^2}{2N}\left[\mathbf{g}^T\mathbf{U}_R\mathbf{B}^{-1*}\mathbf{B}^{-1}\mathbf{U}_R^T\mathbf{g}\right] \text{ or } T(\omega) = \frac{\omega^2}{2N}\sum_n \frac{|r_n|^2}{\left(\omega_n^2-\omega^2\right)^2 + \left(\eta\omega\omega_n\right)^2} \tag{7,8}$$

where $\mathbf{B} = \left[-\omega^2\mathbf{I} + \overline{\mathbf{\Lambda}}_R\right]$ and $\mathbf{r} = \mathbf{U}_R^T\mathbf{g}$.

Eq. (8) yields the kinetic energy density for each member of the ensemble and it is only necessary to define the matrix **A** associated with each member. Thus, based on an ensemble of **A** matrices with controlled statistical inputs, it is possible to calculate the statistics of the energy density. This feature allows a study of the influence of the statistical properties of the system on the natural frequency and mode shape statistics and, consequently, on the energy

density statistics. The procedure is fast and allows the solution of an ensemble of systems with a statistically representative size. Various ways of defining the matrix **A** are discussed in the next section.

2.2 Random Stiffness Matrix

The matrix **A** is taken to have the form

$$\mathbf{A} = \mathbf{A_0} + R\mathbf{A_{ran}}, \tag{9}$$

where $\mathbf{A_0}$ is a diagonal matrix and $\mathbf{A_{ran}}$ is a random symmetric matrix . The matrix $\mathbf{A_0}$ can be interpreted as the stiffness matrix in modal coordinates of the original dynamic system considered in the previous section. The system randomness (the deviation of each member from the original system) is introduced through the matrix $\mathbf{A_{ran}}$ and can be controlled by the constant R. The statistics of the entries of $\mathbf{A_{ran}}$ will determine the statistics of the eigenvalues of **A**, and thus those of the energy density. The entries of $\mathbf{A_{ran}}$ are divided into three groups to allow the randomization of the system in particular ways. The diagonal terms are included in Group A, while the off-diagonal terms are divided into Groups B and C. If the matrix is divided into quadrants, the off-diagonal terms in quadrants 1 and 4 will be included in Group B, while the off-diagonal terms in the other quadrants will constitute group C. For the numerical results shown bellow, each entry of $\mathbf{A_{ran}}$ was considered to be a Gaussian random variable with zero mean and with the variance being dependent on the group (σ_a^2 for group A, σ_b^2 for group B and σ_c^2 for group C). This randomization approach of the matrix $\mathbf{A_{ran}}$ results in an almost constant randomization level for the eigenvalues. This situation is rarely found for a real dynamic system, since the uncertainties from the manufacturing process are likely to have more effect on higher order modes. The randomization level is not exactly constant over the eigenvalues as a result of the limited size of the problem considered. The results presented below will be usually related to the eigenvalues and eigenvalue spacings located in the middle of the eigenvalue range. The diagonal elements of the matrix $\mathbf{A_0}$ are associated with the natural frequencies of the nominal system and were properly defined to provide a constant modal density (allowing the comparison with results for random plates) and to avoid negative values.

More details on the proposed approach can be found in [6].

3. NUMERICAL RESULTS

3.1 Stochastic FE models

The results obtained through the numerical approach described in sections 2.1 and 2.2 are compared bellow with the statistics of random plates modelled using the FE method. The FE models considered only bending waves and the mesh discretization was defined in order to represent the mode shapes with sufficient accuracy. The plates were modelled with free-free boundary conditions and the models were solved to obtain the natural frequencies and the mode shape amplitudes at the excitation point. The energy density was then calculated based in theses variables. The probabilistic models adopted are described in what follows.

3.3 Overall level of randomness

The aim of the present analysis is to verify the capacity of the proposed numerical method of describing the behaviour of the response statistics for different randomization scenarios. Therefore, two situations are investigated: (i) the effect of the overall level of randomness is reduced and (ii) the effect of symmetries on the systems.

The first situation considers the case where the ensemble shifts from an almost deterministic behaviour to a condition of high randomness. This situation can be associated with two distinct scenarios for real systems: an increase of the randomization level used to define the ensemble or an increase in the frequency range, since higher modes are more sensitive to the uncertainties. The matrix $\mathbf{A}_{\mathbf{ran}}$ was defined with $\sigma_a^2 = 2$ and $\sigma_b^2 = \sigma_c^2 = 1$ (giving a GOE matrix) and the parameter R was continuously increased. The results for the eigenvalue statistics are shown in Fig. 1 for an ensemble of 500 members considering the eigenvalue at the centre of the eigenvalue sequence. The curves in the pdf plots are: (----)Normal distribution, (– - –) Exponential distribution, (——) Rayleigh distribution; and in the number variance and Δ_3 plots: (——) GOE statistics, (----) Poisson statistics, (-·-·-) numerical data. Some results are shown in Fig. 2 for a plate with masses attached in random locations (500 members). Three sets of results are shown, representing low, mid and high frequency modes (the results are centred in these modes).The random masses correspond to 15% of the mass of the bare plate. It can be seen that the new approach captures the transition from a Gaussian pdf to a Rayleigh pdf (GOE model) in a very similar way to what is observed for the random plate.

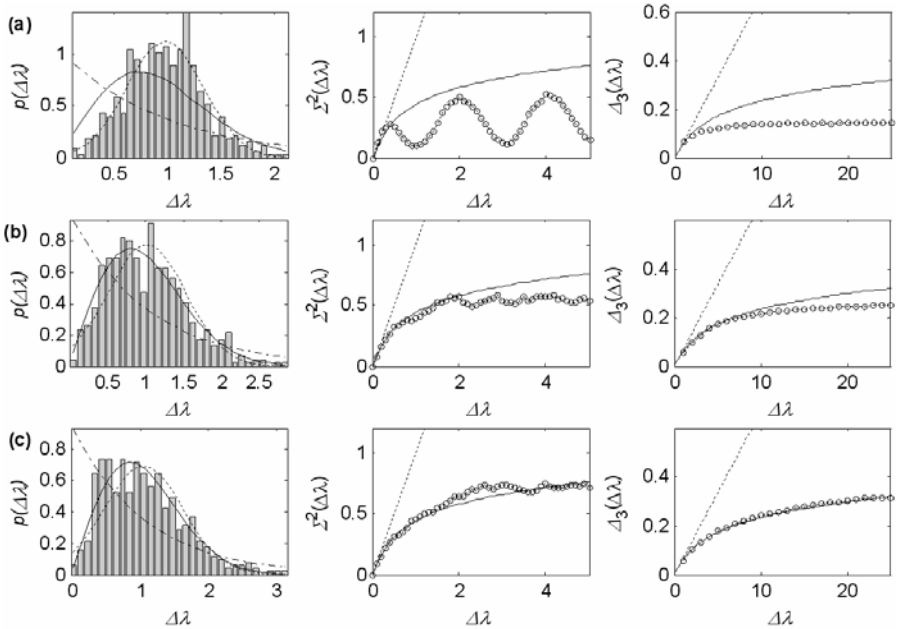

Fig. 1 – Eigenvalues statistics - Numerical approach (500 member ensemble) - varying the overall level of randomness: a) R = 0.2, b) R = 0.5 and c) R = 2.

The results for the energy density variance are shown in Fig. 3 for the plate with random masses (15%) and the numerical data with R = 2 (—— numerical result, ---- GOE theory K = 3, ---- GOE theory K = 2.5, ---- Poisson model K = 3). It can be observed that the results for the random plate converge for the Variance theory prediction with increasing model overlap while the numerical results also agree with the theory (except the values on the extremes,

where the effect of a limited number of eigenvalues can be noted). Reducing the level of randomness on the random plate (by reducing the random masses to 7% of the bare plate) causes the results to diverge from the theory. This behaviour is also well captured by the proposed numerical approach as can be observed in Fig. 4.

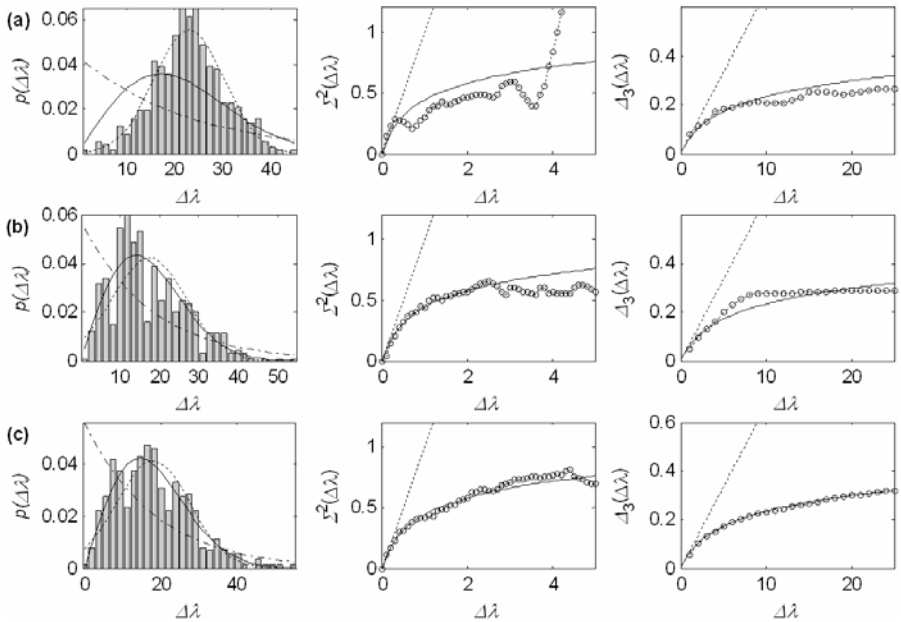

Fig. 2 – Eigenvalue statistics – Plate with random masses (500 member ensemble). a) Mode 30, b) Mode 80 and c) Mode 170.

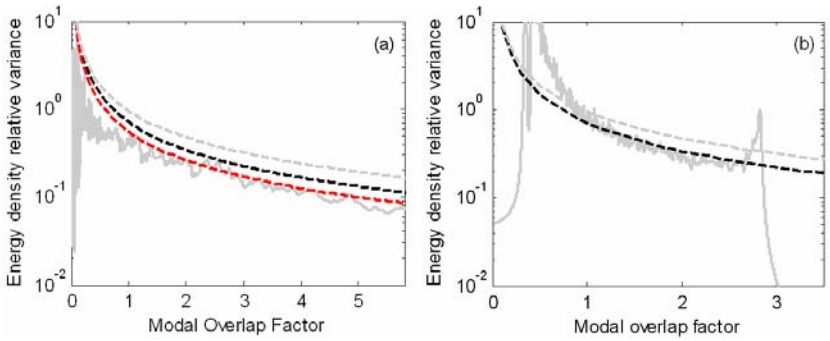

Fig. 3 – Energy density normalized variance. a) Plate with random masses (15%), b) Numerical approach for R = 2.

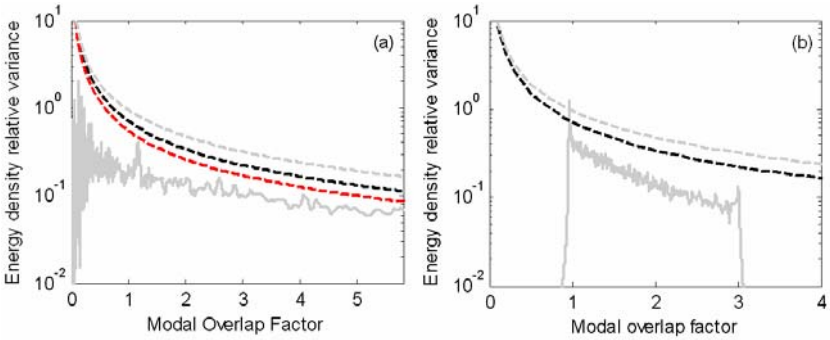

Fig. 4 – Energy density normalized variance. a) Plate with random masses (7%), b) Numerical approach for R = 0.5.

3.4 Symmetries and Poisson statistics

The next situation is related to the occurrence of Poisson statistics and the presence of symmetries. The Poisson model has been previously used for the statistics of the eigenvalues [7] and it has been argued that the model may be applicable in the case of the presence of symmetries. In order to obtain Poisson statistics, the variance of the off-diagonal terms was continuously reduced in the numerical approach and the eigenvalues statistics are shown in Fig. 5. The frequency spacing pdf shifts from a Rayleigh pdf to the Exponential pdf (Poisson model), while the other statistics also converge to the prediction from the Poisson model.

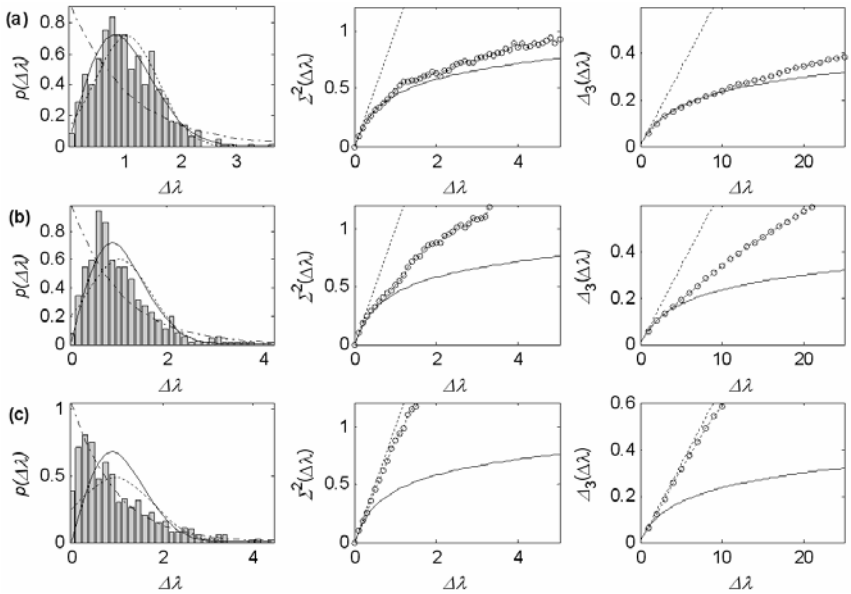

Fig. 5 – Statistics of the eigenvalues of a random matrix (500 member ensemble) – Inducing Poisson statistics: a) $\sigma_b^2 = \sigma_c^2 = 0.5$, b) $\sigma_b^2 = \sigma_c^2 = 0.1$ and c) $\sigma_b^2 = \sigma_c^2 = 0.01$.

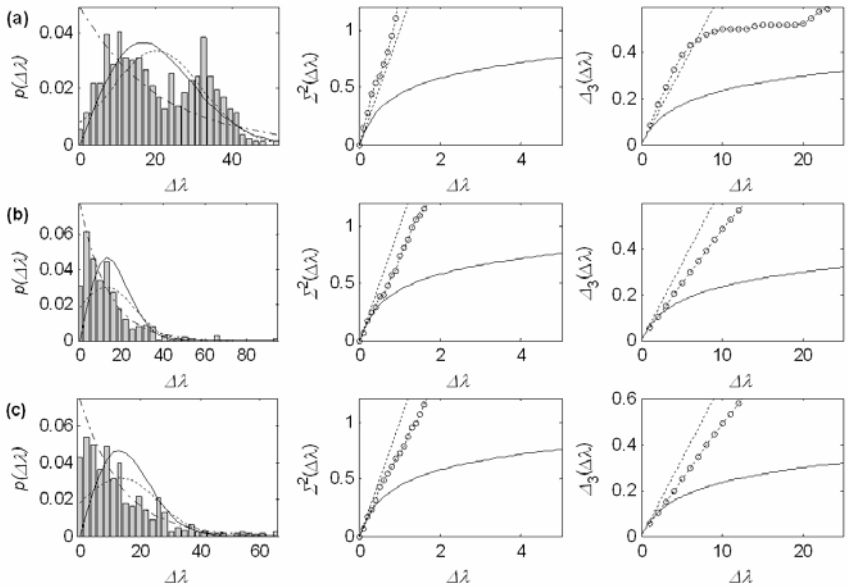

Fig. 6 – Eigenvalue statistics (500 member ensemble) – Rectangular plate with random dimensions. a) Mode 20, b) Mode 70 and c) Mode 200.

The stochastic FE model corresponding to that simulated by the numerical approach considers a rectangular plate with random dimensions. In this case all the members of the ensemble are also rectangular. The results are shown in Fig. 6 and they display the same trend observed through the numerical approach. The results for the energy density variance for both situations are shown in Fig. 7. Once more, the numerical approach is capable of predicting an oscillatory behaviour on the variance results and how well it agrees with the Variance Theory.

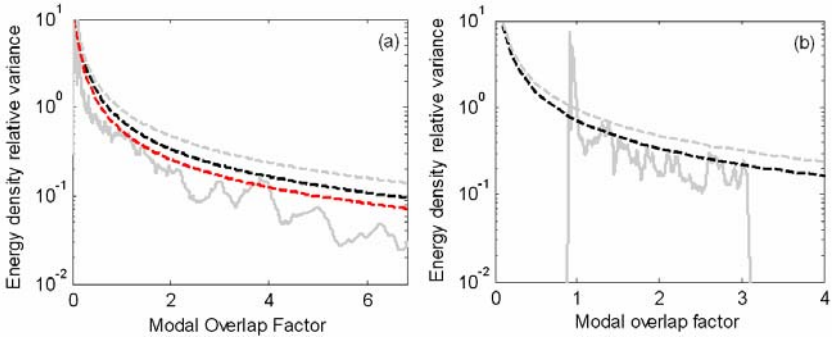

Fig. 7 – Energy density normalized variance. a) Rectangular plate with random dimensions (10%), b) Numerical approach for R = 2, $\sigma_a^2 = 1$ and $\sigma_b^2 = \sigma_c^2 = 0.01$.

4. CONCLUSIONS

A new numerical approach has been proposed for the study of the statistics of random dynamic systems. Results suggest that the main features regarding the statistics of dynamic systems as seen for plates may be reproduced using the new approach. The new approach is based on the analysis of a general dynamic system in modal coordinates where the system randomness is due only to the stiffness matrix. The fact that the random behaviour of the system is determined by the stiffness matrix has been shown not to limit the reproduction of the statistics of real systems. The approach also displayed important characteristics required for the study of the statistics of random systems: fast solution (allowing the analysis of statistically representative ensembles) and easy application of different probabilistic models.

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