



NON-UNIFORM, HARMONIC, EDGE EXCITATION OF A WAVEGUIDE

Baruch Karp

Faculty of Aerospace Engineering, Technion – Israel Institute of Technology Haifa 32000, Israel baruch@tx.technion.ac.il

Abstract

The plane strain response of a semi-infinite, elastic strip to harmonic, symmetric, and nonuniform end excitation is investigated analytically. The solution is obtained as a series expansion of the Rayleigh-Lamb modes of the strip. The variation of the energy partition among the propagating modes with the frequency of the end excitation was found for mixed end conditions prescribed at the excited end (load and displacement types). The biorthogonality relation was employed in deriving the relative amplitudes of each mode to the given excitation. It was found that the far field response of the strip is largely indifferent to whether the excitation is a displacement or stress type. The previously reported phenomenon of the existence of one dominating wave in uniform excitation is shown to extend to moderately non-uniform excitations as well. The phenomenon of complete dominance of one mode for generally non-uniform excitations was exposed.

1. INTRODUCTION

In the present communication the problem of harmonic excitation of an elastic semi-infinite strip, considered as a waveguide, is examined. To the best of the author's knowledge, only uniform end excitations have been addressed analytically. Torvik and McClatchey [1] investigated thoroughly the problem of a semi-infinite strip ($0 \le x < \infty$) in the plane strain condition excited at its end (x = 0) by a harmonic, uniform, axial force with zero transverse traction (pure stress conditions). A similar problem of a strip excited by uniform axial displacement with zero transverse traction at the end (mixed condition) was solved by Gregory and Gladwell [2]. The energy partition among the available propagating modes was calculated as a function of the frequency of the excitation. Both in [1] and [2] it was noticed that most of the energy propagating down the strip axis (approximately 90 percent of the total energy absorbed by the strip) is carried by a single mode having non-dimensional wave number k, which is the closest in its value to the non-dimensional excitation frequency, Ω .

In the present paper, we consider the generation of waves by *non-uniform*, harmonic, and symmetric excitations of two types of mixed end data: axial displacement with zero

transversal traction (*displacement excitation*) and axial load with zero transverse displacement (*stress excitation*). The generality of the previously observed phenomenon of a single dominating wave mode carrying energy into the strip is questioned for different cases of non-uniform excitation forms. To that end, two non-uniform excitations described by simple symmetric functions were examined, as well as one special functions, having distribution identical to one particular wave mode. The far field steady state response of a strip to these non-uniform excitations was found analytically with the aid of the bi-orthogonality condition which was proved to hold for Lamb waves [3]. The analysis closely follows the one given by Gregory and Gladwell [2].

The dominance of one mode is found to extend, in addition to when uniform excitations are applied, also to a half-cosine excitation. Significantly non-uniform excitations, on the other hand, do not appear to expose the same dominance of one wave mode having $k \approx \Omega$. Further inspection of the results suggests that the two types of mixed end conditions (displacement and stress excitations), having same form, expose similar energy partitions among the propagating modes, for excitation forms examined.

Complete dominance of one wave mode (conveying 100 percent of the total energy absorbed) is found to occur for significantly non-uniform excitations. This happens not only when the excitation form matches exactly one of the modes, but also when they differ.

In section 2, the standard series mode solution of a rectangular wave guide is outlined. In Section 3, we derive the expressions for the calculation of the amplitudes of each propagating mode for symmetric modes for the two mixed excitation conditions. The energy partition patterns among the propagating wave modes for various spatial distributions of end excitation for a range of frequencies are given in Section 4.

2. STRIP UNDER HARMONIC EXCITATION

A semi-infinite strip with a thickness 2h consists of homogeneous, isotropic, elastic material and occupies the region $x \ge 0$, $|y| \le h$, $|z| < \infty$ as shown in Figure 1. The strip can be held in the plane strain condition (the *z* coordinate not active) while the faces $y = \pm h$, $x \ge 0$, $|z| < \infty$ are free of tractions. At the end x = 0, a harmonic and symmetric (with reference to the *x*-*z* plane) excitation is applied, with circular frequency ω .



Figure 1 – Schematic view of a semi-infinite strip of thickness 2*h* on which symmetric excitation of form $\mathcal{F}(y)$ is applied at the end x = 0.

The problem is to determine the resulting steady state response of the strip to that excitation and, in particular, to determine the portion of the energy carried by each available

propagating mode as a function of the excitation frequency for several spatial forms of the end excitation.

The standard formulation of the elastic, plane strain problem leads to Rayleigh-Lamb frequency equation (e.g., [4]) that dictates a discrete relation between the frequency ω and the wave number ξ . For a given frequency, the frequency equation can be fulfilled by real, complex or imaginary wave numbers. Real wave numbers represent *propagating waves*. Complex and imaginary wave numbers represent *attenuating* (or *evanescent*) waves. Assuming this set is a complete set of solutions (e.g. [5]), the displacement field will take the form

$$\boldsymbol{u}(x, y, t) = \sum_{n} A_{n} \boldsymbol{U}_{n}(y) e^{i(\xi_{n} x - \omega t)}$$
(1)

for any frequency ω while the summation is taken over the infinite number of all wave numbers. Here ξ_n is the *wave number* of the nth mode, $U_n(y)$ is the associated crosssectional profile for both velocity components, $u = u\mathbf{i} + v\mathbf{j}$, in the x and y directions (*wave mode*), A_n is a complex valued amplitude. The real part of (1) is understood to be the desired solution. At a given frequency, there are a finite number of real wave numbers N and an infinite number of imaginary and complex wave numbers. Then, solution (1) can be rewritten as

$$\boldsymbol{u}(x, y, t) = \sum_{n=1}^{N} A_n \boldsymbol{U}_n(y) e^{i(\xi_n x - \omega t)} + \sum_{n=N+1}^{\infty} A_n \boldsymbol{U}_n(y) e^{i\xi_n x} e^{-i\omega t}$$
(2)

Here, the first summation represents waves propagating without attenuation (elastic material is assumed) while the second term is the sum of the attenuating waves. The response of the strip far from the loaded end (far field) therefore solely consists of the propagating waves, which are the subject of the present investigation.

The actual amplitude of each mode A_n in (2) as well as its very excitation, depends on the particular form of the excitation applied to the strip. We assume here two *types* of mixed conditions prescribed at the end x = 0. Axial traction with no sliding is defined by

$$\sigma_x^0 = \mu A_0 \mathcal{J}(y) e^{-i\omega t} \qquad v^0 = 0 \qquad (a) \quad x = 0 \quad stress \ excitation \tag{3}$$

and axial displacement with no shear is given by

$$u^{0} = A_{0} \mathcal{U}(y)e^{-i\omega t}$$
 $\tau_{xy}^{0} = 0$ @ $x = 0$ displacement excitation (4)

where $\mathcal{J}(y)$ and $\mathcal{U}(y)$ are the *form* functions of the excitation, and A_0 is the amplitude of the excitation. These two types of excitations will be referred to as stress and displacement excitations, respectively.

It is the purpose of the next section to derive the expressions needed to find the amplitudes of the propagating modes A_n . These amplitudes are required for the calculation of the portion of the outgoing energy in every available propagating mode for various forms of excitation. This will further be used to unveil the dependence of this energy partition on the frequency of the excitation for various forms of the excitation, $\mathcal{J}(y)$ or $\mathcal{U}(y)$ (for sake of clarity, the argument y will be omitted in the next section).

3. WAVE AMPLITUDES AND ENERGY PARTITION

Let us begin with the stress excitation given by (3). Using the nondimensional expressions, the axial stress and the transverse displacement at the end (x = 0) can be expressed in terms of a series solution by

$$\sigma_{x}^{0} = \sum_{n} \sigma_{x}^{n} \Big|_{x=0} = \mu \sum_{n} A_{n} S_{x}^{n} e^{-i\omega t} \qquad v^{0} = \sum_{n} v^{n} \Big|_{x=0} = \sum_{n} A_{n} U_{y}^{n} e^{-i\omega t}$$
(5)

Commpletness of the series expansion is recalled here (e.g., [6]) for justification of the general validity of (5). Combining (5) with the end condition (3) will lead to

$$\sum_{n} A_{n} S_{x}^{n} = A_{0} \mathcal{J}(y) \qquad \sum_{n} A_{n} U_{y}^{n} = 0$$
(6)

Now we multiply the first of (6) by U_x^m and the second by T_{xy}^m , where *m* stands for m^{th} mode, followed by integration over the cross-section leading to

$$\int_{-h}^{h} \sum_{n} \left(A_{n} S_{x}^{n} U_{x}^{m} \right) dy = A_{0} \int_{-h}^{h} \left(\mathcal{J} U_{x}^{m} \right) dy \qquad \int_{-h}^{h} \sum_{n} \left(A_{n} T_{xy}^{m} U_{y}^{n} \right) dy = 0$$
(7)

Next, we subtract the first of (7) from the second and interchange the integral and the summation operators to obtain an equation

$$\sum_{n} A_{n} \int_{-h}^{h} \left(T_{xy}^{m} U_{y}^{n} - S_{x}^{n} U_{x}^{m} \right) dy = -A_{0} \int_{-h}^{h} \left(\mathcal{S} U_{x}^{m} \right) dy$$
(8)

The bi-orthogonality property of the wave modes [2] enable one to simplify (8) by retaining on the left hand side only terms with m = n and allowing omission of the summation over n

$$A_{m} \int_{-h}^{h} \left(T_{xy}^{m} U_{y}^{m} - S_{x}^{m} U_{x}^{m} \right) dy = -A_{0} \int_{-h}^{h} \left(\mathcal{L} U_{x}^{m} \right) dy$$
(9)

From (9), the complex coefficient A_m can be deduced directly in the form

$$\frac{A_m}{A_0} = -\frac{1}{J_m} \int_{-h}^{h} \left(\mathcal{S}U_x^m \right) dy \qquad J_m \equiv \int_{-h}^{h} \left(T_{xy}^m U_y^m - S_x^m U_x^m \right) dy \qquad (10a,b)$$

Here J_m and U_x^m are both properties of the m^{th} wave mode. An analogous derivation for the displacement excitation (4) leads to amplitude ratios

$$\frac{A_m}{A_0} = -\frac{1}{J_m} \int_{-h}^{h} (\mathcal{U} S_x^m) dy$$
(11)

The mean total rate of doing work of the external excitation (per unit length in the y direction) is defined by [7]

$$\left\langle P\right\rangle = \frac{1}{T} \frac{1}{2h} \int_{0}^{T} \int_{-h}^{h} (\boldsymbol{\sigma} \cdot \boldsymbol{\dot{u}}) dy dt$$
(12)

with $T=2\pi/\omega$. For harmonic waves, (12) will take the form

$$\langle P \rangle = \sum_{n} \langle P_{n} \rangle = -\frac{1}{4h} \omega \mu \sum_{n} |A_{n}|^{2} \operatorname{Im}\{J_{n}\}$$
 (13)

where the sum is taken over the *n* propagating modes available at the particular frequency. The proportion of the energy E_m communicated by the m^{th} propagating mode is then [2]

$$E_{m} \equiv \frac{\langle P_{m} \rangle}{\langle P \rangle} = \frac{|A_{m}|^{2} \operatorname{Im}\{J_{m}\}}{\sum_{n} |A_{n}|^{2} \operatorname{Im}\{J_{n}\}}$$
(14)

Calculation of energy partition according to (14) requires finding the real valued wave numbers for any frequency by the numerical solution of Rayleigh-Lamb equation (7), evaluation of the integrals (10b), and either (10a) or (11).

4. RESULTS

At first, three simple excitation forms, uniform, half-cosine, and two-cosine, defined by the functions \mathcal{F}

$$\mathscr{F}(y) = A_0$$
, $\mathscr{F}(y) = A_0 \cos\left(\frac{\pi}{2}\frac{y}{h}\right)$, $\mathscr{F}(y) = A_0 \cos\left(2\pi\frac{y}{h}\right)$, (15a,b,c)

respectively, were examined for both stress (3) and displacement (4) excitations. Substitution of each of these functions, in turn, into relation (10) for a stress excitation, followed by integration, lead to explicit expression for the relative amplitudes (10) and (11) for each propagating mode m.

These relative amplitudes, the integral (10b), and the energy partition (14) were calculated for the real roots (wave numbers) of the symmetric Rayleigh-Lamb equation for Poison's ration v = 1/4 and for a non-dimensional frequency range $0 < \Omega < 8$ (the non-dimensional frequency is defined by $\Omega \equiv (2h)/(\pi C_T)\omega$ where C_T is the velocity of a shear wave). The resulting frequency map can be found in any textbook on the subject.

Variation of the energy partition among the propagating modes as a function of frequency for a uniform displacement excitation (15a) reproduces well the previous result given by Gregory and Gladwell [2] and is not shown here.

Energy partition for the case of uniform stress excitation is shown in Fig. 2. Since only one propagating mode is available up to frequency 1.6371 (no energy partition takes place), all graphs begin at frequency 1.6. This result is to be compared (after appropriate frequency scaling) with the results for uniform displacement excitation and with the results for a uniform pure stress excitation given by Torvik and McClatchey [1]. Except for a small frequency range, $1.63 < \Omega < 1.73$, the pattern for uniform displacement closely follows the pattern found in both of the aforementioned cases of uniform excitations. The frequency range of $1.63 < \Omega < 1.73$ is the region where three modes are available while the third mode has a negative phase velocity (see [8] for details and interpretation).



Figure 2 – Energy partition for *uniform stress* excitation for the first six propagating modes.

Fig. 3 traces the energy partition among the available wave modes for stress excitations with the half-cosine form (15b). Although the peaks in Fig. 3 are sharper than the maxima in Fig. 2, dominance of the mode with a wave number equal to the frequency of excitation is preserved for this type of non-uniform excitations, except for a frequency range of $1.63 < \Omega < 2.5$.



Figure 3 – Energy partition for $\cos(\pi/2 y/h)$ stress excitation for the first six propagating modes.

The partition of energy for a non-uniform displacement excitation (15c) is shown in Fig. 4. Here, the pattern of energy partition markedly differs from the pattern found for uniform and half-cosine excitations, with two new features. One feature is a complete dominance of one of the available modes, which conveys 100 percent of the absorbed energy (the third and fourth modes at frequencies 4 and 4.8, respectively). That complete dominance differs from previously observed dominances where the dominant mode delivered up to 90 percent of the energy. The second feature is that, at a few frequencies, dominance sets on from the very frequency at which a new wave mode is available, and is

accompanied by the complete nullification of energy delivered by all other available modes. That phenomenon occurs at frequency 4 in Fig. 6 for a displacement excitation, and for a stress excitation of the same form (not shown here).



Figure 4 – Energy partition for $\cos(2\pi y/h)$ displacement excitation for the first six propagating modes.

For further examination of the phenomenon of complete dominance of a single mode, the strip response to a special excitation functions was calculated. The excitation form is taken to be identical to the second mode $k_2 = 0.72436$ at the frequency $\Omega = 1.66$, the stress distribution of which is given by

$$\mathcal{J}(y) = -1.96793 \cos\left(0.985774 \frac{y}{h}\right) + 1.34602 \cos\left(2.34617 \frac{y}{h}\right)$$
(16)

The frequency of 1.66 is chosen to lie within the range of existence of a backward wave region that appeared to be very sensitive to end data.



Figure 5 – Energy partition for stress excitation (16) – Second mode.

The energy partition for this excitation is given in Fig. 5. As expected, complete dominance of one mode corresponding to its excitation form is observed at $\Omega = 1.66$. This agrees well with our experience with Fourier's theorem. Complete dominance of a single mode at higher frequencies where the form of the dominating wave mode deviates significantly from the excitation was less expected. This probably reflects the more subtle nature of the bi-orthogonality relation. A displacement type excitation for same mode as in (16) reveals the same similarity between energy partition for stress and displacement type excitations. That result, along with analogous similarities for earlier evaluated functions (15), suggests that for mixed end data, both stress and displacement excitations of the same form result in a similar partition of energy.

5. CONCLUSIONS

The sensitivity of the far field response of a waveguide to non-uniform symmetric excitations has been exposed. The energy partition among the available propagating modes is given as a function of the excitation frequency. It was shown that uniform and moderately non-uniform excitations result in similar energy partition patterns, suggesting a low sensitivity of the far field to such variability in excitation. Moreover, two types of mixed end data (axial traction with transverse displacement or axial displacement with transverse traction) induce similar responses in the far field. These findings, together with previously reported results on uniform excitation with pure end data, can be utilized to explain the available experimental data and numerical reports of a transient response of waveguides to non-uniform excitations. This is left for the future to follow the lines given in [9].

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