ICSV14
Cairns • Australia
9-12 July, 2007

14th International Congress on Sound & Vibration

INTEGRATION OF SUB-SYSTEMS ON VEHICLES

Vincent Martin[1] and Saül Mapagha[2]

[1]Institut Jean Le Rond d'Alembert, CNRS/Paris VI
2, place de la gare de Ceinture,78210 St Cyr l'Ecole, France
[2] Centre d'études vibro-acoustiques pour l'automobile,
2, rue joseph Fourier, 76800 Saint Etienne du Rouvray – France

vmartin@ccr.jussieu.fr   s.mapagha@cevaa.com

Abstract

When a host structure is excited by a vibratory system, called sub-system, it also vibrates and may radiate an acoustic pressure. Presently, this subject originates from the car industry, but has in fact a far wider range of applications in other transport industries as well as in the field of musical instruments. The aim of the investigation is first to deduce the vibratory behaviour of the host structure from that of the sub-system observed on a test host structure which is often very rigid and for this reason called marble (or test bench). Second, what constraint should be satisfied on the marble, according to a given constraint on the host structure? And finally, in case the constraint on the test bench is not satisfied, what could be done to reach the goal? These three aspects are developed for harmonic linear vibrations.

The first item is solved by an elementary analytical approach, the second rests on the transfer of mathematical norms, and the third is a problem of optimization for which a geometrical interpretation could provide ideas for the procedure to follow.

1. INTRODUCTION

The car industry (constructors and suppliers) must satisfy physical acoustic and vibratory objectives regarding auditory and/or vibratory perceptions. For example, there are maximum structural displacements and forces that are not to be exceeded concerning the radiating elements such as the chassis. This constraint on the chassis leads to a constraint on the test bench. How can the latter be expressed in terms of the former?

The generic configuration considers a deformable body (the sub-system) linked by at least two elastic suspensions to another deformable vibrating body, both of the flexural beam type. The second one plays the role of the chassis and also of the test bench when its input impedance is infinite. Thanks to globalising notions such as the impedances, the result obtained has the same form as that associated with an elementary configuration with non-deformable bodies.

Then the problem is concerned with deducing the forces entering the chassis from those entering the test bench, also said to be blocked forces when the test bench is rigid. The main difficulty here lies in how to go from the constraint imposed on a mathematical norm on the chassis to the constraint to be satisfied on the test bench. Indeed the first could lead to sufficient conditions and not to necessary conditions on the second. When the coupling occurs through only one elastic suspension, the formulation is reduced to an analytical form to obtain the stiffness of the elastic suspension in order to satisfy the constraint on the chassis, given the observation on the test bench. It will be shown that the optimal stiffness is at the intersection of intervals associated with the frequencies under study. A numerical application with data from the car industry is given. How is it possible to generalize this approach?

2. COUPLING OF TWO VIBRATORY STRUCTURES THROUGH ELASTIC SUSPENSIONS

The main notions of the coupling between a vibratory sub-system and a host structure, also a vibratory system, are well understood with the generic problem of two flexural beams linked with elastic suspensions (fig. 1). Arbitrarily, the beams considered are simply supported.

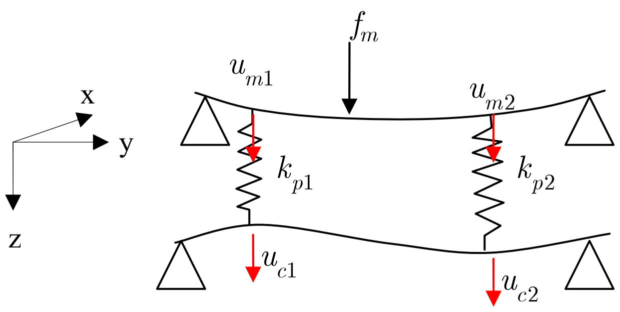


Figure 1. Elementary configuration of a sub-system (above) linked to a host structure (below) through elastic suspensions.

The mechanical impedance seen from the future points that are going to be linked to the elastic suspensions is defined in the case of a single beam. Indeed the well-known flexural wave operator [1,2] concerned with the transverse displacement $\zeta(y,t)$ is

$$\begin{cases} \left(\dfrac{\partial^4}{\partial_y^4} + \dfrac{\rho S}{EI}\partial_{\mathrm{tt}}^2\right)\zeta(y,t) = \dfrac{1}{EI}\sum \text{applied external forces per unit length} \\ \text{Boundary conditions (BC) at } y=0 \text{ et } y=L \text{ and initial conditions} \end{cases}$$

where ρ is the density of the material the beam is made of, E the Young modulus, S the beam cross-section, L the length and I the inertia momentum.

In the frequency domain (harmonic vibrations) and with a single point excitation, the operator is now

$$\begin{cases} \left(\dfrac{d^4}{dy^4} - \gamma^4\right)\zeta(y,\omega) = \dfrac{1}{EI} f_l \delta(y - y_l) \\ + \text{ CL} \end{cases} \tag{1}$$

with the wavenumber γ defined by $\gamma = \sqrt[4]{\dfrac{\rho\ \omega^2}{E\ a_x^2}}$; a_x is called radius of giration.

The solution results from the modal theory having solved first the spectral problem to obtain the modes $\psi_n(y)$ (here with the simply supported beam i.e., $\psi_n(y) = \sin\gamma_n y$) and the eigenwavenumbers γ_n associated to these modes (presently $\gamma_n = \dfrac{n\ \pi}{L}$). This leads to

$$\zeta(y) = \frac{1}{EI}\sum_{n=0}^{\infty}\frac{\psi_n(y)}{\Lambda_n\left(\gamma_n^4 - \gamma^4\right)}f_l\psi_n(y_l) \tag{2}$$

By definition, the mechanical impedance is $Z = \dfrac{f}{i\omega\zeta}$ and, for the sake of simplicity, we introduce $Z' = i\omega Z = \dfrac{f}{\zeta}$. Thus, with a single applied force per unit length, we obtain

$$Z'(y) = \frac{EI}{\displaystyle\sum_{n=0}^{\infty}\frac{1}{\Lambda_n}\frac{1}{\gamma_n^4 - \gamma^4}\psi_n(y_l)\psi_n(y)} \tag{3}$$

With elastic suspensions, each attach point is submitted to restoring forces. By denoting f_m the excitation of the sub-system (the simply supported beam at the top of fig. 1), u_{mi} the transverse displacements of the sub-system at attached point i and u_{ci} the transverse displacements of the host system, the equations for the coupled beams are

$$\begin{cases} \left(\dfrac{d^4}{dy^4} - \gamma_m^4\right)u_m(y) = \dfrac{1}{E_m I_m}\left[-(a,b)^t\cdot\begin{Bmatrix}u_{m1}\\u_{m2}\end{Bmatrix} + (a,b)^t\cdot\begin{Bmatrix}u_{c1}\\u_{c2}\end{Bmatrix} + f_l\delta(y - y_l)\right] \\ \left(\dfrac{d^4}{dy^4} - \gamma_c^4\right)u_c(y) = \dfrac{1}{E_c I_c}\left[-(a,b)^t\cdot\begin{Bmatrix}u_{c1}\\u_{c2}\end{Bmatrix} + (a,b)^t\cdot\begin{Bmatrix}u_{m1}\\u_{m2}\end{Bmatrix}\right] \end{cases} \tag{4}$$

where $(a,b)^t = \left(k_{p1}\delta(y - y_1), k_{p2}\delta(y - y_2)\right)$. The displacements are deduced from (4) in the same way as that to obtain equation (1).

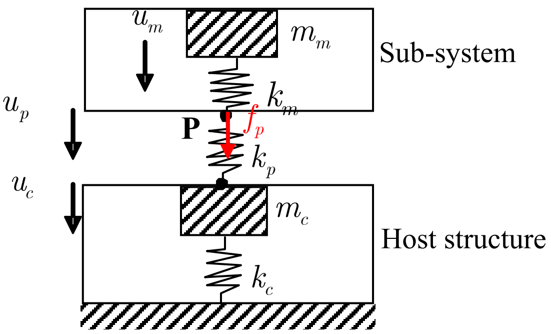
$$\begin{cases} u_m(y) = \dfrac{1}{E_m I_m} \displaystyle\sum_{n=0}^{\infty} \dfrac{1}{\Lambda_n^m\left(\left(\gamma_n^{(m)}\right)^4 - \gamma_m^4\right)} \Bigg[-\left(k_{p1}\psi_{nm}(y_1), k_{p2}\psi_{nm}(y_2)\right) \begin{Bmatrix} u_{m1} \\ u_{m2} \end{Bmatrix} \\ \qquad + \left(k_{p1}\psi_{nm}(y_1), k_{p2}\psi_{nm}(y_2)\right) \begin{Bmatrix} u_{c1} \\ u_{c2} \end{Bmatrix} + f_l\psi_{nm}(y_l) \Bigg] \psi_{nm}(y) \\ u_c(y) = \dfrac{1}{E_c I_c} \displaystyle\sum_{n=0}^{\infty} \dfrac{1}{\Lambda_n^c\left(\left(\gamma_n^{(c)}\right)^4 - \gamma_c^4\right)} \Bigg[-\left(k_{p1}\psi_{nc}(y_1), k_{p2}\psi_{nc}(y_2)\right) \begin{Bmatrix} u_{c1} \\ u_{c2} \end{Bmatrix} \\ \qquad + \left(k_{p1}\psi_{nc}(y_1), k_{p2}\psi_{nc}(y_2)\right) \begin{Bmatrix} u_{m1} \\ u_{m2} \end{Bmatrix} \Bigg] \psi_{nc}(y) \end{cases} \tag{5}$$

or, in matricial form,

$$\begin{cases} \left(\mathbf{Z}_{\mathrm{m}}' + \mathbf{K}_{\mathrm{p}}\right)\mathbf{u_m} - \mathbf{K_p}\mathbf{u_c} = \mathbf{f}_{\mathrm{m}} \\ -\mathbf{K_p}\mathbf{u_m} + \left(\mathbf{Z}_{\mathrm{c}}' + \mathbf{K}_{\mathrm{p}}\right)\mathbf{u_c} = \mathbf{0} \end{cases} \tag{6}$$

where $u_m = \left(u_m(y_1), u_m(y_2)\right)^t$, $u_c = \left(u_c(y_1), u_c(y_2)\right)^t$.The definitions of $\mathbf{Z}_{\mathrm{m}}'$, $\mathbf{K_p}$ and $\mathbf{Z}_{\mathrm{c}}'$ are easily found in equations (5).

It is of interest for further clarification to degenerate configuration 1 in figure 1 to configuration 2 in figure 2 , the operator of which is very simple.



$$\begin{cases} m_m\ddot{u}_m + k_m\left(u_m - u_p\right) = 0 \\ m_p\ddot{u}_p + k_m\left(u_p - u_m\right) + k_p\left(u_p - u_c\right) = f_p \\ m_c\ddot{u}_c + k_p\left(u_c - u_p\right) + k_c u_c = 0 \end{cases}$$

Figure 2. Mechanical vibratory system of 2 dof and associated equations.

In practice, the measurement of u_m is not accessible, contrarily to that of u_p. In these conditions the system is re-written to describe more closely the reality by removing u_m to obtain expression in measurable data. By neglecting m_p this results in

$$\begin{cases} \left(Z_m' + k_p\right)u_p - k_p u_c = f_p \\ -k_p u_p + \left(Z_c' + k_p\right)u_c = 0 \end{cases}$$

and the formal analogy with (6) is clear. It has to be emphasized that $Z_m' = \dfrac{k_m m_m \omega^2}{m_m\omega^2 - k_m}$ is in fact the impedance of the sub-system seen from P. Indeed, it corresponds to the definition of Z_m' trough $Z_m' u_p = f_p - k_m\left(u_p - u_m\right)$ without initial forces.

3. RELATION BETWEEN EFFORTS ENTERING IN TWO DIFFERENT HOST STRUCTURES

When the subsystem is no longer attached to the host structure playing the role of the chassis but to the marble, the writing (6) is still valid just by changing $\mathbf{Z_c}$ in $\mathbf{Z_b}$ (b like bench) and $\mathbf{u_c}$ in $\mathbf{u_b}$. Removing $\mathbf{u_m}$ from (6), the solution is $\mathbf{u_c}{=}(\mathbf{Z'_m K_p^{-1} Z'_c} + \mathbf{Z'_c} + \mathbf{Z'_m})^{-1}\mathbf{f_m}$ or, via $\mathbf{f_c} = \mathbf{Z'_c u_c}$, $\mathbf{f_c}{=}\mathbf{Z'_c}(\mathbf{Z'_m K_p^{-1} Z'_c} + \mathbf{Z'_c} + \mathbf{Z'_m})^{-1}\mathbf{f_m}$, and strictly in the same way , on the test bench, $\mathbf{f_b}{=}\mathbf{Z'_b}(\mathbf{Z'_m K_p^{-1} Z'_b} + \mathbf{Z'_b} + \mathbf{Z'_m})^{-1}\mathbf{f_m}$. Eliminating $\mathbf{f_m}$ between these two matricial equations results in the sought relation

$$\mathbf{f_c}{=}\mathbf{Z'_c}(\mathbf{Z'_m K_p^{-1} Z'_c} + \mathbf{Z'_c} + \mathbf{Z'_m})^{-1}(\mathbf{Z'_m K_p^{-1} Z'_b} + \mathbf{Z'_b} + \mathbf{Z'_m})\left(\mathbf{Z'_b}\right)^{-1}\mathbf{f_b} \tag{7}$$

It should be emphasized that, when $\|\mathbf{Z_b}\| \to \infty$, then $\mathbf{f_{b\infty}}{=}(\mathbf{Z'_m K_p^{-1}} + \mathbf{I})^{-1}\mathbf{f_m}$, and (7) becomes

$$\mathbf{f_c}{=}\mathbf{Z'_c}(\mathbf{Z'_m K_p^{-1} Z'_c} + \mathbf{Z'_c} + \mathbf{Z'_m})^{-1}(\mathbf{Z'_m K_p^{-1}} + \mathbf{I})\mathbf{f_{b\infty}} \tag{8}$$

In the scalar form (cf. fig. 2), (8) reduces to $\dfrac{f_c}{f_{b\infty}} = \dfrac{Z'_c\left(Z'_m + k_p\right)}{Z'_c\left(Z'_m + k_p\right) + Z'_m k_p}$, or

$$\frac{f_c}{f_{b\infty}} = \frac{Z'_c}{Z'_c + Z_M} \tag{9}$$

where $Z_M = \dfrac{Z'_m k_p}{Z'_m + k_p}$. Again the impedance Z_m at the origin of Z'_m is seen from the point of attach and thus is not $i(m_m\omega - k_m/\omega)$. Another remark deserves to be made : when deploying the process, measurements of $\mathbf{f_b}$, $\mathbf{Z_b}$, $\mathbf{Z_c}$ are necessary and, to validate the process, measurement of $\mathbf{f_c}$ is also needed. An indicator of the coherence of the measurements is in $\mathbf{f_{c\infty}}$ deduced from $\mathbf{Z_c}$ that should be equal to $\mathbf{f_{b\infty}}$ deduced from $\mathbf{Z_b}$.

4. CONSTRAINT ON THE EFFORTS ENTERING THE TEST BENCH VERSUS THE CONSTRAINT ON THE EFFORTS ENTERING THE HOST STRUCTURE

From equation (9), a given constraint $|f_c| \le r$ leads directly to the necessary and sufficient condition (NSC) on the force entering the marble

$$|f_{b\infty}| \le r\frac{\left|Z'_c + Z_M\right|}{\left|Z'_c\right|} \tag{10}$$

In the still elementary configuration of figure 1, a constraint of the same type as before on each component of the vector force entering the host structure,

$$\left|f_{cj}\right| \le f_{cj}^{\max} \tag{11}$$

also leads, at that stage of the procedure, to NSC. Indeed, (7) has the form $\mathbf{f}_c = \mathbf{A}.\mathbf{f}_b$ where matrix **A** of dimension 2x2 is made up of coefficients a_{ij}. By supposing that all components are real, condition (11) receives a geometrical translation in the (f_{b1}, f_{b2}) plane. Indeed, the constraint results in $\left|a_{j1}f_{b1} + a_{j2}f_{b2}\right| \le f_{cj}^{\max}$, or $-f_{cj}^{\max} \le a_{j1}f_{b1} + a_{j2}f_{b2} \le f_{cj}^{\max}$. The geometrical equivalent is the area on the (f_{b1}, f_{b2}) plane common to two bands, each between parallel lines of equations $a_{j1}f_{b1} + a_{j2}f_{b2} - f_{cj}^{\max} = 0$ and $a_{j1}f_{b1} + a_{j2}f_{b2} + f_{cj}^{\max} = 0$. How can this remark be extended to cases where the data are complex?

With still the same type of constraint, it must be noted that, in $\mathbf{f}_c = \mathbf{B}\mathbf{f}_{\mathbf{b}\infty}$, with $\mathbf{U}^H\mathbf{B}\mathbf{V} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ the singular value decomposition of **B**, and using $\mathbf{f}_\mathbf{c}^{'}$ and $\mathbf{f}_{\mathbf{b}\infty}^{'}$ denoted by $\mathbf{f}_\mathbf{c}^{'} = \mathbf{U}^\mathbf{H}\mathbf{f}_c$, $\mathbf{f}_{\mathbf{b}\infty}^{'} = \mathbf{V}^\mathbf{H}\mathbf{f}_{\mathbf{b}\infty}$ and the NSC of type (10) applied to a linear equation of the components of $\mathbf{f}_{\mathbf{b}\infty}$ extends to:

$$\left|\left(f_{b\infty}^{'}\right)_j\right| \le \frac{1}{\left|\sigma_j\right|}\left(f_c^{'}\right)_j^{\max} \tag{12}$$

Another type of constraint is

$$\left\|\mathbf{f}_c\right\|_{L^2} \le f_c^{\max} \tag{13}$$

in which case, a first development shows sufficient and not necessary conditions (SC) :

$$\begin{aligned} \left\|\mathbf{f}_c\right\|_{L^2}^2 &= \left\|\mathbf{A}\ .\ \mathbf{f}_b\right\|_{L^2}^2 \\ &\le \max\left(\left|a_{11}\right|^2, \left|a_{12}\right|^2\right) \left|f_{b1}\right|^2 \\ &\quad + \max\left(\left|a_{21}\right|^2, \left|a_{22}\right|^2\right) \left|f_{b2}\right|^2 + 4\ \max\left(\left|a_{11}a_{12}\right|, \left|a_{21}a_{22}\right|\right) \left|f_{b1}f_{b2}\right| \le \left(f_c^{\max}\right)^2 \end{aligned}$$

With the same constraint (13), another development results in a choice between various SCs :

$$\begin{aligned} \left\|\mathbf{f}_c\right\|_{L^2}^2 &= \left\|\mathbf{A}\ .\ \mathbf{f}_b\right\|_{L^2}^2 = \mathbf{f}_b^*\ .\ \mathbf{A}^*.\ \mathbf{A}\ .\ \mathbf{f}_b = \mathbf{f}_b^*\ .\ \mathbf{H}\ .\ \mathbf{f}_b \\ &\le \max\ (h_{ij}) \left|\sum_i f_{bi}\right|^2 \le \max\ (h_{ij})\ \left\|\mathbf{f}_b\right\|_{L^2}^2 \le n \cdot \max\ (h_{ij})\ \max\left(\left|f_{bi}\right|^2\right) \end{aligned}$$

and the choice consists in placing the inequality $\le \left(f_c^{\max}\right)^2$ where we want.

5. ADJUSTMENT OF THE SUSPENSION STIFFNESS IN THE DEGENERATED CONFIGURATION

To satisfy the constraint on $f_{b\infty}$, deduced from that on f_c (eq.10), the parameter chosen here is the suspension stiffness k_p. But the latter can also play a role in moving the resonance frequencies towards a region of low risk from the vibration point of view.

For this objective a numerical application has been carried out. The resonances of (9) are its poles. The denominator of second order in ω^2 leads to two real positive frequencies (without damping). It is well-known that they can be interpreted as the resonance frequencies of the systems before coupling moving away [3]. Choosing as a first resonance of the coupled system a frequency very close to that of the host structure alone, k_p is calculated in order to make the denominator of (10) equal to zero. The stiffness so obtained is quite weak, in accordance with the idea of decoupling the systems. It is well observed that the other resonance is near that of the sub-system alone. This exercise has been carried out with masses and stiffnesses originating from the car industry.

The adjustment of k_p to satisfy the constraint on $f_{b\infty}$ and thus on f_c is now given a geometrical interpretation. The reasoning is applied here to f_c but it is also valid for $f_{b\infty}$; damping is admitted. For each frequency under study, equation (9) and $f_{b\infty}(f_m,k_p) = \dfrac{k_p}{Z'_m + k_p} f_m$ result in $f_c(f_m,k_p) = \dfrac{Z'_c k_p}{(Z'_m + Z'_c)k_p + Z'_m Z'_c} f_m$. Moreover, the measurement on the test bench has provided the entering force $f_{b\infty 0}$ for a given suspension of stiffness k_{p0}; the quantity can thus be deduced and is here a complex data. The latter expression leads to a circle in the complex plane $\left(\Re(f_c),\Im(f_c)\right)$ (cf. figures 3 and 4). From there, the problem with constraint $|f_c| \le r$ has a solution for k_p values such that the arc of circle f_c is included in the constraint circle, centered at the origin. It should be noted that, without damping at all, circle f_c degenerates along the real axis since all quantities are real.

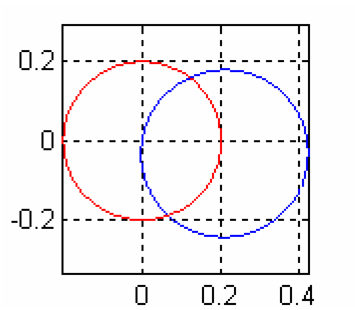


Figure 3. Geometrical representation of the constrained problem with 2x1dof

— $f_c^{\max} = 0.2\,N$
— at $500Hz$, $Z_c = 60. - 20.4i$ N/m/s
$Z_m = 91. + 14.\ i$ N/m/s

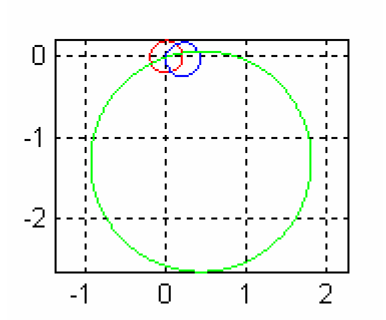


Figure 4. Geometrical representation of the constrained problem with 2x1dof for two frequencies

— $f_c^{\max} = 0.2\,N$
— at $500Hz$, $Z_c = 60. - 20.4i$ N/m/s
$Z_m = 91. + 14.\ i$ N/m/s
— at $850Hz$, $Z_c = 24.6 + 467.\ i$ N/m/s
$Z_m = 142. + 430.\ i$ N/m/s

When dealing with a few frequencies, the admissible k_p values are at the intersections of the k_p intervals for each frequency.

As a numerical application, given the mass of the sub-system of $\mathrm{m_m}$=2.3Kg, the constraint $f_c^{\max} = 0.2\,N$ and, at $500Hz$, $Z_c = 60. - 20.4i$, $Z_m = 91. + 14.\ i$, $k_p \in \left[0,\ 4.3\ 10^5\right]$ in N/m appeared. At $850Hz$, $Z_c = 24.6 + 467.\ i$, $Z_m = 142. + 430.\ i$ and the numerical interval is now $k_p \in \left[0,\ \ 5.4\ 10^5\right]$. The constraint is thus satisfied for both frequencies if and only if $k_p \in \left[0,\ 4.3\ 10^5\right]$.

6. CONCLUSION

In the generic problem of a 1 dof subsystem linked to a 1 dof host structure by an elastic suspension, a predictive approach consists in determining the effort entering the host structure from the forces measured entering the marble. So, by anticipating the behavior of the sub-system when attached to the future host structure, a constraint is given on the effort entering the host structure. In that case, the passage to the constraint on the marble is straightforward and is clearly a necessary and sufficient condition. Now, in presence of a few attach points between two deformable structures, the analysis provides matricial relations through a stiffness matrix linked to the suspensions. It has been shown that the necessary and sufficient conditions are now applied not to each component of the efforts entering the marble, but to their linear combinations. Were we interested in the constraint on each individual component, we would have no choice but to accept sufficient and not necessary conditions with the danger of excessive severization (a neologism denoting a tightening of standards or requirements, difficult to satisfy). Remaining with the necessary and sufficient condition, how is it possible to extract the relation to be satisfied by the stiffness matrix associated with the suspensions, and how can the optimal values of each elastic suspension be deduced? This next step is currently being investigated via the geometrical interpretation.

REFERENCES

[1] P.M. Morse and K.U. Ingard, *Theoretical Acoustics*, Princeton University Press, 1986 (first published in 1968).

[2] M. Roseau, *Vibrations des systèmes mécaniques, Méthodes analytiques et applications,* Masson, 1985.

[3] J.W.S. Rayleigh, *The theory of Sound*, Dover Publications, 1945 (first published in 1877).

The authors thank the Upper-Normandy Regional Council in France for contributing in the funding of the present research emanating from the french car industry.