



A SPECTRAL FINITE ELEMENT MODEL FOR THIN PLATE VIBRATION

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Abstract

The finite element analysis of plate vibration has become one of the classical problems that received a lot of attention from the researchers through the past few decades. Different models were developed including the classical polynomial elements, the hierarchical finite element models, and most recently, the spectral finite element models. Developers of different models introduced there models and performed many studies to prove that the models were suitable and accurate with validations performed against analytical models, when available, and with other numerical models. In this study, the first plate spectral finite element model is presented with a generalized methodology for the derivation element matrices. Three different finite element models will be compared in the study of free vibration characteristics of an isotropic plate with different boundary conditions. The aim of the study is to point out the points of weakness and strength of each model and to emphasise the ease and interchangeability of the models. The models compared are a classical 3rd order/4-node element, a 7th order/ 16 node element, and the proposed spectral finite element model. The models were created using a symbolic manipulator, Mathematica® 4.1, in order to get the elements' matrices in closed form to avoid the errors introduced by the numerical integration that is usually used in creating the element matrices. Results are then compared with those obtained using numerical integration performed by creating a similar code using MATLAB 6.1.

1. INTRODUCTION

The spectral finite element method is a development of the dynamic stiffness matrix approach. The dynamic stiffness matrix approach is distinct from conventional finite element in that it depends on the exact solution of the differential equations involved with the problem [1]; that solution is used instead of the polynomial trial functions to result in a finite-element-like model that could capture the exact dynamics of the structure.

The higher accuracy of the dynamic stiffness matrix method arises from the fact that the higher the frequency of vibration, the shorter the wave length of the propagating wave; thus, to accurately capture the dynamics of the structure, finer mesh is required for conventional finite element models, which in turn means larger model and more computational effort. On the other hand, the nature of the dynamic stiffness matrix is that the dynamic solution is

embedded in the matrix model of the element [2], thus, less elements are required to present the structure with the same or higher accuracy compared to conventional finite element method.

Langley [3] suggested the application of dynamic stiffness matrix method for the study of free and forced vibration of aircraft panels, and later [4], he coupled it with boundary element method to study the acoustic radiations inside an aircraft fuselage. In both studies, the accuracy of the model was illustrated.

The spectral analysis has been a method used for the approximate solution of different types of structural vibrations problems [5]-[11]. Tawfik and Baz [12] resented, for the first time, a spectral finite element model for a general plate. The model used numerical integration for the evaluation of element matrices. They validated their results using classical solutions as well as experimental results for a plate with bonded piezoelectric patches and shunt electric circuits.

The previous survey showed that the spectral finite element model have proven higher accuracy and efficiency in the modelling of the one-dimensional structures. It has also, proven effective in the reduction of the finite element model for structures with viscoeleastic components. On the other hand, the spectral finite element models for plates were only developed for the Levy-type plates or the system matrices had to be evaluated using numerical integration.

In this study, a spectral finite element model is going to be presented together with a 3rd order/4-node and 7th order/16-node models for the analysis of the free vibration characteristics of a plate with different boundary conditions. The main aim of this study is to emphasise the simplicity of developing a generalized finite element model for different structure vibration problems as well as presenting a solid base comparison of the different models.

2. PLATE FINITE ELEMENT MODEL

In this section, the different trial functions used to model the plate are going to be presented, then, the generalized finite element model of a thin plate element, using the classical plate theory, will be presented independent of the trial functions. All the elements presented are assumed rectangular elements that are aligned with the global coordinates.



2.1 Trial Functions

2.1.1 Conventional 3rd order, 16-DOF, element

Figure 1. Node numbering scheme for the 3rd order and spectral finite element models.

The 4-node element could be presented by the sketch in Figure 1. The transverse displacement w(x,y) at any location x and y inside the plate element is expressed by

$$w(x, y) = a_{1} + a_{2}x + a_{3}y + a_{4}xy + a_{5}x^{2} + a_{6}y^{2} + a_{7}x^{2}y + a_{8}xy^{2} + a_{9}x^{3} + a_{10}x^{2}y + a_{11}xy^{2} + a_{12}y^{3} + a_{13}x^{3}y + a_{14}x^{2}y^{2} + a_{15}xy^{3} + a_{16}x^{3}y^{3}$$
(1)

where coefficients a_1 through a_{16} are to be determined in terms of the nodal displacements

2.1.2 7th order, 64-DOF, element

To present the displacement distribution in the 16-node element presented in Figure 2, a 64-term polynomial may be used that contains all the x and y-powers up to the 7th order. The unknown coefficients of the polynomial will extend from a_1 through a_{64} .

2.1.3 Spectral Element Trial Function

The proposed spectral finite element model is based on the use of exponential functions as trial functions instead of the usual polynomials that are used with conventional finite element models. The transverse displacement w(x,y) at any location x and y inside the plate element may be expressed by



Figure 2. Node numbering scheme for the 7th order 16-node element.

$$w(x, y) = a_{1}e^{k_{x}x}e^{k_{y}y} + a_{2}e^{k_{x}x}e^{-k_{y}y} + a_{3}e^{k_{x}x}e^{ik_{y}y} + a_{4}e^{k_{x}x}e^{-ik_{y}y} + a_{5}e^{-k_{x}x}e^{k_{y}y} + a_{6}e^{-k_{x}x}e^{-k_{y}y} + a_{7}e^{-k_{x}x}e^{ik_{y}y} + a_{8}e^{-k_{x}x}e^{-ik_{y}y} + a_{9}e^{ik_{x}x}e^{k_{y}y} + a_{10}e^{ik_{x}x}e^{-k_{y}y} + a_{11}e^{ik_{x}x}e^{ik_{y}y} + a_{12}e^{ik_{x}x}e^{-ik_{y}y} + a_{13}e^{-ik_{x}x}e^{k_{y}y} + a_{14}e^{-ik_{x}x}e^{-k_{y}y} + a_{15}e^{-ik_{x}x}e^{ik_{y}y} + a_{16}e^{-ik_{x}x}e^{-ik_{y}y}$$
(2)

where k_x is the wave number in the x-direction and k_y is the wave number in the y-direction; and:

$$k = \left(\frac{\omega^2 \rho h}{D}\right)^{\frac{1}{4}}$$
, and $k_x = kCos(\theta), k_y = kSin(\theta)$

where k is the wave number of a planar wave propagating at an angle θ measured anticlockwise from the positive x-axis (Figure 3), ρ is the mass density, h plate thickness, and D is the plate flexural rigidity given by

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

Also, the coefficients a_1 through a_{16} are to be determined in terms of the nodal displacements. The above proposed trial function is a generalization of the beam trial function proposed by Doyle [5].



Figure 3. Planar wave propagating in a plate.

2.1.4 Unknown Coefficients in Terms of Nodal Displacement

In general, the displacement trial function could be written as,

$$w(x,y) = \lfloor H_w \rfloor \{a\}$$
(3)

where $\{a\} = [a_1 \cdots a_{16}]$ for 3rd order and spectral element and $\{a\} = [a_1 \cdots a_{64}]$ for the 7th order element.

For the elements under consideration, 4 degrees of freedom are associated with each node; namely, w for the displacement, $w_x & w_y$ for the slope in the x and y-directions respectively, and w_{xy} for the cross derivative of displacement. Thus, we may write,

Where the subscript (,x & ,y) indicate the derivatives in the x and y-directions respectively. Substituting the nodal coordinates into equation (4), we obtain the nodal bending displacement vector $\{w_b\}$ in terms of $\{a\}$ as follows,

$$\{w_b\} = [T_b]\{a\}$$
⁽⁵⁾

where $\{w_b\}$ and $[T_b]$ are the element degrees of freedom and the transformation matrix respectively.

From equation (5), we obtain

$$\{a\} = [T_b]^{-1}\{w_b\}$$
⁽⁶⁾

Substituting equation (6) into equation (5) gives

$$w(x, y) = [H_w][T_b]^{-1} \{w_b\} = [N_w] \{w_b\}$$
(7)

where $[N_w]$ is the shape function for bending given by

$$\begin{bmatrix} N_w \end{bmatrix} = \begin{bmatrix} H_w \end{bmatrix} \begin{bmatrix} T_b \end{bmatrix}^{-1}$$
(8)

2.2 Strain Displacement Relations

Consider the lateral deflection; the classical plate strain-displacement relation for the lateral deflections of thin plates can be written as follows

$$\{\varepsilon\} = \begin{cases} \varepsilon_{x} \\ \varepsilon_{y} \\ \gamma_{xy} \end{cases} = z \begin{cases} -\frac{\partial^{2} w}{\partial x^{2}} \\ -\frac{\partial^{2} w}{\partial y^{2}} \\ -2\frac{\partial^{2} w}{\partial x \partial y} \end{cases} = z \{\kappa\} = z [C_{b}] [T_{b}]^{-1} \{w_{b}\} = z [B_{b}] \{w_{b}\}$$
(9)

Where $\{\varepsilon\}$ is the strain vector, z is the vertical distance measure from the plate mid plane, and the curvature vector $\{\kappa\}$, and

$$\begin{bmatrix} C_b \end{bmatrix} = - \begin{bmatrix} H_{w_{,xx}} \\ H_{w_{,xyy}} \\ 2H_{w_{,xy}} \end{bmatrix}$$
(10)

Thus, the strain-nodal displacement relation can be written as

$$\{\varepsilon\} = z\{\kappa\} = z[B_b]\{w_b\}$$
(11)

2.3 Element Matrices

Principal of virtual work states that

$$\delta \Pi = \delta (U - T) = 0 \tag{12}$$

where Π is the total energy of the system, U is the strain energy, T is the kinetic energy, and $\delta(.)$ denotes the first variation.

2.3.1 The Potential Energy

The variation of the potential energy for a thin plate is given by

$$\delta U = \int_{V} \{\delta \varepsilon\}^{T} \{\sigma\} dV$$
(13)

where $\{\sigma\}$ is the stress vector and V is the volume of the structure. Substituting from equations (15) we get,

$$\delta U = \int_{V} z^{2} \{\delta w_{b}\}^{T} [B_{b}]^{T} [Q] [B_{b}] \{w_{b}\} dV$$

$$= \{\delta w_{b}\}^{T} [T_{b}]^{-T} \frac{h^{3}}{12} \int_{A} [C_{b}] [Q] [C_{b}] dA [T_{b}]^{-1} \{w_{b}\}$$

$$= \{\delta w_{b}\}^{T} [k_{b}] \{w_{b}\}$$
(14)

where z is the vertical distance measured from the plate mid-plane, $[.]^{-T}$ is the transpose of the inverse, $[k_b]$ is the element bending stiffness matrix, and [Q] is the stress strain constitutive elation. In this study, only isotropic plates are going to be used to illustrate the procedure and

the elements' differences, however, the same procedure is applicable to general orthographic plates as well as composite plates.

2.3.2 The Kinetic Energy

The variation of the kinetic energy T of the plate element, ignoring the rotary inertia, is given by,

$$\delta T = \int_{V} \delta w \rho \frac{\partial^2 w}{\partial t^2} dV \tag{15}$$

where ρ is the density. The above equation can be rewritten in terms of nodal displacements as follows

$$\delta T = \int_{V} \delta w \rho \frac{\partial^{2} w}{\partial t^{2}} dV = \int_{V} \{\delta w_{b}\}^{T} [N_{w}]^{T} \rho [N_{w}] \{\ddot{w}_{b}\} dV$$
$$= \{\delta w_{b}\}^{T} [T_{b}]^{-T} \int_{A} [H_{w}]^{T} \rho [H_{w}] dA. \{\ddot{w}_{b}\}$$
$$= \{\delta w_{b}\}^{T} [m_{b}] \{\ddot{w}_{b}\}$$
(16)

where $[m_b]$ is the element bending mass matrix. Finally, the element equation of free vibration can be written as

$$[m_b]\{\ddot{w}_b\} + [k_b]\{w_b\} = \{0\}$$
(17)

Note that the matrices constituting the above equation are all dependent on the driving frequency for the case of the spectral finite element model. Thus, the eigenvalue problem can not be solved directly for the system natural frequencies; rather, an iterative method should be used to obtain the eigenvalues.

2.4 Steps for Evaluating the Spectral Element Matrices

- 1. Determine the frequency at which the system is vibrating.
- 2. Evaluate the interpolation function, H_w , and its derivatives, $H_{w,x}$, $H_{w,y}$, and $H_{w,xy}$, at all the element nodes
- 3. Evaluate the matrix T_b then invert it
- 4. Evaluate the integrals of the system matrices. Note that the matrix T_b is constant with respect to the integrals which enable the simplification of the integral by delaying the multiplication of the matrix until the end of the integral evaluation.

Note that the above steps are generic for any finite element model for the modelling or rectangular plates except for that the frequency loop is must included for the spectral model. Nevertheless, the integral mentioned in the fourth step could be evaluated using analytical or numerical methods.

To get the system Eigenvalues for the proposed spectral finite element model, the problem is solved using any, *reasonable*, initial frequency; then, the resulting eigenvalue should be used in a second iteration. The model usually converges in less than 5 iterations except for very low frequencies (less than 10Hz).

3. NUMERICAL RESULTS

A symbolic manipulator, Mathematics (0, 1) 4.1, was utilized for writing the finite element procedure. The integrals needed for the element matrices where, thus, evaluated analytically for the square plate elements used. Note that, each 7th order element introduces the same number of nodes / DOF's that is introduced by the 3rd order and the spectral element; thus, in all the results presented, the number of elements used for both models is devisable by 9.

3.1 Convergence

The convergence of the different models was tested against analytical results for a simply supported plate as well as classical solutions presented for plates with clamped and free boundary conditions. Table 1 through Table 3 presents the normalized frequency parameter for a simply supported (SSSS) square plate; where the normalized frequency parameter is given by the closed form solution given by

$$\overline{\omega} = \omega \frac{a^2}{\pi^2} \sqrt{\frac{\rho h}{D}} = n^2 + m^2 \tag{18}$$

where ω is the normalized frequency parameter, ω is the natural frequency of the plate, *a* is the plate length, *h* is the plate thickness, and *n* and *m* are the mode number in the *x* and *y* directions respectively.

The results presented in Table 1 through Table 3 show that all the proposed finite element models are capable of converging fast for higher modes of vibration. It is also noticeable that the error of the 3rd order and spectral models are similar for different modes and different mesh sizes. Meanwhile, for the 7th order model, the error is higher in finer mesh sizes!

				3	х3	6>	(6	9>	(9	12x	12	15x	:15
			Exact	Elen	nents	Elem	ients	Elem	ients	Elem	ents	Elem	ents
#	n	М	$\overline{\omega}$	$\overline{\omega}$	Error								
1	1	1	2	2.07	3.7%	2.04	2.0%	2.03	1.4%	2.02	1.1%	2.02	0.9%
2	1	2	5	5.18	3.7%	5.07	1.4%	5.05	0.9%	5.04	0.7%	5.03	0.6%
3	2	1	5	5.18	3.7%	5.07	1.4%	5.05	0.9%	5.04	0.7%	5.03	0.6%
4	2	2	8	8.43	5.4%	8.19	2.4%	8.12	1.5%	8.09	1.2%	8.07	0.9%
5	1	3	10	10.73	7.3%	10.09	0.9%	10.05	0.5%	10.04	0.4%	10.03	0.3%
6	3	1	10	11.03	10.3%	10.14	1.4%	10.07	0.7%	10.05	0.5%	10.04	0.4%
7	2	3	13	14.44	11.1%	13.31	2.4%	13.18	1.4%	13.13	1.0%	13.11	0.8%
8	3	2	13	14.44	11.1%	13.31	2.4%	13.18	1.4%	13.13	1.0%	13.11	0.8%
9	1	4	17	18.92	11.3%	17.22	1.3%	17.09	0.5%	17.05	0.3%	17.04	0.2%
10	4	1	17	18.92	11.3%	17.22	1.3%	17.09	0.5%	17.05	0.3%	17.04	0.2%

Table 1 Normalized frequency parameter convergence for an SSSS plate compared to exact results for spectral finite element model

Table 2 Normalized frequency parameter convergence for an SSSS plate compared to exact results for the 3rd order element.

				3x3	6x6	9x9	12x12	15x15
#	n	m	Exact	Elements	Elements	Elements	Elements	Elements

1	I	1	I —	I —		—	1	—	1	I —	1	I —	1
			ω	ω	Error								
1	1	1	2	2.07	3.7%	2.04	2.0%	2.03	1.4%	2.02	1.1%	2.02	0.9%
2	1	2	5	5.21	4.2%	5.07	1.5%	5.05	1.0%	5.04	0.7%	5.03	0.6%
3	2	1	5	5.21	4.2%	5.07	1.5%	5.05	1.0%	5.04	0.7%	5.03	0.6%
4	2	2	8	8.50	6.3%	8.20	2.5%	8.12	1.5%	8.09	1.2%	8.07	0.9%
5	1	3	10	11.20	12.0%	10.10	1.0%	10.05	0.5%	10.04	0.4%	10.03	0.3%
6	3	1	10	11.58	15.8%	10.15	1.5%	10.07	0.7%	10.05	0.5%	10.04	0.4%
7	2	3	13	15.26	17.4%	13.33	2.5%	13.19	1.4%	13.13	1.0%	13.11	0.8%
8	3	2	13	15.26	17.4%	13.33	2.5%	13.19	1.4%	13.13	1.0%	13.11	0.8%
9	1	4	17	19.82	16.6%	17.29	1.7%	17.10	0.6%	17.06	0.3%	17.04	0.2%
10	4	1	17	19.82	16.6%	17.29	1.7%	17.10	0.6%	17.06	0.3%	17.04	0.2%

Table 3 Normalized frequency parameter convergence for an SSSS plate compared to exact results for the 7th order element.

				1	x1	2>	(2	3>	(3	4x	:4	5x	5
			Exact	Elen	nents	Elem	nents	Elem	ents	Elem	ents	Elem	ents
#	n	m	$\overline{\omega}$	$\overline{\omega}$	Error								
1	1	1	2	2.08	4.0%	2.16	8.0%	2.11	5.5%	2.09	4.3%	2.07	3.6%
2	1	2	5	5.38	7.6%	5.19	3.8%	5.18	3.7%	5.14	2.8%	5.11	2.3%
3	2	1	5	5.38	7.6%	5.19	3.8%	5.18	3.7%	5.14	2.8%	5.11	2.3%
4	2	2	8	8.99	12.3%	8.18	2.2%	8.46	5.8%	8.35	4.3%	8.28	3.5%
5	1	3	10	10.29	2.9%	10.07	0.7%	10.12	1.2%	10.14	1.4%	10.12	1.2%
6	3	1	10	11.02	10.2%	10.68	6.8%	10.17	1.7%	10.18	1.8%	10.14	1.4%
7	2	3	13	14.95	15.0%	13.58	4.5%	13.42	3.2%	13.48	3.7%	13.39	3.0%
8	3	2	13	14.95	15.0%	13.58	4.5%	13.42	3.2%	13.48	3.7%	13.39	3.0%
9	1	4	17	17.96	5.6%	17.44	2.6%	17.24	1.4%	17.12	0.7%	17.14	0.8%
10	4	1	17	17.96	5.6%	17.44	2.6%	17.24	1.4%	17.12	0.7%	17.14	0.8%

Table 4 through Table 6 present the frequency parameter results obtained for a clamped square plate (CCCC) with Poisson's ratio of 0.3 compared to classical results presented in the book by Leissa [13]. The frequency parameter $\overline{\omega}^*$ is given by the equation

$$\overline{\omega}^* = \pi^2 \overline{\omega} \tag{19}$$

It can be easily noticed from the results presented that very reasonable results were obtained for this case with only 36 elements in the cases of 3^{rd} order and spectral elements and only 4 elements for the case of 7^{th} order element.

Table 4 Frequency parameter convergence for CCCC plate compared to classical results [13] for the spectral element

		3>	(3	6x6		9x9		12x12		15x15	
		Elem	ients	Elem	ents	Elem	ents	Elem	ients	Eleme	ents
		*		*		*		*		*	
#	Classical	ω	Error								
1	35.11	36.14	2.9%	36.00	2.5%	35.99	2.5%	35.99	2.5%	35.99	2.5%
2	72.93	74.92	2.7%	73.52	0.8%	73.42	0.7%	73.40	0.6%	73.40	0.6%
3	72.93	74.92	2.7%	73.52	0.8%	73.42	0.7%	73.40	0.6%	73.40	0.6%
4	107.5	111.69	3.9%	108.48	0.9%	108.27	0.7%	108.24	0.7%	108.22	0.7%
5	131.7	156.19	18.6%	132.46	0.6%	131.76	0.1%	131.64	0.0%	131.60	0.0%

6	131.7	156.59	18.9%	133.06	1.1%	132.38	0.6%	132.26	0.5%	132.22	0.4%
7	164.4	189.27	15.2%	166.00	1.0%	165.21	0.5%	165.07	0.4%	165.02	0.4%
8	164.4	189.27	15.2%	166.00	1.0%	165.21	0.5%	165.07	0.4%	165.02	0.4%
9	210.3	300.24	42.7%	214.01	1.7%	211.31	0.5%	210.78	0.2%	210.59	0.1%
10	210.3	300.24	42.7%	214.01	1.7%	211.31	0.5%	210.78	0.2%	210.59	0.1%

Table 5 Frequency parameter convergence for CCCC plate compared to classical results [13] for the 3rd order element

		3>	(3	6x	6	9x	9	12x	:12	15x	15
		Elem	ients	Elem	ents	Eleme	ents	Elem	ents	Eleme	ents
		*		*		*		*		*	
#	Classical	ω	Error								
1	35.11	36.20	3.1%	36.00	2.5%	35.99	2.5%	35.99	2.5%	35.99	2.5%
2	72.93	75.01	2.9%	73.53	0.8%	73.42	0.7%	73.40	0.6%	73.40	0.6%
3	72.93	75.01	2.9%	73.53	0.8%	73.42	0.7%	73.40	0.6%	73.40	0.6%
4	107.5	111.82	4.0%	108.49	0.9%	108.28	0.7%	108.24	0.7%	108.23	0.7%
5	131.7	156.53	18.9%	132.47	0.6%	131.77	0.1%	131.64	0.0%	131.61	0.0%
6	131.7	156.93	19.2%	133.07	1.1%	132.39	0.6%	132.26	0.5%	132.23	0.4%
7	164.4	189.61	15.4%	166.02	1.0%	165.22	0.5%	165.07	0.4%	165.03	0.4%
8	164.4	189.61	15.4%	166.02	1.0%	165.22	0.5%	165.07	0.4%	165.03	0.4%
9	210.3	300.61	42.9%	214.03	1.8%	211.31	0.5%	210.78	0.2%	210.63	0.1%
10	210.3	300.61	42.9%	214.03	1.8%	211.31	0.5%	210.78	0.2%	210.63	0.1%

Table 6 Frequency parameter convergence for CCCC plate compared to classical results [13] for the 7^{th} order element

		3	x3	6x	6	9	x9	12>	(12	15:	x15
		Eler	nents	Eleme	ents	Elen	nents	Elem	ients	Elen	nents
		*	_	*	_	*	_	*	_	*	_
#	Classical	Ø	Error	ω	Error	ω	Error	ω	Error	ω	Error
1	35.11	35.99	2.5%	35.99	2.5%	35.99	2.5%	35.99	2.5%	35.99	2.5%
2	72.93	73.42	0.7%	73.39	0.6%	73.39	0.6%	73.39	0.6%	73.39	0.6%
3	72.93	73.42	0.7%	73.39	0.6%	73.39	0.6%	73.39	0.6%	73.39	0.6%
4	107.5	108.26	0.7%	108.22	0.6%	108.22	0.6%	108.22	0.6%	108.22	0.6%
5	131.7	137.29	4.3%	131.59	0.0%	131.58	-0.1%	131.58	-0.1%	131.58	-0.1%
6	131.7	138.07	4.9%	132.22	0.4%	132.21	0.4%	132.21	0.4%	132.21	0.4%
7	164.4	168.81	2.7%	165.01	0.4%	165.00	0.4%	165.00	0.4%	165.00	0.4%
8	164.4	168.82	2.7%	165.01	0.4%	165.00	0.4%	165.00	0.4%	165.00	0.4%
9	210.3	230.45	9.6%	210.58	0.1%	210.52	0.1%	210.52	0.1%	210.52	0.1%
10	210.3	231.15	9.9%	210.58	0.1%	210.52	0.1%	210.52	0.1%	210.52	0.1%

Table 7 through Table 9 present the frequency parameter results for a free square plate. It should be noticed that 0% relative error was obtained for the case of 144 spectral elements for almost all the modes. It should also be noticed that the 9th and 10th modes had negative relative error which is not seen as a mistake in the finite element model since the results are already compared to classical *approximate solution* of the problem. Similar results were obtained for the case of the 3rd order element model while only 4 elements were sufficient for the 7th order model.

Table 7 Frequency parameter convergence for FFFF plate compared to classical results [13] for the spectral element.

#	Classical					
		3x3	6x6	9x9	12x12	15x15

		Eler	nents	Elem	ents	Elen	nents	Elem	ents	Elem	ents
		*		*		*		*		*	
		ω	Error	ω	Error	ω	Error	ω	Error	ω	Error
1	13.47	28.37	110.6%	14.81	10.0%	13.75	2.0%	13.47	0.0%		
2	19.6	31.13	58.9%	20.51	4.7%	19.78	0.9%	19.60	0.0%	19.60	0.0%
3	24.27	34.09	40.5%	24.98	2.9%	24.41	0.6%	24.27	0.0%	24.27	0.0%
4	34.8	40.91	17.5%	35.21	1.2%	34.88	0.2%	34.80	0.0%	34.80	0.0%
5	34.8	40.91	17.5%	35.21	1.2%	34.88	0.2%	34.80	0.0%	34.80	0.0%
6	61.09	64.47	5.5%	61.26	0.3%	61.12	0.0%	61.10	0.0%	61.10	0.0%
7	61.09	64.47	5.5%	61.26	0.3%	61.12	0.0%	61.10	0.0%	61.10	0.0%
8	63.69	65.72	3.2%	63.80	0.2%	63.71	0.0%	63.69	0.0%	63.69	0.0%
9	69.5	71.29	2.6%	69.37	-0.2%	69.28	-0.3%	69.27	-0.3%	69.27	-0.3%
10	77.59	78.88	1.7%	77.25	-0.4%	77.19	-0.5%	77.18	-0.5%	77.17	-0.5%

Table 8 Frequency parameter convergence for FFFF plate compared to classical results[13] for the 3rd order element.

		3	x3	6>	(6	9	x9	12>	(12	15x	(15
		Elen	nents	Elem	ients	Elen	nents	Elem	ients	Elem	ents
	<u>.</u>	*	_	*	_	*	_	*	_	*	_
#	Classical	ω	Error								
1	13.47	13.49	0.2%	13.47	0.0%	13.47	0.0%	13.47	0.0%	13.47	0.0%
2	19.6	19.64	0.2%	19.60	0.0%	19.60	0.0%	19.60	0.0%	19.60	0.0%
3	24.27	24.36	0.4%	24.28	0.0%	24.27	0.0%	24.27	0.0%	24.27	0.0%
4	34.8	35.01	0.6%	34.82	0.1%	34.81	0.0%	34.80	0.0%	34.80	0.0%
5	34.8	35.01	0.6%	34.82	0.1%	34.81	0.0%	34.80	0.0%	34.80	0.0%
6	61.09	61.45	0.6%	61.20	0.2%	61.12	0.0%	61.10	0.0%	61.10	0.0%
7	61.09	61.45	0.6%	61.20	0.2%	61.12	0.0%	61.10	0.0%	61.10	0.0%
8	63.69	64.55	1.3%	63.77	0.1%	63.71	0.0%	63.69	0.0%	63.69	0.0%
9	69.5	69.58	0.1%	69.39	-0.2%	69.29	-0.3%	69.27	-0.3%	69.27	-0.3%
10	77.59	77.63	0.1%	77.35	-0.3%	77.21	-0.5%	77.19	-0.5%	77.18	-0.5%

Table 9 Frequency parameter convergence for FFFF plate compared to classical results [13] for the 7th order element

		1:	x1	2>	(2	3:	x3	4>	(4	5×	(5
		Elen	nents	Elem	ients	Elen	nents	Elem	ents	Elem	ents
		*		*		*		*		*	
#	Classical	ω	Error	ω	Error	ω	Error	ω	Error	ω	Error
1	13.47	13.47	0.0%	13.47	0.0%	13.47	0.0%	13.47	0.0%	13.47	0.0%
2	19.6	19.60	0.0%	19.60	0.0%	19.60	0.0%	19.60	0.0%	19.60	0.0%
3	24.27	24.27	0.0%	24.27	0.0%	24.27	0.0%	24.27	0.0%	24.27	0.0%
4	34.8	34.80	0.0%	34.80	0.0%	34.80	0.0%	34.80	0.0%	34.80	0.0%
5	34.8	34.80	0.0%	34.80	0.0%	34.80	0.0%	34.80	0.0%	34.80	0.0%
6	61.09	61.11	0.0%	61.09	0.0%	61.09	0.0%	61.09	0.0%	61.09	0.0%
7	61.09	61.11	0.0%	61.09	0.0%	61.09	0.0%	61.09	0.0%	61.09	0.0%
8	63.69	63.69	0.0%	63.69	0.0%	63.69	0.0%	63.69	0.0%	63.69	0.0%
9	69.5	69.28	-0.3%	69.27	-0.3%	69.27	-0.3%	69.27	-0.3%	69.27	-0.3%
10	77.59	77.20	-0.5%	77.17	-0.5%	77.17	-0.5%	77.17	-0.5%	77.17	-0.5%

3.2 Wave Direction

An important parameter of the spectral finite element model derived in section 2 is the wave direction angle θ . That parameter seems to affect all the equations derived for the spectral

finite element model. In all the previous results, θ was taken as 45°. It seemed natural that the effect of this parameter should be tested.

At this point, it should be noticed that all the simply supported symmetricantisymmetric modes (e.g. mode (1,4) and (4,1)) are the ones that give exactly the same values for the normalized frequency parameters (see Table 1); in contrast with all other symmetric-symmetric and anisymmetric-antisymmetric modes (e.g. mode (1,3), (3,1) and (2,4), (4,2)).

Table 10 presents the values obtained for the normalized frequency parameter for a square simply supported plate with different values of the direction angle θ . The number of elements was taken to be 81 elements. It can be noticed that as the angle is deviated from the symmetric 45°, the previous observation does not hold true anymore which can be understood easily to be a result of the broken symmetry. Nevertheless, the normalized frequency parameter results deviate slightly as the angle is changed from 5° to 85°. It should also be noticed that the spectral finite element model proposed here fails as the angle becomes 0° or 90°.

Figure 4 presents the variation of the normalized frequency parameter of modes (1,3) and (3,1) for the plate with the change in the wave angle from 5° to 85°. A distinction can not be made between both curves since, for a square plate, both should be equal theoretically. On the other hand, it is noticeable that the difference between both corves is less than 0.3%. Figure 5 presents contour shading for the mode shape of the plate for both modes when the wave propagation angle is 5° (note that 5° and 85° are similar cases). Note that there is no significant difference between Figure 5 and



Figure 4. Variation of the normalized frequency parameter with the wave direction angle for modes (1,3) and (3,1) of a simply supported square plate.

the plot in Figure 6 which presents the plots for the modes shapes at wave propagation angle of 45° . Nevertheless, the mode shapes shown match the theoretical expectations.

#	Ν	m	Exact	$\theta = 5^{\circ}$	$\theta = 15^{\circ}$	θ =25°	$\theta = 35^{\circ}$	θ =45°	$\theta = 55^{\circ}$	θ =65°	$\theta = 75^{\circ}$	$\theta = 85^{\circ}$
1	1	1	2	2.03	2.03	2.03	2.03	2.03	2.03	2.03	2.03	2.03
2	1	2	5	5.05	5.05	5.05	5.05	5.05	5.05	5.05	5.05	5.05
3	2	1	5	5.05	5.05	5.05	5.05	5.05	5.05	5.05	5.05	5.05
4	2	2	8	8.13	8.13	8.12	8.12	8.12	8.12	8.12	8.13	8.13
5	1	3	10	10.05	10.05	10.05	10.05	10.05	10.05	10.05	10.05	10.05
6	3	1	10	10.08	10.08	10.07	10.07	10.07	10.07	10.07	10.08	10.08
7	2	3	13	13.19	13.19	13.18	13.18	13.18	13.18	13.18	13.19	13.19
8	3	2	13	13.21	13.20	13.19	13.19	13.18	13.19	13.19	13.20	13.21
9	1	4	17	17.07	17.06	17.07	17.07	17.09	17.07	17.07	17.06	17.07
10	4	1	17	17.16	17.14	17.12	17.10	17.09	17.10	17.12	17.14	17.16

Table 10 Change of the resulting normalized frequency parameter with the wave direction angle θ



Figure 5. Mode shapes of modes (a) (3,1) and (b) (1,3) for a simply supported square plate. Wave propagation angle 5° .



(a)

(b)

Figure 6. Mode shapes of modes (a) (3,1) and (b) (1,3) for a simply supported square plate. Wave propagation angle 45°.

Figure 7 presents the variation of the normalized frequency parameter of modes (1,4) and (4,1) as obtained by the spectral finite element model. Note the intersection of both curves at propagation angle of 45° . Figure 8 presents the contour shading plots of the mode shape at wave propagation angle of 5° . Note the distinct shape of the modes (1,4) and (4,1) that appear in those plots. On the other hand, Figure 9 presents the plots for the modes (1,4) and (4,1) with propagation angle of 45° , those plots match the results mentioned by Leissa [13].



Figure 7. Variation of the normalized frequency parameter with the wave direction angle for modes (1,4) and (4,1) of a simply supported square plate.



(a)

(b)

Figure 8. Mode shapes of modes (a) (1,4) and (b) (4,1) for a simply supported square plate. Wave propagation angle 5° .



(a)

(b)

Figure 9. Mode shapes of modes (a) (4,1) and (b) (1,4) for a simply supported square plate. Wave propagation angle 45°.

3.3 Analytical vs. Numerical Integration

A similar code was written using MATLAB® 6.1 script. Numerical integration was used for the evaluation of the element matrices. The numerical integration used Gauss-Legendre quadrature method for numerical integration [14]. Different number of quadrature points were used. In all the above mentioned models, the results obtained using numerical integration was not of significant difference that those obtained by analytical integration from as little as 5 integration points in each direction.

4. CONCLUDING REMARKS

The main objective of this study was to show the possibility of preparing a generic finite element model for the solution of the plate vibration problem. The study used three different models to emphasize the possibility of the generalization

In this study, a new spectral finite element model is proposed, namely, using exponential functions as trial functions instead of the conventional polynomial trial functions which have proven superior in the one-dimensional structures. The model integrals and routines for the calculations of the normalized frequency parameter were performed symbolically using Mathematics® version 4.1.

The effect of the estimated wave direction has shown to be of minor effect on the resulting frequency parameter. Thus, the use of 45° angle is a reasonable choice for most problems except for those with indicated excitation direction. Also, the effect of the direction on mode shape has shown to be of minor.

The results obtained from the different models emphasised that the use of sophisticated models is not of great significance on the results obtained for square plate vibration problem.

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