



HIGH FREQUENCY APPROXIMATIONS FOR TREATING ACOUSTIC RADIATION OR SCATTERING PROBLEMS WITH THE BOUNDARY ELEMENT METHOD

¹Martin Ochmann, ²Bodo Nolte, ¹Ralf Burgschweiger, ²Ingo Schäfer

¹University of Applied Sciences Berlin Department of Mathematics – Physics - Chemistry Luxemburger Str. 10, 13353 Berlin, Germany ochmann@tfh-berlin.de

²Forschungsanstalt der Bundeswehr für Wasserschall und Geophysik – Federal Armed Forces Underwater Acoustics and Marine Geophysics Research Institute, Klausdorfer Weg 2-24, D-24148 Kiel, Germany

Abstract

The numerical prediction of sound fields radiated or scattered from complex shaped structures into the three-dimensional space can be performed effectively by using the boundary element method (BEM). However, at high frequencies, it is often necessary to deal with discretized structures consisting of many thousand finite surface elements, in order to ensure a sufficient number of elements per wavelength. Consequently, the computational cost of the method grows considerably with increasing frequency. Thus, suitable approximations tailored to the high frequency range can reduce computational time drastically. Several such techniques are known. In this paper, the theoretical framework of the Rayleigh integral and the plane wave approximation (PWA) for the radiation problem will be explained and compared with each other. For the scattering problem, the so-called Kirchhoff approach is often used. On the other hand, the Rayleigh integral and the PWA can also be applied to the scattering problem, since it can be formulated as an equivalent radiation problem. The theory of the three approximations will be presented and compared with respect to the scattering problem, too. In the present and in an accompanying paper [1], the Kirchhoff approach and the PWA will be applied to the scattering of plane waves from cylindrical shells located in free space for a frequency range up to 100 kHz.

1. INTRODUCTION

High frequency approximations combined with the BEM are very useful for obtaining fast numerical results of radiation or scattering problems. However, since several different but similar approaches exist, based on various simplifications, it is sometimes not easy to decide which method is most suitable for a particular problem. For that reason, the purpose of the present paper mainly is to review some important high frequency approximations like the PWA, the Rayleigh integral, and the Kirchhoff approximation, in order to explain the corresponding theoretical background descending from the theory of integral equations, and to discuss deviating numerical results for a specific scattering problem.

2. THE BOUNDARY INTEGRAL EQUATIONS FOR THE RADIATION PROBLEM

A vibrating structure radiates sound into the surrounding space. The radiated sound field is characterized by sound pressure p, sound velocity \vec{v} , and derived quantities such as the sound intensity \vec{I} , the radiated sound power P, the radiated efficiency σ etc., which shall be calculated by numerical methods.

As shown in Fig. 1, the bounded volume of the radiating structure in three-dimensional space is denoted by B (like Body). The interior of B is called B_i and the exterior B_e . The surface normal n should be directed into the exterior B_e .



Fig. 1: Geometry of the radiation and scattering problem

A compilation of the formulas given below can be found in the book "Formulas of Acoustics" [2]. The most frequently used integral equation formulation in acoustics is the well-known Helmholtz integral equation (abbreviation HIE) for exterior field problems. The HIE is obtained by applying Green's second theorem to the Helmholtz equation (see for example [3-5]). Depending on the location of the field point *x*, the HIE takes the form

$$\iint \left[p(x) \partial_{g}(x, y) - \partial_{p}(y) \right]_{d_{g}} = \begin{cases} p(x), x \in B_{e} \\ 1 - p(x) - x \in S \end{cases}$$
(1a)

$$\iint_{S} \left[p(y) \frac{\partial g(x, y)}{\partial n(y)} - \frac{\partial p(y)}{\partial n(y)} g(x, y) \right] ds = \begin{cases} \frac{1}{2} p(x), x \in S \\ 0, x \in B \end{cases}$$
(1b)

$$(1c)$$

where

$$g(x, y) = \frac{1}{4\pi r} e^{-jkr}$$
 with $r = r(x, y) = ||\vec{r}||, \ \vec{r} = y - x$, (1d)

is the free-space Green's function, and y is a spatial point on the structural surface S. The geometrical notations are chosen as shown in Fig. 1. Eqs. (1a), (1b), and (1c) are called exterior HIE, surface HIE, and interior HIE, respectively.

For simplicity, we only consider the most important Neumann boundary value problem, which describes a body, that vibrates with normal velocity v. Therefore, the pressure gradient

$$\frac{\partial p}{\partial n} = -j\omega\rho v \tag{2}$$

is prescribed on S. Here, ρ is the fluid density and $\partial / \partial n$ is the derivative in the direction of the outward normal *n*.

For calculating the quantities of the sound field using the complete BEM, two steps are necessary. First, the surface HIE (1b) has to be solved which gives the pressure on the surface of the structure. This procedure requires the main effort, since a complex, fully populated, and unsymmetrical system of linear equations has to be solved. Second, the sound field in the whole outer space can be calculated with the help of the exterior HIE by an integration over the surface *S*. The main advantage of the high frequency approximations presented below is, that the first time-consuming step becomes redundant, since the surface pressure – or in the general case both field quantities on the surface - will be estimated approximately, so that only the exterior integral equation (1a) has to be evaluated.

3. THE BOUNDARY INTEGRAL EQUATIONS FOR THE SCATTERING PROBLEM

The scattering problem can be formulated as an equivalent radiation problem by the following procedure (see [2]): Considering, for example, the sound-hard scatterer, the normal velocity v_{in} of the incident pressure wave p_{in} will be evaluated at the surface S where the scatterer is assumed (for the moment) to be sound transparent. If B is now vibrating with the negative normal velocity $(-v_{in})$, the radiated sound pressure is identical to the pressure p_s scattered from B due to the incident wave p_{in} . Hence, instead of (2), we simply have

$$\frac{\partial p_s}{\partial n} = -j\omega\rho(-v_{in}) \tag{3}$$

for the scattering problem, which again is an inhomogeneous boundary condition like (2).

However, sometimes it is more convenient to have an explicit boundary integral equation for the total pressure $p = p_T = p_s + p_{in}$ as starting point for a numerical calculation. The scattered wave p_s has to fulfill the exterior Helmholtz formula (1). The incident pressure p_{in} is assumed to have no singularities in B_i , and hence, it must satisfy the interior Helmholtz formula

$$\iint \left[p(y) \frac{\partial g(x, y)}{\partial (x, y)} - \frac{\partial p(y)}{\partial (x, y)} g(x, y) \right] ds = \begin{cases} 0, & x \in B_e \\ -\frac{1}{2} p(x), x \in S \end{cases}$$
(4a) (4b)

$$\int \int \int [1 (y) \partial n(y) \partial n(y) \partial n(y) \partial n(y) \partial n(y) dn(y) dn(y$$

By adding both Eqs. (1) and (4), one gets the Helmholtz formula

Г

$$\begin{bmatrix} p(x), & x \in B_e \end{bmatrix}$$
(5a)

$$p_{in} + \iint_{S} \left[p(y) \frac{\partial g(x, y)}{\partial n(y)} - \frac{\partial p(y)}{\partial n(y)} g(x, y) \right] ds = \begin{cases} \frac{1}{2} p(x), & x \in S \\ 0, & x \in B_i \end{cases}$$
(5b)
(5c)

for the total pressure *p*. Assuming that the surface of the scatterer is rigid, the pressure gradient $\partial p / \partial n = -j\omega\rho v$ on the surface is zero. Hence, for a rigid scatterer the boundary integral equation (5a) can be written as

$$p_{s} = p(x) - p_{in} = \iint_{s} p(y) \frac{\partial g(x, y)}{\partial n(y)} ds(y).$$
(6)

4. THE RALEIGH INTEGRAL AND THE PWA

For deriving the Rayleigh integral (see [5], [6]), it is assumed that the vibrating part of the radiating body is embedded in a plane rigid baffle. Such an asumption is approximately valid for high frequencies and a weak curvature of the surface B. By using the half-space Green's function over an infinite rigid plane, Eq. (1a) simplifies to the Rayleigh integral

$$p(x) = 2j\omega\rho \iint_{S} v_n g(x, y) ds, \quad x \in B_e , \qquad (7)$$

which gives an exact solution for a radiator in a plane rigid baffle.

On the other hand, if we assume that pressure and normal velocity on the surface of the radiatior are satisfying

$$p = \rho c v_n , \qquad (8)$$

we get from Eqs. (1a) und (2) the so-called plane wave approximation (PWA)

$$p(x) = \iint_{S} \left[\rho c v_n \left(\frac{\partial g(x, y)}{\partial n(y)} + j k g(x, y) \right) \right] ds, \quad x \in B_e$$
(9)

and due to

$$\frac{\partial g}{\partial n} = -g\left(jk + \frac{1}{r}\right)\left(e_r, n\right) \tag{10}$$

 $(e_r \text{ is the unit vector in the direction } \vec{r})$, we obtain

$$p(x) = j\omega\rho \iint_{S} \left[v_n g(x, y) \left(1 - \left(1 - \frac{j}{kr} \right) (e_r, n) \right) \right] ds, \quad x \in B_e \quad .$$

$$\tag{11}$$

As observed by Herrin et al. [6], the PWA is identical with the Rayleigh integral in the highfrequency range where $kr \gg 1$, but only when we assume, that $(e_r, r) = -1$. This assumption means that only the vibrating part of the surface *S*, which can be seen for the receiver point *x*, should be taken into account when performing the integration over *S*. Hence, for applying the Rayleigh integral, we have to differentiate between the "visible" and the "invisible" part of the structure (see [6]), which is not necessary when using the PWA.

5. THE KIRCHHOFF APPROXIMATION

There are several slightly different representations of the Kirchhoff approximation in the literature. Originally, the approach was suggested by Kirchhoff for treating the diffraction of light when passing through apertures. In order to calculate the light intensity behind the aperture, Kirchhoff assumed that the field u and its normal derivative are zero on the back side of the screen and are equal to the values of the unperturbed incident wave on a surface overstretching the aperture in first approximation (more details can be found in [7]). By inserting these values into the KIE (1a), a first approximation of the light intensity behind the aperture can be computed.

For acoustics, the same procedure is explained in ([8], end of page 169), if there is an aperture D. However, if there is only a screen without a hole, it is recommended that the total pressure on the front side of the screen is substituted by the incident field. However, we are of the opinion that only the scattered pressure has to be substituted. Also, Skudrzik [5] mentioned that one has to substitute the incident field in the aperture and on the illuminated surface. However, the total sound pressure on the screen depends on the boundary condition on the screen. For example, if the screen, which can be identified with the scattering object, is rigid, the reflection coefficient R becomes 1, and hence a suitable approximation would be to assume that the pressure on the boundary is twice the incident pressure, just as it occurs in the Rayleigh integral. Taking this fact into account, it was recommended in [9] to introduce R explicitly by writing

$$p_s = R p_{in} \tag{12}$$

and due to $(e_{in}, n) \approx -1$ for illuminated elements, we get approximately for plane waves

$$\frac{\partial p_s}{\partial n} = -R \frac{\partial p_{in}}{\partial n} . \tag{13}$$

Here, e_{in} is the unit normal vector in the incidence direction of the incident wave. Inserting (12) and (13) into (5a) and performing the surface integral only over the illuminated part S_{ill} , which means that $(e_r, r) < 0$, of the surface we get (see [9])

$$p_{s}(x) = \iint_{S_{iii}} \left[(1+R)p_{in} \frac{\partial g(x,y)}{\partial n(y)} - (1-R) \frac{\partial p_{in}}{\partial n} g(x,y) \right] ds, \quad x \in B_{e} \quad .$$
(14)

For R = 1, we obtain

$$p_{s}(x) = 2 \iint_{S_{in}} p_{in} \frac{\partial g(x, y)}{\partial n(y)} ds, \quad x \in B_{e} \quad .$$
(15)

On the other hand, by inserting the assumption of the PWA for an incident plane wave

$$p = p_{in} + p_s = p_{in} + \rho c(-v_{in}) = p_{in} - p_{in}(e_{in}, n) = (1 - (e_{in}, n))p_{in}$$
(16)

into Eq. (6), we get

$$p_{s} = \iint_{s} p_{in}(y)(1 - (e_{in}, n))\frac{\partial g(x, y)}{\partial n(y)}ds(y).$$

$$(17)$$

If we again restrict the surface integral only to the illuminated part of the surface with $(e_{in}, n) = -1$, instead of $(e_r, r) < 0$, the formula for the PWA becomes identical with the Kirchhoff approximation (15). The part of the structure, where $(e_{in}, n) = -1$ holds, is of course the dominant source for scattering, especially in the high frequency range. In summary, the Kirchhoff approximation and the PWA are leading to the same result, if plane wave scattering from a rigid structure and only the illuminated part of the surface is considered.

A similar statement can be found in [10, p. 325]: In the Kirchhoff approach, only the iluminated region of the scatterer is taken into account and every element of this area is considered as embedded in an planar infinite rigid baffle.

6. COMPARISON BETWEEN THE PWA AND THE KIRCHOFF APPROACH -NUMERICAL RESULTS

In Fig 2, the scattering object is shown. It is assumed that a plane wave travelling along the negative y-axis (from point 1 to point 2) is impinging on the rigid structure.



Fig. 2: Scattering object

In Fig. 3 and Fig. 4, the target strength

$$TS = 20 \lg (r \cdot p_s / p_{in}) \tag{18}$$

in point 1 and point 2, respectively, calculated with the PWA and with the Kirchhoff approximation (KIA) is shown.



Fig. 3: Frequency curves of the TS in dB at point 1; red PWA, green KIA



Fig. 4: Frequency curves of the TS in dB at point 2; red PWA, green KIA

Fig. 3 shows that the agreement between both methods is very good for a field point lying on the illuminated side of the target. However, if the field point is on the backside of the scattering object, a difference of about 2 dB can be observed over nearly the whole frequency range (see Fig. 4). More numerical results up to 100 kHz can be found in the accompanying paper [1].

7. CONCLUSIONS

The correlations between the different high-frequency approximation PWA, Rayleigh integral, and the Kirchhoff approximation are discussed. It is shown that the PWA leads to the Rayleigh integral for radiation problems, if kr >> 1 and if the integration is only performed over the visible part of the structure. The PWA leads to the Kirchhoff approximation for scattering problems, if only the illuminated part of the surface is considered for plane wave scattering from a rigid structure. The numerical scattering results from a cylindrical shell confirm these observations.

REFERENCES

- [1] R. Burgschweiger, M. Ochmann, B. Nolte, I. Schäfer, A boundary element package containing approximate solvers for treating high frequency acoustic scattering, Proceedings of the "14th International Congress on Sound and Vibration (ICSV14)", July 2007, Cairns, Australia (on CD-ROM).
- [2] M. Ochmann and F. P. Mechel: Analytical and numerical methods in acoustics. Chap. O in: F. P. Mechel (Ed.), Formulas of Acoustics, Springer-Verlag Berlin Heidelberg, 930 – 1023, 2002.
- [3] H. A. Schenck, Improved integral formulation for acoustic radiation problems, J. Acoust. Soc. Amer., 44, pp. 41-58, 1968.
- [4] D. Colton and R. Kress, *Integral equations in scattering theory*, Wiley-Interscience Publication, New York, 1983.
- [5] E. Skudrzyk, *The foundations of acoustics*, Springer-Verlag, Wien New York, 1971.
- [6] D. W. Herrin, F. Martinus, T. W. Wu, A. F. Seybert, An assessment of the high frequency boundary element and Rayleigh integral approximations, Applied Acoustics, 67, 819-833, 2006.
- [7] M. Born, *Optik*, Springer-Verlag, Berlin Heidelberg New York, 3rd Ed., 1972.
- [8] P. Filippi, D. Habault, J.-P. Lefebvre, A. Bergassoli, *Acoustics: Basic physics, theory and methods*, Academic Press, San Diego London, 1999.
- [9] B. Nolte, C. Fiedler, A. Wendlandt, I. Schäfer, Numerical methods for scattering phenomena, Proceedings for the Forum Acusticum, Budapest, 2005.
- [10] M. C. Junger and D. Feit, *Sound, structures, and their interaction*, Acoustical Society America, American Inst. Physics, 1993.