ICSV14
Cairns • Australia
9-12 July, 2007

14th International Congress on Sound & Vibration

NATURAL FREQUENCY STATISTICS OF PLATES WITH BOUNDED UNCERTAIN PROPERTIES USING INTERVAL ANALYSIS

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Abstract

The vibrational characteristics of a dynamic system can be sensitive to variations in parameters such as its material properties and this sensitivity increases as the frequency increases. Since, for most practical structures, their geometric or material properties are not exactly known, prediction methods for the dynamic response of a structure with uncertain properties generally requires some model for the statistics of its natural frequencies. In this paper, the effect of uncertainty in the material parameters and dimensions of a plate on the variability in its vibrational characteristics is presented using a technique called the interval factor method. With this method, the lower bound, upper bound and mean values of the natural frequencies and modeshapes have been determined where the density, Young's modulus and plate thickness were allowed to vary within a predefined band.

1. INTRODUCTION

In the mid to high frequency ranges, the dynamic characteristics and responses of structures can be greatly affected by small changes in parameters such as material properties and structural dimensions, as well as variation in operating conditions. For many dynamic systems there is a degree of uncertainty about these parameters. Deterministic modelling techniques such as finite element analysis (FEA) are limited in the mid to high frequency range by the computational expense required to model a structure at these frequencies. A very large number of degrees of freedom are needed for complex models to accurately capture the short wavelength deformation at high frequencies. A number of techniques have been developed to extend the use of FEA to the mid frequency region. Monte Carlo simulations have been used to this affect however they are restricted by the computational expense required to calculate a solution [1]. A quasi-Monte Carlo method can be used to reduce the time taken in computation [2]. Higher order perturbations methods [3, 4] are similarly restricted by computational time and power as well as the amount of information required to model joints between subsystems [5]. An analysis of the uncertainty of a system has been completed using Latin hypercube sampling [6]. This method was found to have advantages over random sampling for complex systems.

Interval analysis was first developed by Moore [7] in the 1960's. The application of interval properties to various systems has been discussed by Moore [8] and Alefeld and Herzberger [9]. The basic idea of this method, applied to the structural response of a system, is to determine the band that a particular response might fall into. This is dependent on the structural parameters which are allowed to vary within a fixed range, where this range is denoted by an upper and lower limit. The interval analysis method has been applied to determine the eigenvalues and eigenvectors of truss structures with uncertain material properties [10]. The method has been shown to be robust. Unlike various stochastic methods, it also allows the upper and lower response limits to be calculated for the case where the probability distributions of the uncertain parameters are not known [11, 12]. Research has also been conducted into the static displacement of structures using a combination of interval analysis and matrix perturbation methods [13-16].

This paper presents the use of the interval factor method to determine the eigenvalues and eigenvectors of a plate in flexure, in which the material and geometric properties were allowed to vary within fixed intervals. In this method, each parameter variation is expressed in terms of interval factors. Results are presented for the variation in the natural frequencies and modeshapes due to uncertainty in any of the individual parameters. The benefit of this method is that it allows the upper and lower limits of the natural frequencies and modeshapes to be determined by solving only one finite element solution.

2. INTERVAL FACTOR METHOD

In this section, a brief review of the interval analysis method is presented. $X^I = \left[\underline{x}, \overline{x}\right]$ is as a real number where $\underline{x}$ and $\overline{x}$ are respectively the lower and upper limits of the closed interval in which X^I lies. X^I can be written in the form [7, 8, 13]

$$X^I = \left[x^c - \Delta x,\ x^c + \Delta x\right] \tag{1}$$

where x^c is the mean value of X^I and Δx is the uncertainty of X^I. These are determined by the following equations

$$x^c = \frac{\overline{x} + \underline{x}}{2}, \quad \Delta x = \frac{\overline{x} - \underline{x}}{2} \qquad (2), (3)$$

By factoring out x^c, equation (1) can be written as

$$X^I = x^c\left[1 - \frac{\Delta x}{x^c},\ 1 + \frac{\Delta x}{x^c}\right] = x^c\left[1 - \frac{\overline{x} - \underline{x}}{2x^c},\ 1 + \frac{\overline{x} - \underline{x}}{2x^c}\right] \tag{4}$$

Introducing the interval factor X_f^I

$$X_f^I = \left[\underline{x}_f,\ \overline{x}_f\right] = \left[1 - \Delta x_f, 1 + \Delta x_f\right] = \left[1 - \frac{\overline{x} - \underline{x}}{2x^c},\ 1 + \frac{\overline{x} - \underline{x}}{2x^c}\right] \tag{5}$$

X^I can now be defined as

$$X^I = x^c X_f^I \tag{6}$$

From equations (4) and (5), the interval ratio is given by

$$\Delta x_f = \Delta x / x^c \tag{7}$$

3. FINITE ELEMENT ANALYSIS OF A PLATE IN FLEXURE

In order to analyse the deterministic vibrational characteristics of a plate in flexure, a finite element approach has been chosen. For this purpose, a 4 node element has been used. This element has 12 degrees of freedom corresponding to one translational and two rotational degrees of freedom per node. The mass and stiffness matrices for each individual element are calculated by Petyt [17] and are given by equations (8) and (9), respectively. E_e is the Young's modulus, ρ_e is the density, ν_e is Poisson's ratio and h_e is the thickness of the element. $[\mathbf{n}]$ and $[\mathbf{d}]$ are sub matrices which are functions of the element dimensions, *a* and *b*. These sub matrices are described in the Appendix.

$$[\mathbf{m}_e] = \frac{\rho_e h_e ab}{6300}[\mathbf{n}], \qquad [\mathbf{k}_e] = \frac{E_e h_e^3}{48(1-\nu^2)ab}[\mathbf{d}] \tag{8), (9}$$

4. INTERVAL EIGENVALUE ANALYSIS OF A PLATE IN FLEXURE

In the following analysis, the variations in the Young's modulus E, density ρ and thickness h of a thin plate structure are considered. Each parameter is assumed to vary within the interval ratio which is constant across each element of the plate. The interval variables for the individual parameters can be described in terms of the interval factor and the deterministic or mean value

$$E_e^I = E_f^I E^c, \qquad \rho_e^I = \rho_f^I \rho^c, \qquad h_e^I = h_f^I h^c \tag{10)-(12}$$

where E_f^I, ρ_f^I and h_f^I are the interval factors which represent the variation in the parameters for all the elements of the plate. The mass and stiffness matrices of the elements can now be described in terms of a deterministic component and the interval factors.

$$[\mathbf{m}_e]^I = \rho_f^I h_f^I \rho^c h^c \frac{ab}{6300}[\mathbf{n}] \tag{13}$$

$$[\mathbf{k}_e]^I = E_f^I (h_f^I)^3 E^c (h^c)^3 \frac{1}{48ab(1-\nu^2)}[\mathbf{d}] \tag{14}$$

Equations (13) and (14) can be written more simply as equations (15) and (16). The deterministic matrices are of the same form as equations (8) and (9) and are constructed by using the mean properties, E^c, ρ^c and h^c.

$$[\mathbf{m}_e]^I = \rho_f^I h_f^I [\mathbf{m}_{e\det}], \qquad [\mathbf{k}_e]^I = E_f^I (h_f^I)^3 [\mathbf{k}_{e\det}] \tag{15), (16}$$

The mass and stiffness matrices of each element can be combined for the whole plate by use of the transformation matrix $[\mathbf{T}_e]$ which locates the element matrix within the global coordinates.

$$[\mathbf{m}]^I = \sum_{e=1}^{m} [\mathbf{T}_e]^T [\mathbf{m}_e]^I [\mathbf{T}_e], \qquad [\mathbf{k}]^I = \sum_{e=1}^{m} [\mathbf{T}_e]^T [\mathbf{k}_e]^I [\mathbf{T}_e] \tag{17), (18}$$

The natural frequencies can now be found in terms of the deterministic natural frequencies and the interval factors. The subscript j represents the degrees of freedom. Again the deterministic components have the mean properties, E^c, ρ^c and h^c.

$$(\omega_j^I)^2 = \frac{\{\phi_j\}^{I^T}[\mathbf{k}]^I\{\phi_j\}^I}{\{\phi_j\}^{I^T}[\mathbf{m}]^I\{\phi_j\}^I} = \frac{\phi_{jf}^I E_f^I (h_f^I)^3 \phi_{jf}^I}{\phi_{jf}^I \rho_f^I h_f^I \phi_{jf}^I} \frac{\{\phi_j\}_{\det}^T[\mathbf{k}_{\det}]\{\phi_j\}_{\det}}{\{\phi_j\}_{\det}^T[\mathbf{m}_{\det}]\{\phi_j\}_{\det}} = \frac{E_f^I (h_f^I)^2}{\rho_f^I}(\omega_{j\det})^2 \tag{19}$$

Thus the lower and upper limits and the mean of the natural frequencies can be calculated in terms of the interval ratios ΔE_f, $\Delta\rho_f$ and Δh_f, and are given by equations (20) to (22), respectively.

$$\underline{\omega}_j = \frac{(1-\Delta E_f)^{1/2}(1-\Delta h_f)}{(1+\Delta\rho_f)^{1/2}}\omega_{j\det}, \qquad \overline{\omega}_j = \frac{(1+\Delta E_f)^{1/2}(1+\Delta h_f)}{(1-\Delta\rho_f)^{1/2}}\omega_{j\det} \tag{20), (21}$$

$$\omega_j^c = \frac{\overline{\omega}_j + \underline{\omega}_j}{2} \tag{22}$$

Similarly, the modeshapes can be found in terms of the interval factors. The decoupled mass and stiffness matrices are [18]

$$\{\phi\}^T[\mathbf{m}]^I\{\phi\} = [I], \qquad \{\phi\}^T[\mathbf{k}]^I\{\phi\} = diag\left[\omega^2\right] \tag{23), (24}$$

Equation (24) can be described in terms of the interval factors and the deterministic components

$$\phi_f^I E_f^I \left(h_f^I\right)^3 \phi_f^I \{\phi_j\}_{\det}^T [\mathbf{k}_{\det}]\{\phi_j\}_{\det} = diag\left[\left(\omega^I\right)^2\right] \tag{25}$$

Substituting equation (19) into the right hand side of equation (25) yields

$$\phi_f^I = \frac{1}{(h_f^I \rho_f^I)^{1/2}}, \quad \text{where} \quad \{\phi_j\} = \phi_f^I \{\phi_j\}_{\det} \tag{26), (27}$$

Equations (26) and (27) are used to express the lower and upper bounds of the modeshapes in terms of the interval ratios and are given by equations (28) and (29), respectively. These lower and upper bounds represent the amplitude change in the eigenvectors of the system, not a physical change in shape of the mode. It should be noted that the upper and lower values of the modeshape interval do not correspond to the upper and lower values of the natural frequency interval.

$$\{\underline{\phi}_j\} = \frac{1}{(1+\Delta\rho_f)^{1/2}(1+\Delta h_f)^{1/2}}\{\phi_j\}_{\det}, \quad \{\overline{\phi}_j\} = \frac{1}{(1-\Delta\rho_f)^{1/2}(1-\Delta h_f)^{1/2}}\{\phi_j\}_{\det} \quad (28), (29)$$

5. COMPUTATIONAL RESULTS

The interval analysis method has been applied to an aluminium plate. The plate has a nominal thickness of 2 mm, Young's modulus of 70 GPa, density 2800 kg/m^3 and Poisson's ratio of 0.3. The dimensions of the plate are 600 mm x 900 mm and it has been meshed using elements with a maximum length of 10 mm such that the results are accurate to 4000 Hz. Simply supported boundary conditions have been applied to all edges of the plate.

The intervals for the natural frequencies and the modeshapes have been calculated by varying the properties of the plate by ±1%. The results are presented in Table 1 for the 73rd mode which corresponds to a natural frequency of 919.2 Hz. It can be seen that the variation in frequency is more sensitive to uncertainty in the plate thickness than density or Young's modulus. As expected, if more than one parameter is allowed to vary, the frequency deviation increases.

Table 2 shows the ratio change in the j^{th} natural frequency from the original deterministic natural frequency. The ratio change in the modeshapes is given in Table 3. The results show a noticeable variation in the natural frequency even with a relatively small percentage change in the parameters. In addition, the variation increases with frequency. For example, if all three parameters are allowed to vary by 1%, a natural frequency at 2000 Hz could vary by 40 Hz, but at 4000 Hz the difference is as much as 80 Hz. In Table 3, it can be observed that the variation in the amplitude of the modeshape is only dependent on the density and plate thickness. For each of these parameters, the modeshape ratio is the same.

Table 1. The frequency bounds for a natural frequency at 919.2 Hz with parameters varying by 1%.

Properties	$\underline{\omega}_{73}$ (Hz)	ω_{73}^{c} (Hz)	$\overline{\omega}_{73}$ (Hz)
$\rho = \pm1\%, h = 0, E = 0$	914.6	919.2	923.8
$\rho = 0, h = \pm1\%, E = 0$	910.0	919.2	928.4
$\rho = 0, h = 0, E = \pm1\%$	914.6	919.2	923.8
$\rho = \pm1\%, h = \pm1\%, E = 0$	905.5	919.4	933.1
$\rho = \pm1\%, h = 0, E = \pm1\%$	910.0	919.3	928.4
$\rho = 0, h = \pm1\%, E = \pm1\%$	905.4	919.3	933.0
$\rho = \pm1\%, h = \pm1\%, E = \pm1\%$	900.9	919.5	937.7

Table 2. The natural frequency ratio for parameters varying by 1%.

Properties	$\Delta\omega_j / \overline{\omega}_j$
$\rho = \pm1\%, h = 0, E = 0$	0.005
$\rho = 0, h = \pm1\%, E = 0$	0.01
$\rho = 0, h = 0, E = \pm1\%$	0.005
$\rho = \pm1\%, h = \pm1\%, E = 0$	0.015
$\rho = \pm1\%, h = 0, E = \pm1\%$	0.01
$\rho = 0, h = \pm1\%, E = \pm1\%$	0.015
$\rho = \pm1\%, h = \pm1\%, E = \pm1\%$	0.02

Table 3. The modeshape ratio for parameters varying by 1%.

Properties	$\{\Delta\phi_j\}/\{\bar{\phi}_j\}$
$\rho = \pm 1\%, h = 0$	0.005
$\rho = 0, h = \pm 1\%$	0.005
$\rho = \pm 1\%, h = \pm 1\%$	0.01

6. CONCLUSIONS

The interval factor method has been used to determine the effect of parameter uncertainty on the natural frequencies and modeshapes of a simply supported plate. Using this method, the lower and upper bounds and the mean value of the vibrational characteristics have been found. As expected, as the frequency increased the affect of the uncertainty on the vibrational characteristics also increased. It is expected that similar results could be obtained for a panel with any boundary conditions. The significant advantage of this method is the short time required to compute the results.

This work was carried out by funding provided by the Australian Research Council under an ARC Linkage grant LP0454302.

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APPENDIX

The following equations are the sub matrices for the mass and stiffness matrices for each individual element, given by equations (7) and (8). *a* and *b* refer to the width and length of each plate element.

$$[\mathbf{n}] = \begin{bmatrix} \mathbf{n}_{11} & \mathbf{n}_{21}^T \\ \mathbf{n}_{21} & \mathbf{n}_{22} \end{bmatrix}, \qquad [\mathbf{d}] = \begin{bmatrix} \mathbf{d}_{11} & & sym & \\ \mathbf{d}_{21} & \mathbf{d}_{22} & & \\ \mathbf{d}_{31} & \mathbf{d}_{32} & \mathbf{d}_{33} & \\ \mathbf{d}_{41} & \mathbf{d}_{42} & \mathbf{d}_{43} & \mathbf{d}_{44} \end{bmatrix} \qquad \text{(A1), (A2)}$$

$$\mathbf{n}_{11} = \begin{bmatrix} 3454 & & & & & \\ 922b & 320b^2 & & & sym & \\ 922a & 252ab & 320a^2 & & & \\ 1226 & 398b & 548a & 3454 & & \\ 398b & 160b^2 & 168ab & 922b & 320b^2 & \\ -548a & -168ab & -240a^2 & -922a & -252ab & 320a^2 \end{bmatrix} \qquad \text{(A3)}$$

$$\mathbf{n}_{21} = \begin{bmatrix} 394 & 232b & 232a & 1226 & 548b & -398a \\ -232b & -120b^2 & -112ab & -548b & -240b^2 & 168ab \\ -232a & -112ab & -120a^2 & -398a & -168ab & 160a^2 \\ 1226 & 548b & 398a & 394 & 232b & -232a \\ -548b & -240b^2 & -168ab & -232b & -120b^2 & 112ab \\ 398a & 168ab & 160a^2 & 232a & 112ab & -120a^2 \end{bmatrix} \qquad \text{(A4)}$$

$$\mathbf{n}_{22} = \begin{bmatrix} 3454 & & & & & \\ -922b & 320b^2 & & & sym & \\ -922a & 252ab & 320a^2 & & & \\ 1226 & -398b & -548a & 3454 & & \\ -398b & 160b^2 & 168ab & -922b & 320b^2 & \\ 548a & -168ab & -240a^2 & 922a & -252ab & 320a^2 \end{bmatrix} \qquad \text{(A5)}$$

$$\mathbf{d}_{11} = \begin{bmatrix} \frac{4(a^4+b^4)}{a^2b^2} + \frac{2}{5}(7-2\nu) & & sym \\ \frac{4a^2}{b} + \frac{2}{5}(b+4b\nu) & \frac{16}{15}\left(5a^2 - b^2(\nu-1)\right) & \\ \frac{4b^2}{a} + \frac{2}{5}a(1+4\nu) & 4ab\nu & \frac{16}{15}\left(5b^2 - a^2(\nu-1)\right) \end{bmatrix} \tag{A6}$$

$$\mathbf{d}_{21} = \begin{bmatrix} \frac{2a^2}{b^2} - \frac{4b^2}{a^2} + \frac{2}{5}(2\nu-7) & \frac{2a^2}{b} - \frac{2b}{5}(1+4\nu) & -\frac{4b^2}{a} + \frac{2}{5}a(\nu-1) \\ \frac{2a^2}{b} - \frac{2b}{5}(1+4\nu) & \frac{8}{15}\left(5a^2 + 2b^2(\nu-1)\right) & 0 \\ \frac{4b^2}{a} - \frac{2}{5}a(\nu-1) & 0 & \frac{4}{15}\left(10b^2 + a^2(\nu-1)\right) \end{bmatrix} \tag{A7}$$

$$\mathbf{d}_{31} = \begin{bmatrix} \frac{14}{5} - \frac{2(a^4+b^4)}{a^2b^2} - \frac{4\nu}{5} & -\frac{2a^2}{b} - \frac{2b}{5}(\nu-1) & -\frac{2b^2}{a} - \frac{2a}{5}(\nu-1) \\ \frac{2a^2}{b} + \frac{2b}{5}(\nu-1) & \frac{4}{3}a^2 - \frac{4}{15}b^2(\nu-1) & 0 \\ \frac{2b^2}{a} + \frac{2a}{5}(\nu-1) & 0 & \frac{4}{3}b^2 - \frac{4}{15}a^2(\nu-1) \end{bmatrix} \tag{A8}$$

$$\mathbf{d}_{41} = \begin{bmatrix} -\frac{4a^2}{b^2} + \frac{2b^2}{a^2} + \frac{2}{5}(2\nu-7) & -\frac{4a^2}{b} + \frac{2}{5}b(\nu-1) & \frac{2b^2}{a} - \frac{2}{5}a(1+4\nu) \\ \frac{4a^2}{b} - \frac{2}{5}b(\nu-1) & \frac{8a^2}{3} + \frac{4b^2}{15}(\nu-1) & 0 \\ \frac{2b^2}{a} - \frac{2}{5}a(1+4\nu) & 0 & \frac{8b^2}{3} + \frac{16a^2}{15}(\nu-1) \end{bmatrix} \tag{A9}$$

The remaining stiffness sub-matrices are described by the following matrix multiplications

$$\mathbf{d}_{22} = \mathbf{I}_3\mathbf{d}_{11}\mathbf{I}_3 \tag{A10}$$

$$\mathbf{d}_{32} = \mathbf{I}_3\mathbf{d}_{41}\mathbf{I}_3, \qquad \mathbf{d}_{33} = \mathbf{I}_1\mathbf{d}_{11}\mathbf{I}_1 \qquad \text{(A11), (A12)}$$

$$\mathbf{d}_{42} = \mathbf{I}_3\mathbf{d}_{31}\mathbf{I}_3, \qquad \mathbf{d}_{43} = \mathbf{I}_1\mathbf{d}_{21}\mathbf{I}_1, \qquad \mathbf{d}_{44} = \mathbf{I}_2\mathbf{d}_{11}\mathbf{I}_2 \qquad \text{(A13)-(A15)}$$

where

$$\mathbf{I}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{I}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{(A16)-(A18)}$$