



WAVE PROPAGATION, REFLECTION AND TRANSMISSION IN CURVED BEAMS

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Abstract

Wave motion in thin, uniform, curved beams with constant curvature is considered. The beams are assumed to undergo only in-plane motion, which is described by the sixth-order coupled differential equations based on Flügge's theory. In the wave domain the motion is associated with the three independent wave modes. A systematic wave approach based on reflection, transmission and propagation of waves is presented for the analysis of structures containing curved beam elements. Displacement, internal force and propagation matrices are derived. These enable transformations to be made between the physical and wave domains and provide the foundation for systematic application of the wave approach to the analysis of waveguide structures with curved beam elements. The energy flow associated with waves in the curved beam is also discussed. It is seen that energy can be transported independently by the propagating waves and also by the interaction of a pair of positive and negative going wave components which are non-propagating, i.e. their wavenumbers are imaginary or complex. A further transformation can be made to power waves, which can transport energy independently.

1. INTRODUCTION

Curved beams are used widely in built-up structures and hence their dynamic behaviour is of interest. Previous work in this area has been summarised in several articles, for example [1-3]. Wu and Lundberg [4] have investigated the transmission of energy through a curved section connecting two straight beams. They presented numerical results in the form of polar radiation diagrams for beams with different curvatures. Walsh and White [5] considered the energy flow associated with a single propagating wave component in a curved beam based on four different theories – Love's theory, Flügge's theory and the corrections for rotary inertia and shear deformation. They derived expressions which relate the power to the extensional, bending and shear waves. Kang *et al.* [6] applied the wave approach based on the reflection, transmission and propagation of waves to obtain the natural frequencies of finite curved beams.

The main aim of this paper is to describe a systematic wave approach based on reflection, transmission and propagation of waves and to use this to determine the energy flow

characteristics of waves in a thin, curved beam. The approach is also valid when rotary inertia, shear deformation and damping are important, but these effects are neglected here. Attention is focused on in-plane motion and Flügge's theory is used. The motion is described in terms of six independent (or uncoupled) wave components.

In section 2, the dispersion relation and the ratio of tangential displacement to radial displacement for the six wave components are obtained. In section 3 displacement, internal force and propagation matrices are derived. These enable transformations to be made between the physical and wave domains and provide the foundation for systematic application of the wave approach [7] to waveguide structures with curved elements. In section 4, the energy flow associated with the wave components is obtained in a systematic way. Their contributions are classified according to different conditions for the wavenumbers. The energy flow paths at a given frequency are identified. Energy can be transported independently by propagating waves or by pairs of wave components with imaginary or complex wavenumbers. A further transformation is found to power wave components – these propagate energy independently through the curved beam.

2. IN-PLANE WAVE MOTION IN CURVED BEAMS

Consider the in-plane motion of a thin, uniform, curved beam with constant curvature. Neglecting the effects of shear deformation and rotary inertia, the governing equations for free vibration in the radial and tangential directions are given by [5]

$$-EI\left(\frac{\partial^4 w}{\partial s^4} + \frac{2}{R^2}\frac{\partial^2 w}{\partial s^2} + \frac{w}{R^4}\right) - \frac{EA}{R}\left(\frac{w}{R} + \frac{\partial u}{\partial s}\right) = \rho A \frac{\partial^2 w}{\partial t^2},$$

$$EA\left(\frac{\partial^2 u}{\partial s^2} + \frac{1}{R}\frac{\partial w}{\partial s}\right) = \rho A \frac{\partial^2 u}{\partial t^2}$$
(1a,b)

where *E* is the Young's modulus, *I* the second moment of area, *A* the cross-sectional area, ρ the density, *s* the circumferential coordinate along the centerline, *t* time, and *w* and *u* the displacements of the centerline in the radial and tangential directions respectively. The rotation φ of the cross-section and the normal force *N*, bending moment *M*, and shear force *Q* are given by [5]

$$\varphi = -\frac{u}{R} + \frac{\partial w}{\partial s}, \qquad N = EA\left(\frac{w}{R} + \frac{\partial u}{\partial s}\right) + \frac{EI}{R}\left(\frac{w}{R^2} + \frac{\partial^2 w}{\partial s^2}\right),$$

$$M = EI\left(\frac{w}{R^2} + \frac{\partial^2 w}{\partial s^2}\right), \qquad Q = -EI\left(\frac{1}{R^2}\frac{\partial w}{\partial s} + \frac{\partial^3 w}{\partial s^3}\right)$$
(2a-d)

Equations (1) and (2) are based on Flügge's theory. When R tends to infinity, the radial and tangential displacements decouple and the equations become those for a uniform, straight beam.

2.1 Dispersion relations

The radial and tangential displacements satisfying equation (1) are assumed to be time harmonic and of the form

$$w(s,t) = C_w e^{i(\omega t - ks)}, \qquad u(s,t) = C_u e^{i(\omega t - ks)}$$
(3a,b)

where C_w and C_u are constants, ω the angular frequency and k the wavenumber for the curved beam. Substituting equations (3a,b) into equation (1) gives

$$\begin{bmatrix} \frac{I}{AR^{2}} (k^{2}R^{2} - 1)^{2} + 1 - \frac{\rho}{E} R^{2} \omega^{2} & -ikR \\ ikR & k^{2}R^{2} - \frac{\rho}{E} R^{2} \omega^{2} \end{bmatrix} \begin{bmatrix} C_{w} \\ C_{u} \end{bmatrix} = 0$$
(4)

Setting the determinant of the matrix in equation (4) to zero gives the dispersion equation

$$k^{6} - \left(k_{L}^{2} + 2\kappa^{2}\right)k^{4} + \left(\kappa^{4} - k_{B}^{4} + 2\kappa^{2}k_{L}^{2}\right)k^{2} - \left(\kappa^{4}k_{L}^{2} + \kappa^{2}k_{B}^{4} - k_{L}^{2}k_{B}^{4}\right) = 0$$
(5)

where $k_L = \sqrt{\rho \omega^2 / E}$ and $k_B = \sqrt[4]{\rho A \omega^2 / EI}$ are the longitudinal and bending wavenumbers for a straight beam, respectively, and $\kappa = 1/R$ is the curvature. The beam is assumed to be undamped so that k_L and k_B are real. Equation (5) is a cubic equation in k^2 so that there are three pairs of solutions at any given frequency, three for positive-going waves and three for negative-going waves. The wavenumbers of positive-going waves are defined to be such that

$$\operatorname{Im}\{k\} \le 0, \qquad \operatorname{Re}\{\partial k/\partial \omega\} > 0 \text{ if } \operatorname{Im}\{k\} = 0 \qquad (6a,b)$$

Equation (6a) indicates that, if the imaginary value of the wavenumber of a positive-going wave is non-zero, the amplitude of the wave decays in the positive *s* direction. If the imaginary value is zero, equation (6b) indicates that the energy transport velocity associated with a positive-going wave should be positive.

The non-dimensional radius of gyration, wavenumber and frequency are introduced and are respectively given by

$$\chi = \sqrt{\frac{I}{AR^2}}, \qquad \xi = kR, \qquad \Omega = \frac{\omega R}{c_L}$$
(7a-c)

where $c_L = \sqrt{E/\rho}$ is the longitudinal phase velocity. Figure 1 shows the wavenumbers ξ_1 , ξ_2 and ξ_3 for the positive-going waves in the curved beam with $\chi^2 = 1/1200$ which corresponds to h/R = 0.1 if the beam is rectangular. In the figure the frequency range is divided into 4 regions



Figure 1. Dispersion relations for positive-going waves in the curved beam with $\chi^2 = 1/1200$.

by the bifurcation points. In region I, the wavenumbers are all purely real so that all the wave modes propagate along the curved beam. One interesting feature is that the (real) wavenumber ξ_2 for the second mode is negative in this region. Thus the phase velocity of the wave mode is negative while the energy is transported in the positive *s* direction, i.e. a wave transports energy in the direction opposite to the direction of the phase velocity. In region II, ξ_2 is complex and, since $\xi_2 = -(\xi_3)^*$, this represents a spatially decaying standing wave. Only the first mode can propagate. In region III, also, only the first mode propagates. The other wave modes are both evanescent, i.e., they decay without a change in phase. In region IV, ξ_3 becomes purely real, representing a propagating wave. In this region the wavenumbers are broadly analogous to those of bending (ξ_1, ξ_2) and extensional waves (ξ_3) in a straight beam.

2.2 Displacement ratio

The radial and tangential displacements of the curved beam are not independent of each other. From equation (4), the ratio $\alpha = C_u/C_w$ is given by

$$\alpha_{i} = \frac{i\kappa k_{i}}{k_{L}^{2} - k_{i}^{2}}; \qquad i = 1, 2, \dots, 6$$
(8)

where i = 1, 2, 3 denote the three positive-going waves, respectively, and i = 4, 5, 6 denote the corresponding negative-going waves. Note that $\alpha_{4,5,6} = -\alpha_{1,2,3}$ since $k_{4,5,6} = -k_{1,2,3}$.

Figure 2 shows the displacement ratio for the three positive-going waves for the curved beam with $\chi^2 = 1/1200$. The four regions shown in Figure 1 are not marked for clarity, but can be inferred from the discontinuous behaviour of the curves. It can be seen that the radial motion is dominant for the first wave mode since $|\alpha_1| < 1$ in the frequency range considered. In region II $|\alpha_2| = |\alpha_3|$. In regions III and IV, the radial motion is dominant for the second mode. Near the ring frequency $\Omega = 1$, the radial motion is dominant for the third mode (the magnitude of α_3 is zero at the ring frequency) but, as frequency increases, the tangential motion becomes dominant. The phase difference between the displacement components is between $\pi/2$ and $-\pi/2$.



Figure 2. Displacement ratio $\alpha = C_u/C_w$ for the curved beam with $\chi^2 = 1/1200$.

3. MATRIX REPRESENTATION OF WAVE MOTION

A systematic methodology for wave analysis based on reflection, transmission and propagation of waves is provided by the definition of displacement, internal force and propagation matrices [7]. In this section, the matrices for the curved beam are presented. Since the curved beam is a three-mode system, the relevant vectors and matrices are of order 3×1 and 3×3 , respectively.

Assuming the displacements to be of the form given by equation (3), the radial and tangential displacements of the beam are given respectively by

$$w(s) = C_1 e^{-ik_1 s} + C_2 e^{-ik_2 s} + (\alpha_3)^{-1} C_3 e^{-ik_3 s} + C_4 e^{-ik_4 s} + C_5 e^{-ik_5 s} + (\alpha_6)^{-1} C_6 e^{-ik_6 s},$$

$$u(s) = \alpha_1 C_1 e^{-ik_1 s} + \alpha_2 C_2 e^{-ik_2 s} + C_3 e^{-ik_3 s} + \alpha_4 C_4 e^{-ik_4 s} + \alpha_5 C_5 e^{-ik_5 s} + C_6 e^{-ik_6 s}$$
(9a,b)

The generalized displacements and corresponding internal forces can be grouped in the vectors

$$\mathbf{w} = \begin{bmatrix} w & \varphi & u \end{bmatrix}^T, \qquad \mathbf{f} = \begin{bmatrix} Q & M & N \end{bmatrix}^T$$
(10a,b)

where the superscript *T* denotes the transpose. Note that the rotation φ and internal forces *Q*, *M* and *N* are obtained from equations (2) and (9). The wave vectors consisting of the amplitudes of the waves are defined by

$$\mathbf{a}^{+}(s) = \begin{bmatrix} C_1 e^{-ik_1 s} & C_2 e^{-ik_2 s} & C_3 e^{-ik_3 s} \end{bmatrix}^T, \qquad \mathbf{a}^{-}(s) = \begin{bmatrix} C_4 e^{-ik_4 s} & C_5 e^{-ik_5 s} & C_6 e^{-ik_6 s} \end{bmatrix}^T$$
(11a,b)

The displacement and internal force vectors are related to the vectors of wave amplitudes by [7]

$$\begin{cases} \mathbf{w} \\ \mathbf{f} \end{cases} = \begin{bmatrix} \Psi^+ & \Psi^- \\ \Phi^+ & \Phi^- \end{bmatrix} \begin{cases} \mathbf{a}^+ \\ \mathbf{a}^- \end{cases}$$
(12)

where the matrices Ψ and Φ define the transformation from the wave domain to the physical domain. They are given by

where the column vectors $\boldsymbol{\psi}_i$ and $\boldsymbol{\phi}_i$ for i = 1, 2, 4, 5 are

$$\Psi_{i} = \begin{cases} 1\\ -(\kappa\alpha_{i} + ik_{i})\\ \alpha_{i} \end{cases}, \qquad \Phi_{i} = \begin{cases} iEIk_{i} (\kappa^{2} - k_{i}^{2})\\ EI(\kappa^{2} - k_{i}^{2})\\ EA(\kappa - ik_{i} \alpha_{i}) + EI\kappa(\kappa^{2} - k_{i}^{2}) \end{cases}$$
(14a,b)

and ψ_i and ϕ_i for i = 3, 6 are

$$\Psi_{i} = \frac{1}{\alpha_{i}} \begin{cases} 1\\ -(\kappa\alpha_{i} + ik_{i})\\ \alpha_{i} \end{cases}, \quad \varphi_{i} = \frac{1}{\alpha_{i}} \begin{cases} iEIk_{i} (\kappa^{2} - k_{i}^{2})\\ EI(\kappa^{2} - k_{i}^{2})\\ EA(\kappa - ik_{i}\alpha_{i}) + EI\kappa(\kappa^{2} - k_{i}^{2}) \end{cases}$$
(15a,b)

Using these matrices, the reflection and transmission matrices for arbitrary discontinuities or for boundaries can be found in a simple manner [7].

The propagation matrix \mathbf{F} , describing propagation of waves over a distance *L* along the curved beam, is given by

$$\mathbf{F}(L) = \begin{bmatrix} e^{-ik_1L} & 0 & 0\\ 0 & e^{-ik_2L} & 0\\ 0 & 0 & e^{-ik_3L} \end{bmatrix}$$
(16)

Note that the propagation matrix is diagonal (i.e. the waves are not coupled during propagation) and the diagonal elements are independent of position.

4. ENERGY FLOW IN CURVED BEAMS

The time-averaged power Π associated with waves in one-dimensional structures can be expressed as [7]

$$\Pi = \frac{1}{2} \mathbf{a}^H \mathbf{P} \mathbf{a} \tag{17}$$

where the superscript *H* denotes the Hermitian, $\mathbf{a} = \left[\left(\mathbf{a}^+ \right)^T \quad \left(\mathbf{a}^- \right)^T \right]^T$ and the power matrix **P** is given by

$$\mathbf{P} = \frac{i\omega}{2} \left[\begin{bmatrix} \left(\Psi^{+} \right)^{H} \Phi^{+} & \left(\Psi^{+} \right)^{H} \Phi^{-} \\ \left(\Psi^{-} \right)^{H} \Phi^{+} & \left(\Psi^{-} \right)^{H} \Phi^{-} \end{bmatrix} - \begin{bmatrix} \left(\Phi^{+} \right)^{H} \Psi^{+} & \left(\Phi^{+} \right)^{H} \Psi^{-} \\ \left(\Phi^{-} \right)^{H} \Psi^{+} & \left(\Phi^{-} \right)^{H} \Psi^{-} \end{bmatrix} \right]$$
(18)

Substituting equation (13) into equation (18) gives the power matrix \mathbf{P} for the curved beam. In the four frequency regions the power matrix is given, respectively, by

<u>region I</u> P =	$ \begin{bmatrix} P_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} $	$egin{array}{c} 0 \ P_{22} \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ $	$ \begin{array}{c} 0 \\ 0 \\ P_{33} \\ 0 \\ 0 \\ 0 \\ 0 \end{array} $	$ \begin{array}{c} 0 \\ 0 \\ -P_{11} \\ 0 \\ 0 \end{array} $	$ \begin{array}{c} 0 \\ 0 \\ 0 \\ -P_{22} \\ 0 \end{array} $	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -P_{33} \end{bmatrix},$	$\underline{\operatorname{region II}}_{\mathbf{P}} = \begin{bmatrix} P_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0$	(19a-d)
<u>region III</u> P =	$ \begin{bmatrix} P_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} $	$\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ P_{25}^{*} \\ 0 \end{array}$	$egin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ P_{36}^{*} \end{array}$	$ \begin{array}{c} 0 \\ 0 \\ -P_{11} \\ 0 \\ 0 \end{array} $	$ \begin{array}{c} 0 \\ P_{25} \\ 0 \\$	$\begin{bmatrix} 0 \\ 0 \\ P_{36} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\frac{\text{region IV}}{\mathbf{P}} = \begin{bmatrix} P_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{25} & 0 \\ 0 & 0 & P_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & -P_{11} & 0 & 0 \\ 0 & P_{25}^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -P_{33} \end{bmatrix}$	

where the elements are

$$P_{11} = \omega EIk_{1} \left\{ 2\left(k_{1}^{2} - \kappa^{2}\right) + \frac{k_{B}^{4}\kappa^{2}}{\left(k_{L}^{2} - k_{1}^{2}\right)^{2}} \right\}, \qquad P_{22} = \omega EIk_{2} \left\{ 2\left(k_{2}^{2} - \kappa^{2}\right) + \frac{k_{B}^{4}\kappa^{2}}{\left(k_{L}^{2} - k_{2}^{2}\right)^{2}} \right\},$$
(20a-f)
$$P_{33} = \omega EA \frac{k_{L}^{2}}{k_{3}} \left\{ 1 + \frac{2\left(k_{3}^{2} - \kappa^{2}\right)\left(k_{L}^{2} - k_{3}^{2}\right)^{2}}{k_{B}^{4}\kappa^{2}} \right\}, \qquad P_{26} = -\frac{i\omega EI}{\kappa} \left(k_{2}^{2} - \left(k_{2}^{*}\right)^{2}\right)\left(\left(k_{2}^{*}\right)^{2} - k_{1}^{2}\right),$$
$$P_{25} = -\omega EIk_{2} \left\{ 2\left(k_{2}^{2} - \kappa^{2}\right) + \frac{k_{B}^{4}\kappa^{2}}{\left(k_{l}^{2} - k_{2}^{2}\right)^{2}} \right\}, \qquad P_{36} = -\omega EA \frac{k_{L}^{2}}{k_{3}} \left\{ 1 + \frac{2\left(k_{3}^{2} - \kappa^{2}\right)\left(k_{L}^{2} - k_{3}^{2}\right)^{2}}{k_{B}^{4}\kappa^{2}} \right\}$$

It can be noticed that an element of \mathbf{P} is non-zero, i.e., energy can be transported, in three cases: by a single wave with real wavenumber (i.e., a propagating wave); by interaction of two opposite-going waves of one mode, for which the wavenumber is purely imaginary (i.e., two opposite-going nearfield waves); or by interaction of two opposite-going waves from different modes, for which the wavenumbers are a complex conjugate pair. These results are consistent with the work by Langley [8] for a general one-dimensional dynamic system.

Figure 3 shows the magnitudes of the non-zero elements of **P** for the curved beam with $\chi^2 = 1/1200$ as a function of a frequency. Note that there are always six energy transport paths (i.e., six non-zero elements in the power matrix) at any frequency. At high frequencies, the powers associated with the waves tend to those of the straight beam: it is seen that the normalised magnitudes of P_{11} , P_{25} and P_{33} tend to unity above the ring frequency $\Omega = 1$.



Figure 3. Non-zero elements of the power matrix for the curved beam with $\chi^2 = 1/1200$: $|P_{11}|$ (______), $|P_{22}|$ (______), $|P_{33}|$ (______), $|P_{25}|$ (______), $|P_{26}| = |P_{35}|$ (______), $|P_{36}|$ (______). In the figure $|P_{33}|$ and $|P_{36}|$ are normalised with respect to ωEAk_L and the others are normalised with respect to $2\omega EIk_B^3$.

The power matrix is not diagonal except for the frequency region I. A further transformation can be defined using a power basis, where energy is transported independently by a single component, using the eigenvalues and eigenvectors of the power matrix. Let **V** be the diagonal matrix consisting of the (real) eigenvalues and **E** be the (unitary) matrix whose columns are the eigenvectors of **P**. Since $\mathbf{P} = \mathbf{EVE}^{-1}$, equation (17) can be written as

$$\Pi = \frac{1}{2} \mathbf{p}^H \mathbf{V} \mathbf{p} \tag{21}$$

where $\mathbf{p} = \mathbf{E}^{-1}\mathbf{a}$ is a vector of power wave amplitudes. Since V is diagonal, equation (21) indicates that energy is transported independently by the individual power wave components of **p**. For example, V and E in the frequency region II are given by

$$\mathbf{V} = \begin{bmatrix} P_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & |P_{26}| & 0 & 0 & 0 & 0 \\ 0 & 0 & |P_{26}| & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -P_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -|P_{26}| & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -|P_{26}| \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{\phi^*}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{\phi^*}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{\phi^*}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\phi^*}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$
(22a,b)

where $\phi = P_{26}/|P_{26}|$. Similar transformation into the power wave domain in the frequency regions III and IV can also be made [7].

5. CONCLUDING REMARKS

This paper concerned in-plane motion of curved beams based on Flügge's theory. Displacement and force matrices were derived – these allow transformations to be made between the physical and wave domains enabling a systematic analysis to be made of waveguide structures with curved components. The energy flow associated with waves in the curved beam was also obtained in a systematic way. It was seen that energy is transported independently by propagating waves or by the interaction of two wave components, for which the wavenumbers are a complex conjugate pair. A further transformation to power wave components was found – these components transport power independently.

REFERENCES

- [1] J. P. Charpie and C. B. Burroughs, "An analytical model for the free in-plane vibration of beams of variable curvature and depth", *Journal of the Acoustical Society of America* **94**, 866-879 (1993).
- [2] P. Chidamparam and A. W. Leissa, "Vibrations of planar curved beams, rings and arches", *Applied Mechanics Reviews* **46**, 467-483 (1993).
- [3] N. M. Auciello and M. A. De Rossa, "Free vibrations of circular arches: a review", *Journal of Sound and Vibration* **176**, 433-458 (1994).
- [4] C. M. Wu and B. Lundberg, "Reflection and transmission of the energy of harmonic elastic waves in a bent bar", *Journal of Sound and Vibration* **190**, 645-659 (1996).
- [5] S. J. Walsh and R. G. White, "Vibrational power transmission in curved beams", *Journal of Sound and Vibration* **233**, 455-488 (2000).
- [6] B. Kang, C. H. Riedel and C. A. Tan, "Free vibration analysis of planar curved beams by wave propagation", *Journal of Sound and Vibration* **260**, 19-44 (2003).
- [7] S.-K. Lee, Wave reflection, transmission and propagation in structural waveguides, PhD Thesis, University of Southampton (2006).
- [8] R. S. Langley, "A transfer matrix analysis of the energetics of structural wave motion and harmonic vibration", *Proceedings of the Royal Society of London A* **452**, 1631-1648 (1996).