



# DYNAMIC ANALYSIS OF VEHICLES WITH UNCERTAIN PARAMETERS

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### Abstract

A quarter car model is used to investigate the dynamic characteristic and random vibration response of cars with uncertainty in this paper. The sprung mass, unsprung mass, suspension damping, suspension and tyre stiffness are considered as random variables. The road irregularity is considered a Gaussian random process and modeled by means of a simple exponential power spectral density. The numerical characteristics including mean value and standard deviation of the natural frequencies, mode shapes and root mean square random response of vehicles are obtained by using the Monte-Carlo simulation method. The influences of the randomness of the vehicle's parameters on the dynamic characteristic and random response are investigated in detail using a practical example.

### **1. INTRODUCTION**

The vibration of an on-road vehicle is predominantly excited by the unevenness of the road surface on which the vehicle travels. Vehicle dynamic analysis has been a hot research topic for many years due to its important role in ride comfort, vehicle safety and overall vehicle performance. Numerous papers about the theoretical and experimental investigation on the dynamic behaviour of passively and actively suspended road vehicles have been published [1-3]. The quarter-car model [4], half-car model [5] and full-vehicle model [6] have been developed with researches related to the dynamic behaviour of vehicle and its vibration control. The simplest representation of a ground vehicle is a quarter-car model with a spring and a damper connecting the body to a single wheel which is in turn connected to the ground via the tire spring. The mass of the body usually is described as sprung mass, the mass representing the wheel, tire, brakes and part of the suspension linkage mass is referred to as the unsprung mass.

Although mathematical modelling tools for analysis/computation have experienced a tremendous growth, most research in vehicle dynamics was based on the assumption that all parameters of vehicle systems are deterministic. Actually, the spring stiffness and damping rate may vary with respect to the nominal value due to production tolerances and/or wear, ageing... etc. The vehicle body mass and the tyre radial stiffness can have stochastic variations due to the variety of possible vehicle loading conditions and to the uncertainty of the inflating pressure of poorly maintained tyres [7]. In cars and buses, weight and placements of passengers often

exhibit significant variability. In addition, even same brand and type vehicles leaving the production line may have uncertainties in size, mass and performance and so on. Hence, the problem of vehicle vibration subject to uncertain parameters is of great significance in realistic engineering applications.

Vibration analysis for deterministic vehicle systems under stochastic road excitations can be accomplished by structural random vibration theory. For the dynamic analysis of structures with uncertainties, stochastic analytical methods such as the Monte-Carlo simulation method (MCSM) and perturbation method (PM) are widely used. In the MCSM, the values of the structural parameters are changed within a given range. A large amount of dynamic analyses on the same structure is then performed, and the statistical data (mean value and standard deviation) of the natural frequencies, mode shapes and dynamic response are obtained [8]. In practice, this is the method of last resort since the attendant computational cost can be prohibitive for systems modelled using a large number of degrees of freedom. The PM uses a combination of matrix perturbation theory, finite element method and Taylor series expansion to obtain the dynamic characteristics of stochastic structures [9]. The major drawback of such local approximation techniques is that the results become highly inaccurate when the coefficients of variation of the input random variables are increased. In order to investigate the effect of individual parameters on the structural system response expediently and reduce the computational work, the random factor method (RFM) has been proposed and developed to analyze the dynamic response of structures with random parameters recently [10]. The main limitation of the RFM is that the randomness of each kind of parameter over all elements must be totally correlated.

In this paper, a two-degree-of-freedom quarter car model is used to investigate the vibration response of cars with uncertainty under random road input excitations. The vehicle's parameters are considered as random variables and the road unevenness is considered a Gaussian random process and modelled by means of a simple exponential power spectral density (PSD), the so-called "one slope PSD". The first two statistical moments of the dynamic characteristic and response are obtained by using conventional Monte-Carlo simulation method. A practical example is used to investigate the influences of the uncertainty of the vehicle's parameters on the vehicle's dynamic behaviour.

# 2. VEHICLE MODEL AND RANDOM VIBRATION ANALYSIS

A two-degree-of-freedom quarter-car model is shown in Figure 1. In this model, the sprung and unsprung masses corresponding to the one corner of the vehicle are denoted respectively by  $m_s$  and  $m_u$ . The suspension system is represented by a linear spring of stiffness  $k_s$  and a linear damper with a damping rate  $c_s$ , while the tyre is modelled by a linear spring of stiffness  $k_r$ . Since damping in the tyre is typically very small, it is neglected in this study.  $x_r$  is the road displacement input. The model is generally reputed to be sufficiently accurate for capturing the essential features related to discomfort, road holding and working space.

The linear equations of motions of the vehicle system model are

$$m_{s}\ddot{x}_{s} + c_{s}(\dot{x}_{s} - \dot{x}_{u}) + k_{s}(x_{s} - x_{u}) = 0$$
<sup>(1)</sup>

$$m_{u}\ddot{x}_{u} - c_{s}(\dot{x}_{s} - \dot{x}_{u}) - k_{s}(x_{s} - x_{u}) + k_{t}(x_{u} - x_{r}) = 0$$
(2)

By using a vector matrix form, equations (1) and (2) can be rewritten as

$$[M] \{ \ddot{X} \} + [C] \{ \dot{X} \} + [K] \{ X \} = \{ P \}$$
(3)

$$\begin{bmatrix} M \end{bmatrix} = \begin{bmatrix} m_s & 0 \\ 0 & m_u \end{bmatrix}, \begin{bmatrix} C \end{bmatrix} = \begin{bmatrix} c_s & -c_s \\ -c_s & c_s \end{bmatrix}, \begin{bmatrix} K \end{bmatrix} = \begin{bmatrix} k_s & -k_s \\ -k_s & k_s + k_t \end{bmatrix},$$
$$\{P\} = \begin{cases} 0 \\ k_t x_r \end{cases}, \{X\} = \begin{cases} x_s \\ x_u \end{cases}$$
(4)



Figure 1. The quarter-car model of the vehicle.

The displacement  $x_r$  (road irregularity) may be represented by a random variable defined by a stationary and ergodic stochastic process with zero mean value. The power spectral density of the process may be determined on the basis of experimental measurements and in the literature there are many different formulations for it. In this paper for sake of simplicity, the following spectrum [7] is considered

$$S_{x_r}(\omega) = \frac{A_b v}{\omega^2}$$
(5)

From equations (4) and (5), the power spectral density matrix  $[S_P(\omega)]$  of  $\{P\}$  can be obtained

$$\begin{bmatrix} S_P(\omega) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & k_t^2 S_{x_r}(\omega) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & k_t^2 \frac{A_b v}{\omega^2} \end{bmatrix}$$
(6)

Equation (3) presents a set of coupled differential equations. If the vehicle is initially considered at rest, then its solution can be obtained in terms of the decoupling transform and Duhamel integral [10]

$$\left\{u(t)\right\} = \int_{0}^{t} \left[\phi\right] \left[h(t-\tau)\right] \left[\phi\right]^{T} g(\tau) \left\{P(\tau)\right\} d\tau$$
(7)

where  $[\phi]$  is the normal modal matrix of the vehicle and can be expressed as

$$\begin{bmatrix} \phi \end{bmatrix} = \begin{bmatrix} \{\phi_1\}, \{\phi_2\} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$
(8)

[h(t)] is the impulse response function matrix of the vehicle, and can be expressed as

$$[h(t)] = diag\{h_j(t)\}, \ h_j(t) = \begin{cases} \frac{1}{\omega_{jd}} \exp(-\zeta_j \omega_j t) \sin \omega_{jd} t & t \ge 0\\ 0 & t < 0 \end{cases}, \ (j = 1, 2) \tag{9}$$

$$\zeta_{j} = \frac{1}{2\omega_{j}} \frac{\left\langle \phi_{j} \right\rangle^{T} [C] \left\langle \phi_{j} \right\rangle}{\left\langle \phi_{j} \right\rangle^{T} [M] \left\langle \phi_{j} \right\rangle} \qquad (10)$$

where  $\omega_j$  and  $\zeta_j$  are respectively the  $j^{\text{th}}$  natural frequency and modal damping of the vehicle, and  $\omega_{jd} = \omega_j (1 - \zeta_j^2)^{1/2}$ .

From equation (7), the correlation function matrix of the displacement response of the vehicle  $[R_u(\varepsilon)] = E(\{u(t)\}\{u(t+\varepsilon)\}^T)$  can be obtained

$$\left[R_{u}(\varepsilon)\right] = \int_{0}^{t} \int_{0}^{t} \left[\phi\right] \left[h(\tau)\right] \left[\phi\right]^{T} \left[R_{P}(\tau - \tau_{1} + \varepsilon)\right] \left[\phi\right] \left[h(\tau_{1})\right]^{T} \left[\phi\right]^{T} d\tau d\tau_{1}$$

$$(11)$$

where  $[R_P(\tau - \tau_1 + \varepsilon)]$  is the correlation function matrix of the  $\{P(t)\}$ .

By performing a  $[R_u(\varepsilon)]$  Fourier transformation, the power spectral density matrix of the displacement response  $[S_u(\omega)]$  is

$$[S_u(\omega)] = [\phi][H(\omega)][\phi]^T [S_p(\omega)][\phi][H^*(\omega)][\phi]^T$$
(12)

where  $[H^*(\omega)]$  is the conjugate matrix of  $[H(\omega)]$ ,  $[H(\omega)]$  is the frequency response function matrix of the vehicle and can be expressed as

$$[H(\omega)] = diag[H_j(\omega)], H_j(\omega) = \frac{1}{\omega_j^2 - \omega^2 + i \cdot 2\xi_j \omega_j \omega} \quad (i = \sqrt{-1}, j = 1, 2)$$
(13)

Integrating  $[S_u(\omega)]$  within the frequency domain, the mean square value matrix of the vehicle's displacement response, that is,  $[\psi_u^2]$  can be obtained

$$\left[\psi_{u}^{2}\right] = \int_{-\infty}^{\infty} \left[S_{u}(\omega)\right] d\omega = \int_{-\infty}^{\infty} \left[\phi\right] \left[H(\omega)\right] \left[\phi\right]^{T} \left[S_{P}(\omega)\right] \left[\phi\right] \left[H^{*}(\omega)\right] \left[\phi\right]^{T} d\omega$$
(14)

Then the mean square value of the *kth* degree of freedom of the vehicle's dynamic displacement response can be expressed as

$$\psi_{uk}^{2} = \vec{\phi}_{k} \cdot \int_{-\infty}^{\infty} [H(\omega)] [\phi]^{T} [S_{P}(\omega)] [\phi] [H^{*}(\omega)] d\omega \cdot \vec{\phi}_{k}^{T} \quad (k = 1, 2)$$
(15)

where  $\vec{\phi}_k$  is the *kth* line vector of the matrix  $[\phi]$ .

# 3. RANDOM RESPONSE ANALYSIS OF VEHICLE WITH UNCERTAIN PARAMETERS

The vehicle's parameters corresponding to  $m_s$ ,  $m_u$ ,  $k_s$ ,  $c_s$  and  $k_t$  are simultaneously considered as random variables. The randomness of vehicle's parameters will result in randomness of the matrices [M] and [K] and [C], and consequently the natural frequencies  $\omega_j$ , mode matrix  $[\phi]$  and modal damping  $\zeta_j$ . The random variables are each given a mean value ( $\mu$ ) and standard deviation ( $\sigma$ ), for example,  $m_s = \mu_{m_s} \pm \sigma_{m_s}$ . A further parameter used in this paper is the variation coefficient v, defined by the ratio of the standard deviation to the mean value, that is  $v = \sigma/\mu$ .

In the MCSM, N samples of the random variables are generated in given ranges. The implementation of the method consists in the numerical simulation of these samples associated to the random quantities of the physical problem, the procedure used for a deterministic analysis is repeated for each sample of the simulation process, obtaining then N responses that are computed to get the first two statistical moments of the response. For the two-degree-freedom system, the computational effort is acceptable for analysis of the mean value and standard deviation of vehicle's dynamic characteristics and random response. By using the conventional Monte-Carlo simulation method,  $\mu_{\omega_j}$ ,  $\sigma_{\omega_j}$ ,  $\mu_{[\phi]}$ ,  $\sigma_{[\phi]}$ ,  $\mu_{\zeta_j}$ ,  $\sigma_{\zeta_j}$ ,  $\mu_{\psi_{uk}^2}$ 

and  $\sigma_{\psi_{uk}^2}$  can be obtained.

# **4. NUMERICAL EXAMPLE**

The mean values of vehicle's parameters for this study are given in Table 1, which are typical for a lightly damped passenger car [4]. In the following simulations,  $A_b = 1.4e - 5(m)$  and v = 50(m/s) are taken into consideration. In order to investigate the effect of the uncertainty of random variables  $m_s$ ,  $m_u$ ,  $k_s$ ,  $c_s$  and  $k_t$  on the vehicle's dynamic characteristics and responses, the values of variation coefficient  $v_{m_s}$ ,  $v_{m_u}$ ,  $v_{k_s}$ ,  $v_{c_s}$  and  $v_{k_t}$  of random vehicle's parameters  $m_s$ ,  $m_u$ ,  $k_s$ ,  $c_s$  and  $k_t$  are respectively taken as different groups. The computational results of natural frequencies, mode shapes and mean square displacement responses are respectively given in Tables 2, 3 and 4, in which 10000 simulations are used. In Table 4,  $\psi_{usm}^2$  and  $\psi_{uum}^2$  respectively denote the mean square random displacement response of sprung and unsprung masses.

Parameters	Mean values
Sprung mass $m_s$	$\mu_{m_s}$ =240kg
Unsprung mass $m_u$	$\mu_{m_u}$ =36kg
Secondary suspension stiffness $k_s$	$\mu_{k_s}=980 \mathrm{Ns/m}$
Damping coefficient $c_s$	$\mu_{c_s} = 16,000 \mathrm{N/m}$
Primary suspension stiffness $k_t$	$\mu_{k_r} = 160,000 \text{N/m}$

Table 1. The mean values of vehicle system parameters for the quarter-car model

Model	$\mu_{\omega_1}$	$\sigma_{_{\omega_{\mathrm{l}}}}$	$\mu_{\omega_2}$	$\sigma_{\scriptscriptstyle \omega_2}$
Deterministic model	7 7901	0	60.0645	0
$V_{m_s} = V_{m_u} = V_{k_s} = V_{k_t} = 0$	7.7801	0	09.9045	0
$v_{m_s} = 0.1 \ v_{m_u} = v_{k_s} = v_{k_t} = 0$	7.8138	0.4002	69.9650	0.0047
$v_{m_u} = 0.1 \ v_{m_s} = v_{k_s} = v_{k_t} = 0$	7.7801	4.9532e-4	70.2139	3.6088
$v_{k_s} = 0.1 \ v_{m_s} = v_{m_u} = v_{k_t} = 0$	7.7711	0.3582	69.9665	0.3292
$V_{k_t} = 0.1  V_{m_s} = V_{m_u} = V_{k_s} = 0$	7.7768	0.0379	69.9084	3.1882
$V_{m_s} = V_{m_u} = V_{k_s} = V_{k_t} = 0.1$	7.8003	0.5388	70.1663	4.8127

Table 2. The computational results of natural frequencies (unit: rad/s)

Model	$\mu_{\phi_{11}}$	$\sigma_{\scriptscriptstyle{\phi_{11}}}$	$\mu_{\phi_{22}}$	$\sigma_{_{\phi_{22}}}$
Deterministic model	0.0645	0	0 1666	0
$V_{m_s} = V_{m_u} = V_{k_s} = V_{k_t} = 0$	-0.0043	0	0.1000	0
$v_{m_s} = 0.1 \ v_{m_u} = v_{k_s} = v_{k_t} = 0$	-0.0648	0.0033	0.1666	1.1392e-5
$v_{m_u} = 0.1 \ v_{m_s} = v_{k_s} = v_{k_t} = 0$	-0.0645	4.2097e-6	0.1672	0.0086
$v_{k_s} = 0.1 \ v_{m_s} = v_{m_u} = v_{k_t} = 0$	-0.0645	7.5966e-6	0.1666	1.9614e-5
$v_{k_t} = 0.1 \ v_{m_s} = v_{m_u} = v_{k_s} = 0$	-0.0645	8.1962e-6	0.1666	2.1163e-5
$v_{m_s} = v_{m_u} = v_{k_s} = v_{k_t} = 0.1$	-0.0648	0.0033	0.1672	0.0086

Table 3. The computational results of mode shapes

Table 4. The computational results of mean square displacement (unit: mm<sup>2</sup>)

Model	$\mu_{\!\psi^2_{usm}}$	$\sigma_{\!\psi^2_{usm}}$	$\mu_{\!\psi^2_{uum}}$	$\sigma_{\!\psi^2_{uum}}$
Deterministic model	438.2249	0	1.0259e+3	0
$V_{m_s} = V_{m_u} = V_{k_s} = V_{c_s} = V_{k_t} = 0$	130.2219	Ŭ	1.0237013	0
$v_{m_s} = 0.1 \ v_{m_u} = v_{k_s} = v_{c_s} = v_{k_t} = 0$	430.1550	36.13936	1.0146e+3	10.82472
$v_{m_u} = 0.1 \ v_{m_s} = v_{k_s} = v_{c_s} = v_{k_t} = 0$	426.8979	13.23232	1.0141e+3	59.23112
$v_{k_s} = 0.1 \ v_{m_s} = v_{m_u} = v_{c_s} = v_{k_t} = 0$	429.3036	31.69608	1.0142e+3	14.612
$v_{c_s} = 0.1 \ v_{m_s} = v_{m_u} = v_{k_t} = v_{k_s} = 0$	427.4127	0.0608	1.0141e+3	0.0344
$v_{k_t} = 0.1 \ v_{m_s} = v_{m_u} = v_{k_s} = v_{c_s} = 0$	436.2534	24.11112	1.0210e+3	47.258
$v_{m_s} = v_{m_u} = v_{k_s} = v_{c_s} = v_{k_t} = 0.1$	440.3322	61.32608	1.0205e+3	84.52896

### **5. CONCLUSIONS**

(1) The uncertainty of the vehicle's natural frequencies is dependent on the uncertainty of sprung mass, unsprung mass, suspension damping, suspension and tyre stiffness. The randomness of sprung mass produce the greatest effect on the vehicle's first natural frequency, however, the randomness of unsprung mass produce the greatest effect on the vehicle's second natural frequency.

(2) The randomness of the mode shapes is almost dependent on the uncertainty of sprung mass and unsprung mass, and is almost independent of suspension and tyre stiffness.

(3) The uncertainty of sprung mass and unsprung produce the greatest effect on the mean square displacement of them respectively.

(4) Comparing with the case that only one of the uncertainty of sprung mass, unsprung mass, suspension damping, suspension and tyre stiffness is taken into account, the change of the

vehicle's dynamic characteristics and response are greater when their uncertainty are considered simultaneously. This kind of result is coincident with the conclusions in references [10,11].

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