



THE RIJKE TUBE: GREEN'S FUNCTION APPROACH IN THE TIME AND FREQUENCY DOMAIN

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Abstract

The Rijke tube is modelled as an open-ended tube with a blockage, jump in cross-section and jump in mean temperature. Its Green function is known in the frequency domain as well as in the time domain. For the time-domain problem, the heat release characteristic has the form of a real function (giving heat release rate in terms of the acoustic velocity at the heat source). An integral equation is derived, which involves this heat release characteristic and the time-domain Green function. The integral equation is solved numerically by an iteration, stepping forward in time to give the time history of the acoustic velocity and of the heat release rate. Both linear and nonlinear heat release characteristics can be studied by this method. For the frequency-domain problem, the heat release characteristic has the form of a transfer function (relating complex velocity amplitudes to complex heat release amplitudes). An equation is derived for the complex eigenfrequencies of the heat-driven oscillation in the Rijke tube. This equation involves the transfer function and the frequency-domain Green function. It is solved numerically to give the frequency and growth rate of any mode in the Rijke tube.

1. INTRODUCTION

1.1 Rijke tube configuration

We consider a Rijke tube with axisymmetric geometry; a cross-section between the tube axis and the tube wall is shown in Figure 1. The ends are open with pressure nodes just outside the tube at $x = \ell_1$ and $x = \ell_2$ (Rayleigh end correction). There is a blockage, a change of crosssectional area from \mathcal{A}_1 to \mathcal{A}_2 and a jump in mean temperature from \overline{T}_1 to \overline{T}_2 . The speed of sound jumps from c_1 to c_2 due to the temperature jump.

The blockage, which is assumed to be compact, is modelled by an incompressible "airplug" of effective length L_{eff} oscillating parallel to the x-axis. L_{eff} depends on the geometry of the

flame holder and has to be calculated numerically (see [1]). In the region around the blockage, the acoustic field is three-dimensional, but in the upstream region between x = 0 and X_1 , as well as in the downstream region between $x = X_2$ and L, the field is one-dimensional.



Figure 1 - Rijke tube with a point source at $x = x_q$

The jump in mean temperature is caused by a steady heat source (marked by a solid grey line in Figure 1) which is situated near the downstream edge of the flame holder; the unsteady heat source (marked by a broken grey line) is assumed to be just downstream of this point, at $x = x_q$; its rate of heat release per unit mass of air (from the heat source to the air) is

$$q'(x,t) = q(t)\delta(x - x_a).$$
⁽¹⁾

1.2 The Green function

An important element of our theoretical model is the exact acoustic Green function $G(\mathbf{x}, \mathbf{x}', t, t')$. This is the velocity potential in the tube at position \mathbf{x} and time t, created by an impulsive point source at position \mathbf{x}' and time t'. The exact Green function is the solution of

$$\frac{1}{c^2} \frac{\partial^2 G}{\partial t^2} - \nabla^2 G = \delta(\mathbf{x} - \mathbf{x}') \delta(t - t'), \qquad (2)$$

inside the tube (*c* is the speed of sound, taking values c_1 and c_2 in the upstream and downstream region, respectively) and satisfies the following boundary conditions: It is zero at $x = \ell_1$ and $x = \ell_2$ (this neglects losses from the ends), and it has a normal derivative equal to zero on all internal surfaces and on the tube axis. It also satisfies the conditions of reciprocity and causality. For the case where x' is in one of the one-dimensional regions, Green's function has the form

$$G(x, x', t-t') = \sum_{n=1}^{\infty} g_n(x, x') H(t-t') \sin \omega_n(t-t').$$
(3)

H is the Heaviside function, ω_n are the eigenfrequencies of the Rijke tube (with steady heating) and g_n are the modal amplitudes. Green's function is an impulse response: it is zero before the impulse (at t = t') and consists of a superposition of eigenmodes (numbered by the index *n*) thereafter.

The frequency-domain equivalent of G(x, x', t-t') is the time-harmonic Green function $\hat{G}(x, x', \omega)$, which satisfies

$$\frac{\partial^2 \hat{G}}{\partial x^2} + k^2 \hat{G} = \delta(x - x'), \qquad (4)$$

and the same boundary conditions as G. The wave number k takes the values $\frac{\omega}{c_1}$ and $\frac{\omega}{c_2}$ in the upstream and downstream region, respectively.

Analytical expressions for ω_n , $g_n(x, x')$ and $\hat{G}(x, x', \omega)$ have been derived in [1] and are listed in the Appendix for the case where x' is in the downstream region.

2. TIME DOMAIN PROBLEM

2.1 Governing equation

The velocity potential ϕ in the Rijke tube is governed by the nonhomogeneous wave equation (see [2] p. 508; γ is the specific heat ratio),

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = -\frac{\gamma - 1}{c^2}q'(x,t).$$
(5)

We will derive an integral equation from (5) and (2) by performing the following steps: Equation (5) and the one-dimensional form of (2) are written in terms of the source variables x' and t'; (2) is multiplied by $\phi(x',t')$, (5) by G(x,x',t-t'), and the resulting equations are subtracted. This gives

$$\phi(x',t')\delta(x-x')\delta(t-t') + \frac{\gamma-1}{c^2}Gq'(x',t') = \frac{1}{c^2} \left(\phi \frac{\partial^2 G}{\partial t'^2} - G \frac{\partial^2 \phi}{\partial t'^2}\right) - \left(\phi \frac{\partial^2 G}{\partial x'^2} - G \frac{\partial^2 \phi}{\partial x'^2}\right).$$
(6)

This equation is integrated with respect to x' (from ℓ_1 to ℓ_2) and t' (from 0 to t). For a heat release of the form (1), the result can be simplified (using the boundary conditions at the tube ends) to give

$$\phi(x,t) = -\frac{\gamma - 1}{c^2} \int_{t'=0}^{t} G(x, x_q, t - t') q(t') dt' + \int_{x'=\ell_1}^{\ell_2} \frac{1}{c^2} \left[\phi \frac{\partial G}{\partial t'} - G \frac{\partial \phi}{\partial t'} \right]_{t'=0}^{t} dx'.$$
(7)

We assume the following initial conditions:

$$\left. \phi(x',t') \right|_{t'=0} = 0 \quad \text{and} \quad \left. \frac{\partial \phi(x',t')}{\partial t'} \right|_{t'=0} = 0 \quad \text{for all } x' \in (\ell_1,\ell_2) \,, \tag{8}$$

i.e. the velocity and acceleration are zero throughout the tube at the time t'=0. Combined with causality of the Green function $(G=0 \text{ and } \frac{\partial G}{\partial t'}=0 \text{ for } t' \le t)$, equation (7) then simplifies, due to the second integral on the right band side becoming zero.

The remainder of (7) can be turned into an equation for the velocity by differentiating with respect to x. Evaluation at $x = x_q$ leads to an integral equation giving the velocity at the heat source in terms of the heat release,

$$u_q(t) = -\frac{\gamma - 1}{c^2} \int_{t'=0}^t \left. \frac{\partial G(x, x', t-t')}{\partial x} \right|_{\substack{x=x_q \\ x'=x_q}} q(t') \, \mathrm{d}t'; \tag{9}$$

the abbreviation $u_q(t) = \frac{\partial \phi(x,t)}{\partial x} \Big|_{x=x_q}$ has been introduced for the velocity at the heat source.

2.2 Numerical solution

The *x*-derivative of Green's function is calculated from (3) and inserted into (9). This gives (using complex notation)

$$u_q(t) = \frac{\gamma - 1}{c^2} \operatorname{Im} \left[\sum_{n=1}^{\infty} \left. \frac{\partial g_n(x, x')}{\partial x} \right|_{\substack{x = x_q \\ x' = x_q}} e^{-i\omega_n t} \int_{t'=0}^{t} e^{i\omega_n t'} q(t') \, \mathrm{d}t' \right].$$
(10)

The integral in (10), $I_n(t) = \int_{t'=0}^{t} e^{i\omega_n t'} q(t') dt'$, can be split into two parts, one over the

interval $(0, t - \Delta t)$, and the other over an interval of width Δt , where Δt is a small time step,

$$I_n(t) = \int_{t'=0}^{t-\Delta t} e^{i\omega_n t'} q(t') dt' + \int_{t'=t-\Delta t}^{t} e^{i\omega_n t'} q(t') dt'.$$
 (11)

The first integral represents $I_n(t - \Delta t)$. The second integral can be approximated: the time interval Δt is assumed to be very small, and therefore the heat release rate in this interval is nearly constant and equal to $q(t - \Delta t)$. With these results, (11) can be written as

$$I_n(t) = I_n(t - \Delta t) + q(t - \Delta t) \frac{e^{i\omega_n t}}{i\omega_n} \left(1 - e^{-i\omega_n \Delta t}\right),$$
(12)

and this leads with equation (10) to

$$u_{q}(t) = \frac{\gamma - 1}{c^{2}} \operatorname{Im}\left[\sum_{n=1}^{\infty} \left. \frac{\partial g_{n}(x, x')}{\partial x} \right|_{\substack{x = x_{q} \\ x' = x_{q}}} e^{-i\omega_{n}t} I_{n}(t) \right].$$
(13)

The heat release rate q generally depends on aerodynamic fluctuations in the tube; this dependence is called the "heat release characteristic". In many cases, this dependence is of the form

$$q(t) = q(u_q(t-\tau)), \qquad (14)$$

i.e. q is a function of the acoustic velocity at the heat source, delayed by a time lag τ .

Equations (14) (evaluated at $t - \Delta t$), (12) and (13) comprise an iteration procedure for the time history of the velocity $u_q(t)$, and also that of the heat release q(t). They are solved by stepping forward in time with steps Δt . The starting point of the iteration is provided by the initial condition $u_q(t-\tau)\Big|_{t=0} = u_0$. This precedes the initial conditions specified in (8).

2.3 Numerical results and discussion

The iteration described by equations (12) and (13) was performed numerically for a tube with the following properties:

$$\begin{split} L &= 1 \text{ m (tube length)}, \ \ell_1 = -0.014 \text{ m}, \ \ell_2 = 1.014 \text{ m}, \\ \frac{\mathcal{A}_2}{\mathcal{A}_1} &= 1.128, \ L_{eff} = 0.093 \text{ m}, \\ \overline{T}_1 &= 288 \text{ K (room temperature)}, \ \overline{T}_2 = 488 \text{ K}, \\ c_1 &= 342 \text{ m s}^{-1}, \ c_2 = 446 \text{ m s}^{-1}. \end{split}$$

The heat source was located at $x_q = 0.3L$, a position where the fundamental mode in the tube is unstable and the second mode is stable. The eigenfrequencies of the first two modes are $\omega_1 = 1235 \text{ s}^{-1}$ and $\omega_2 = 2722 \text{ s}^{-1}$.

The heat release rate known from hot wire theory is given by $a + b\sqrt{|\overline{u} + u_q(t-\tau)|}$ [3], where a, b are positive constants and \overline{u} is the velocity of the mean flow. The fluctuating part is

$$q(t) = b\sqrt{|\overline{u} + u_q(t-\tau)|} - b\sqrt{\overline{u}} .$$
(15)

Figures 2 and 3 show the time histories of $u_q(t)$ and q(t), respectively, for the heat release characteristic (15) with

$$b = 5 \times 10^5 \text{ m}^{3/2} \text{s}^{-7/2}, \ \overline{u} = 1.0 \text{ m s}^{-1}, \ \tau = 0.0002 \text{ s}.$$



Figure 3 - Time history of the heat release rate q(t)

In the early stages of the time histories, the exponential increase of the amplitudes is evident. This is expected at low amplitudes, where a linearized version of (15) is valid. When the velocity amplitude approaches the mean velocity of 1.0 ms^{-1} , the rate of increase slows down. Flow reversal leads to a period doubling of the heat release rate in the last third of the time history. The velocity amplitude continues to grow in this nonlinear regime, without reaching a limit cycle.

3. FREQUENCY DOMAIN PROBLEM

3.1 Equation for the eigenfrequencies of the heat-driven oscillations

The steps described in section 2.1 and leading up to the governing equation (9) can be followed in the frequency domain to give

$$\hat{u}_{q}(\omega) = \frac{\gamma - 1}{c^{2}} \frac{\partial \hat{G}(x, x', \omega)}{\partial x} \bigg|_{\substack{x = x_{q} \\ x' = x_{q}}} \hat{q}(\omega).$$
(16)

 $\hat{u}_q(\omega)$ and $\hat{q}(\omega)$ are the Fourier transform of $u_q(t)$ and q(t), respectively, e.g.

$$q(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{q}(\omega) e^{-i\omega t} d\omega.$$
(17)

The frequency domain equivalent of the heat release characteristic is the transfer function $T(\omega)$, which relates the heat release $\hat{q}(\omega)$ to the velocity $\hat{u}_{q}(\omega)$,

$$\hat{q}(\boldsymbol{\omega}) = T(\boldsymbol{\omega})\,\hat{u}_{q}(\boldsymbol{\omega})\,. \tag{18}$$

This can be used to substitute for $\hat{q}(\omega)$ in (16) to give

$$1 = \frac{\gamma - 1}{c^2} \frac{\partial \hat{G}(x, x', \omega)}{\partial x} \bigg|_{\substack{x = x_q \\ x' = x_q}} T(\omega)$$
(19)

This equation determines the (generally complex) eigenfrequencies of the heat-driven oscillations in the Rijke tube.

3.2 Numerical results and discussion

Equation (19) can be solved by the Newton/Raphson method to obtain the complex eigenfrequencies. This was done for the transfer function $T(\omega) = \frac{b}{2\sqrt{u}} e^{i\omega\tau}$ (frequency-domain equivalent of the linearised version of (15)) and for the parameter values listed in section 2.3. The time lag τ was varied and the imaginary part of the complex eigenfrequencies, which indicates the stability behaviour of a mode, was noted. The first mode was found to be unstable in the range $0 < \omega_1 \tau < \pi$, and approximately the same range of instability was found for the second mode. These results are in line with those reported by earlier authors [4].

4. CONCLUSIONS AND OUTLOOK

A Green function approach has been presented to model the behaviour of a Rijke tube in the time domain as well as in the frequency domain. The time-domain problem gives the time history of the oscillation in the tube and allows one to study the effect of nonlinear heat

release characteristics. Features, such as limit cycles, period-doubling, etc. can be predicted. The frequency-domain problem gives the frequencies and growth rates of the modes in the tube. The effect of different flame transfer functions can be examined with this approach.

APPENDIX

The eigenfrequencies ω_n are the roots of $f(\omega) = 0$, where

$$f(\omega) = \frac{\mathcal{A}_1}{\mathcal{A}_2} \frac{1}{c_1} \left[-\cos\omega\tau_1 \sin\omega\tau_2 + \frac{\overline{\rho}_1}{\overline{\rho}_2} \cos\omega\tau_2 \left(\frac{\mathcal{A}_2}{\mathcal{A}_1} \frac{c_1}{c_2} \sin\omega\tau_1 + \frac{L_{eff}}{c_2} \omega\cos\omega\tau_1 \right) \right]. \quad (20)$$

 τ_1 and τ_2 are wave travel times: $\tau_1 = \frac{X_1 - \ell_1}{c_1}$ and $\tau_2 = \frac{X_2 - \ell_2}{c_2}$.

The Green function amplitudes in (3) are given by

$$g_n(x, x') = 2 \frac{C(x, \omega_n) D(x', \omega_n)}{\omega_n f'(\omega_n)},$$
(21)

where f' is the derivative of the function $f(\omega)$ in (3); C and D are given by

$$C(x,\omega) = \sin \frac{\omega(x-\ell_2)}{c_2},$$
(22)

$$D(x',\omega) = \frac{\overline{\rho}_1}{\overline{\rho}_2} \left(\sin \omega \tau_1 + \frac{\mathcal{A}_1}{\mathcal{A}_2} \frac{L_{eff}}{c_1} \omega \cos \omega \tau_1 \right) \cos \frac{\omega(x' - X_2)}{c_2} + \frac{\mathcal{A}_1}{\mathcal{A}_2} \frac{c_2}{c_1} \cos \omega \tau_1 \sin \frac{\omega(x' - X_2)}{c_2}.$$
 (23)

The time-harmonic Green function is given by

$$\hat{G}(x, x', \omega) = \frac{C(x, \omega)D(x', \omega)}{\omega f(\omega)}.$$
(24)

The expressions (21) to (24) are for the case where x' is in the one-dimensional region downstream of the flame holder.

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