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# ANALYSIS OF 1/2-ORDER SUBHARMONIC RESONANCE IN A HORIZONTALLY SUPPORTED JEFFCOTT ROTO 

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#### Abstract

The $1 / 2$-order subharmonic resonance occurs when the rotational speed is in the vicinity of twice the natural frequency. The Jeffcott rotor used basically and widely in the analysis of resonancies, is a two-degree-of-freedom model with a disk at the midspan of a massless-shaft. For the above condition on the rotational speed, we clarify the nonlinear characteristics of the resonance in a horizontally supported Jeffcott rotor. Applying the method of multiple scales, we directly derive the amplitude equations for the horizontal direction and vertical direction and depict the relationship between the rotational speed and the response amplitude. Furthermore, experiments are performed and the results are compared with the theoretical ones.


## 1. INTRODUCTION

Rotating machineries, such as steam turbines, gas turbines, motors and so on, often become the main source of vibration. When the machineries are operated at high velocities with the developments in science, occurrence of nonlinear oscillations can be easily predicted. When the rotational speed is in the vicinity of twice the natural frequency, the $1 / 2$-order subharmonic resonance occurs. The Jeffcott rotor used basically and widely in the analysis of resonancies, is a two-degree-of-freedom model with a disk at the midspan of a massless-shaft [1][2]. Ishida and Inoue investigated nonlinear phenomena in the neighborhood of the rotational speed of twice the major critical speed [3]. In a horizontally supported Jeffcott rotor, the effect of gravity influences the rotor of whirling motion. The equilibrium position differs from the bearing centerline by the static displacement due to gravity. The restoring force of the shaft has nonlinearity due to the extension of the shaft center line and angular clearance of bearing. We consider the asymmetrical nonlinearity of the gravity and the restoring force. For the above condition, we clarify
the nonlinear characteristics of the resonance. We apply the method of multiple scales, which is widely applied to analysis of reciprocation, to analysis of whirling motion. We directly derive the amplitude equations for the horizontal and vertical direction and depict the relationship between the rotational speed and the response amplitude. Furthermore, experiments are performed and the results are compared with the theoretical ones.

## 2. ANALYTICAL MODEL AND EQUATION OF MOTION



Figure 1. Analytical model

The analytical model of a horizontally supported Jeffcott rotor is shown in Fig. 1. A rigid disk is mounted at the mid-span of a massless elastic shaft which is supported at both ends with ball bearings. We introduce the static coordinate system as shown in Fig. 1(b). The origin O of the coordinate system $\mathrm{O}-x y$ coincides with the bearing centerline connecting the centers of the right and left bearings. The disk of the rotor has mass $m$ and its center of gravity G deviates slightly $e_{d}$ from the geometrical center M. Furthermore, assuming the cubic nonlinearity in the stiffness of the shaft and the bearings of the supportiong points, which is the most fundamental symmetric nonlinearity [4], the equations of motion of the rotor system can be written as follows:

$$
\begin{gathered}
m \frac{d^{2} x}{d t^{2}}+c_{d} \frac{d x}{d t}+k x+\beta_{d}\left(x^{2}+y^{2}\right) x=m e_{d} \omega^{2} \cos \omega t \\
m \frac{d^{2} y}{d t^{2}}+c_{d} \frac{d y}{d t}+k y+\beta_{d}\left(x^{2}+y^{2}\right) y=m e_{d} \omega^{2} \sin \omega t-m g_{d}
\end{gathered}
$$

where $\omega, c_{d}, k, \beta_{d}$, and $g_{d}$ are the angular velocity of the shaft, the viscous damping coefficient, the linear spring constant of the elastic shaft, the cubic nonlinear spring constant, and the gravity acceleration, respectively. The equilibrium position differs from the bearing centerline by the static displacement $y_{s t}$ due to the gravity. Denoting the equilibrium point by $y=y_{s t}$ and considering Eq. (2), we conclude that the equilibrium positions satisfies the equation

$$
\begin{equation*}
k y_{s t}+\beta_{d} y_{s t}^{3}=-m g . \tag{1}
\end{equation*}
$$

Substituting $y=y_{s t}+\Delta y$ into Eqs. (2) and (2), we obtain equations of motion with respect to $x$ and $y$. All lengths are nondimensionalized using the static displacement $y_{s t}$ and the time is nondimensionalized using $T=\sqrt{\left(k+\beta_{d} y_{s t}{ }^{2}\right) / m}$. We denote the resulting dimensionless
quantities corresponding to $x, y$, and $t$ by $x^{*}, y^{*}$, and $t^{*}$, respectively. We introduce the following dimensionless parameters:

$$
\begin{equation*}
e=\frac{e_{d}}{y_{s t}}, c=\frac{c_{d}}{\sqrt{m\left(k+\beta_{d} y_{s t}^{2}\right)}}, 1+\omega_{-}=\sqrt{\frac{k+3 \beta_{d} y_{s t}^{2}}{k+\beta_{d} y_{s t}^{2}}, \beta=\frac{\beta_{d} y_{s t}^{2}}{k+\beta_{d} y_{s t}^{2}}, \nu=\frac{\omega}{T} . . . . . ~} \tag{2}
\end{equation*}
$$

In this way, we can obtain the following dimensionless equations of motion.

$$
\begin{gather*}
\ddot{x}+c \dot{x}+x+2 \beta x \Delta y+\beta x^{3}+\beta x \Delta y^{2}=e \nu^{2} \cos \nu t  \tag{3}\\
\Delta \ddot{y}+c \Delta \dot{y}+\left(1+\omega_{-}\right)^{2} \Delta y+\beta x^{2}+3 \beta \Delta y^{2}+\beta x^{2} \Delta y+\beta \Delta y^{3}=e \nu^{2} \sin \nu t \tag{4}
\end{gather*}
$$

where ( ${ }^{\circ}$ ) represents the derivative with respect to $t^{*}$.

## 3. THEORETICAL ANALYSIS

In this section, we dinote averaged equations from the dimensionless equations of motion, Eqs. (3) and (4), by using the method of multiple scales [5]. As a result, the approximate solutions are expressed as follows:

$$
\begin{align*}
x & =a_{x} \cos \left(\frac{\nu}{2} t+\varphi_{x}\right)+O\left(\epsilon^{2}\right)+O\left(\epsilon^{3}\right)  \tag{5}\\
\Delta y & =a_{y} \cos \left(\frac{\nu}{2} t+\varphi_{y}\right)+O\left(\epsilon^{2}\right)+O\left(\epsilon^{3}\right) . \tag{6}
\end{align*}
$$

where the time variations of $a_{x}, a_{y}, \varphi_{x}$, and $\varphi_{y}$ are governed with the following equations:

$$
\begin{gather*}
\frac{d}{d t} a_{x}=\frac{1}{2}\left\{-c a_{x}+X_{3} a_{x} a_{y}{ }^{2} \sin \left(-2 \varphi_{x}+2 \varphi_{y}\right)-4 f a_{x} \cos \left(2 \varphi_{x}\right)+4 f a_{y} \sin \left(-\varphi_{x}-\varphi_{y}\right)\right\}  \tag{7}\\
a_{x} \frac{d}{d t} \varphi_{x}=-\frac{1}{2}\left\{\sigma a_{x}+X_{1} a_{x}{ }^{3}+X_{2} a_{x} a_{y}{ }^{2}+X_{3} a_{x} a_{y}{ }^{2} \cos \left(-2 \varphi_{x}+2 \varphi_{y}\right)\right. \\
\left.-4 f a_{x} \sin \left(-2 \varphi_{x}\right)+4 f a_{y} \cos \left(-\varphi_{x}-\varphi_{y}\right)\right\}  \tag{8}\\
\frac{d}{d t} a_{y}=\frac{1}{2}\left\{-c a_{y}+Y_{3} a_{x}{ }^{2} a_{y} \sin \left(2 \varphi_{x}-2 \varphi_{x}\right)+4 f a_{x} \sin \left(-\varphi_{x}-\varphi_{y}\right)-12 f a_{y} \cos \left(2 \varphi_{y}\right)\right\}  \tag{9}\\
a_{y} \frac{d}{d t} \varphi_{y}=-\frac{1}{2}\left\{\sigma a_{y}-2 \hat{\omega}_{-} a_{x}+Y_{1} a_{y}{ }^{3}+Y_{2} a_{x}{ }^{2} a_{y}+Y_{3} a_{x}{ }^{2} a_{y} \cos \left(2 \varphi_{x}-2 \varphi_{y}\right)\right. \\
\left.+4 f a_{x} \sin \left(-\varphi_{x}-\varphi_{y}\right)+4 f a_{y} \cos \left(2 \varphi_{y}\right)\right\} \tag{10}
\end{gather*}
$$

where

$$
\begin{gather*}
X_{1}=\frac{10}{3} \beta^{2}-3 \beta, X_{2}=\frac{44}{3} \beta^{2}-2 \beta, X_{3}=2 \beta^{2}-\beta, \\
Y_{1}=30 \beta^{2}-3 \beta, Y_{2}=\frac{44}{3} \beta^{2}-2 \beta, Y_{3}=2 \beta^{2}-\beta, f=\frac{\beta \hat{e}}{\nu^{2}-1} . \tag{11}
\end{gather*}
$$

Because the solutions that amplitude equal 0 have indefiniteness phase angle $\varphi_{x}$ and $\varphi_{y}$, we analyze the equation in Cartesian coordinate as follows:

$$
\begin{align*}
& a_{x} e^{i \varphi_{x}}=a_{x r}+i a_{x i}  \tag{12}\\
& a_{y} e^{i \varphi_{y}}=a_{y r}+i a_{y i}, \tag{13}
\end{align*}
$$

where $a_{x r}, a_{x i}, a_{y r}$ and $a_{y i}$ are real number and imaginary number in horizontal direction and vertical direction. The time variations of $a_{x r}, a_{x i}, a_{y r}$ and $a_{y i}$ are governed with the following equations:

$$
\begin{align*}
& \frac{d}{d t} a_{x r}=\frac{1}{2}\left\{\sigma a_{x i}-c a_{x r}+X_{1}\left({a_{x r}}^{2}+{a_{x i}}^{2}\right) a_{x i}+X_{2}\left(a_{y r}{ }^{2}+a_{y i}{ }^{2}\right) a_{x i}\right. \\
& \left.+X_{3}\left(2 a_{x r} a_{y r} a_{y i}-a_{x i} a_{y r}^{2}+a_{x i} a_{y r}^{2}\right)-4 f a_{x r}-4 f a_{y i}\right\}  \tag{14}\\
& \frac{d}{d t} a_{x i}=-\frac{1}{2}\left\{\sigma a_{x r}+c a_{x i}+X_{1}\left({a_{x r}}^{2}+{a_{x i}}^{2}\right) a_{x r}+X_{2}\left(a_{y r}{ }^{2}+a_{y i}{ }^{2}\right) a_{x r}\right. \\
& \left.+X_{3}\left(2 a_{x i} a_{y r} a_{y i}+a_{x r} a_{y r}^{2}-a_{x r} a_{y i}{ }^{2}\right)-4 f a_{x r}-4 f a_{y i}\right\}  \tag{15}\\
& \frac{d}{d t} a_{y r}=\frac{1}{2}\left\{\sigma a_{y i}-c a_{y r}-2 \hat{\omega}_{-} a_{y i}+Y_{1}\left(a_{y r}{ }^{2}+a_{y i}{ }^{2}\right) a_{y i}+Y_{2}\left(a_{x r}{ }^{2}+a_{x i}{ }^{2}\right) a_{y i}\right. \\
& \left.+Y_{3}\left(2 a_{x r} a_{x i} a_{y r}-a_{x i}{ }^{2} a_{y i}+a_{x i}{ }^{2} a_{y i}\right)-4 f a_{x i}-12 f a_{y r}\right\}  \tag{16}\\
& \frac{d}{d t} a_{y i}=-\frac{1}{2}\left\{\sigma a_{y r}+c a_{y i}-2 \hat{\omega}_{-} a_{y r}+Y_{1}\left(a_{y r}^{2}+a_{y i}{ }^{2}\right) a_{y r}+Y_{2}\left(a_{x r}{ }^{2}+a_{x i}{ }^{2}\right) a_{y r}\right. \\
& \left.+Y_{3}\left(2 a_{x r} a_{x i} a_{y i}+{a_{x r}}^{2} a_{y r}-a_{x i}{ }^{2} a_{y r}\right)-4 f a_{x r}-12 f a_{y i}\right\} . \tag{17}
\end{align*}
$$

From Eqs. (7)-(10) and Eqs. (14)-(17), we have frequency response curves for x and y directions, as Fig. 2, where $e=3.6 \times 10^{-2}, c=4.0 \times 10^{-3}, \beta=1.2 \times 10^{-1}, \hat{\omega}_{-}=1.1 \times 10^{-1}$. The solid and dashed lines denote stable and unstable steady state amplitude, respectively. In the $x$ direction, the trivial equilibrium point changes to unstable from stable in the vicinity of twice the natural frequency. With increasing the rotational speed, the amplitude becomes bigger. Also in the $y$ direction, we observe oscillation together with the excitation in the $x$ direction. But the amplitude is smaller than that of the $x$ direction. The symbol of are simulation results of applying Runge-Kutta method to Eqs. (3) and (4). The results of the theoretical analysis corresponds with the results of the simulations.


Figure 2. Frequency response curve

## 4. EXPERIMENT

We show experimental setup in Fig. 3. The span and diameter of the shaft are 12 mm and 700 mm , respectively. The mass and diameter of the disk are 8.21 kg and 0.3 mm , respectively. Displacement of the disk in $x$ and $y$ directions are measured by laser sensors. Figure 4 shows experimentally obtained frequency response curves. As theoretically predicted, the resonance is generated in $x$ direction. On the other hand in $y$ direction, the resonance is bigger than analytical result.


Figure 3. Experimental setup


Figure 4. Experimental frequency response curve

## 5. SUMMARY

In this papare, we analyzed equations of motion about $1 / 2$-order subharmonic resonance in a horizontally supported Jeffcott rotor considering the cubic nonlinearity. We theoretically clarified the vibration characteristic by frequency response curves. And we compared analytical results and experimental results. We summarize conclusions below.
(1) By using the method of multiple scales, we analyzed nonlinear dynamics of a horizontally supported Jeffcott rotor, and we obtained the averaged equations of the $x$ direction and the $y$
direction.
(2) We theoretically predicted occurrence of $1 / 2$-Order Subharmonic Resonance due to the nonlinearity and the gravity by frequency response curves.
(3) We confirmed $1 / 2$-Order Subharmonic Resonance experimentaly and compared with the theoretical ones. The quantitative difference exist from frequency respoce curve theoretically depicted.

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