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NONLINEAR VIBRATION ANALYSIS IN ACOUSTICS

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Abstract

Mechanisms of nonlinear coupling between the normal modes of vibrating mechanical structures have been explored, previously particularly in the context of struck idiophones (percussive instruments). However, the analysis of the vibrations resulting from impulsive loads has been restricted by our ability to solve the nonlinear equations of motion. In recent times, numerical methods for solving the equations have burgeoned, and we are now in a position to apply the methods to structures of interest such as musical plates and gongs. In this paper, we consider the nonlinear coupling of modes of vibration whereby energy is transferred between the normal modes of a kinked bar. A finite element analysis package is employed to analyse the response. The results confirm that initially missing modes of vibration are generated through nonlinear coupling mechanisms.

1. INTRODUCTION

The analysis of many musical instruments begins with the vibrational behaviour of strings, bars, plates or shells. The linear vibrational theory of such simple structures is generally well-understood and documented. The complexity of the analysis obviously increases with the complexity of the structure, and for many structures analytical solutions of their governing equations are not possible. In such situations, numerical techniques may be employed to solve the equations.

Nonlinear analysis of simple structures is also well-advanced. For musical structures, nonlinear analysis is employed to predict the hardening and softening behaviours with increased amplitude of vibration, as well as coupling mechanisms between modes of vibration [1]. However, the inclusion of nonlinear terms in the governing equations for a structure results in non-trivial equations and severely limits our ability to solve them analytically. Instead, a variety of approximations and implementations have been employed in the analysis of nonlinear phenomena.

The authors have been interested for some time in the validity of various nonlinear numerical techniques adopted by structural engineers [2], as well as the application of structural engineering techniques to the analysis and design of musical structures [3, 4]. In the last decade, nonlinear analysis of structures has become more common due to changes in the design codes that govern structural engineering practice. This has led to the establishment of more robust numerical techniques for solving the nonlinear equations of motion for complex structures. From this vantage point, we now re-visit the nonlinear phenomena of mode coupling that is exhibited in some vibrating structures, and consider the application of modern numerical methods to this problem.

2. MODE COUPLING IN MUSICAL STRUCTURES

Standard linear analysis of simple structures predicts that the natural modes of vibration are independent. Furthermore, the removal of a particular mode of vibration, accomplished by exciting the structure at one of the nodes of that mode, should ensure that the mode does not re-appear. However, for systems with complicated boundary conditions, there can be a transfer of energy from the modes initially excited to others that may have originally been missing. An example of this effect is the shimmer of high frequency modes on the tam-tam (large gong) that appear in the sound spectrum in a time order of up to one second after the initial low frequency thump [5].

Investigations of such mode coupling have been undertaken using a simplified, one-dimensional analogue of a tam-tam, namely a symmetrically kinked bar [6]. The bar was designed with particular geometrical parameters such that the natural frequencies of the second and third symmetric modes were harmonically related, namely $\omega_3 = 2\omega_2$. The results demonstrated that by striking a symmetrically kinked bar on a node of the third mode, that mode began with a zero amplitude and rose to a maximum amplitude in a time order of 0.1 seconds. Hence, there was a coupling mechanism between modes that inherently allowed transfer of energy from one mode to another.

The theoretical analysis of the vibration indicated that the nonlinear coupling is a result of two separate mechanisms. The first is the amplitude-dependent tension generated in the bar by its motion and the coupling of shear forces at the kinks. This predicts that the amplitude of the third mode will be proportional of the square of the amplitude of the second mode, that is

$$a_3 \propto a_2^2 \tag{1}$$

where a denotes the amplitude. The second, less significant, mechanism arises from the internal bending moments about either side of the kink. The semi-quantitative agreement between theory and experiment suggested that the major physical mechanisms producing the mode coupling had been successfully identified. The analysis was performed prior to the advent of fast personal computers and, more particularly, nonlinear numerical techniques. In this paper, we study the same problem to investigate the application of finite elements to this phenomenon.

3. FINITE ELEMENT ANALYSIS

In essence, the finite element method is a computational technique based on an extension of matrix structural analysis [7]. The method models a structure through an assemblage of elements connected at discrete points. The dynamic analysis of a system produces a set of natural frequencies based on the system's geometrical characteristics.

The problem at hand involves causing a bar to vibrate as a result of the application of a single non-periodic impulse. The response will therefore be transient, and the initial amplitude of vibration will depend on the size of the applied impulse. Hence, the analysis must take into account that the applied load is a function of time.

Transient dynamic analysis entails the direct numerical integration of the equations of motion to calculate nodal displacements, velocities and accelerations. Using the finite element method, the governing differential equations are converted to a discrete set of matrix equations in terms of the nodal displacements. These discrete equations can be expressed in the form

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t) \tag{2}$$

where **M** is the mass matrix, **C** is the damping matrix, **K** is the stiffness matrix, **F** is the applied load vector, **u** is the vector of nodal displacements and a dot over a symbol denotes differentiation with time. The finite element calculations reported below were carried out using the commercial package Strand7 [8].

4. MODELLING ISSUES

4.1. Geometry

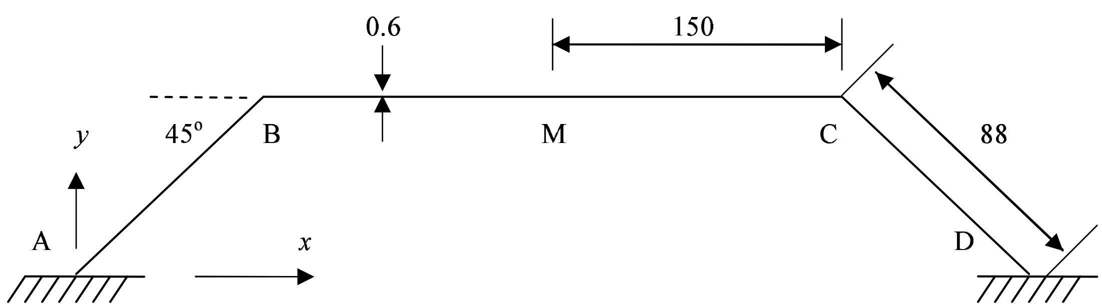

Figure 1. Geometry of the kinked bar (units = mm).

The kinked bar shown in Figure 1 was modelled on the dimensions and physical characteristics used by Legge and Fletcher [6]. The values used for the current paper are given in Table 1. The transverse displacement of point M was calculated, and only symmetrical modes were considered. This allowed the analysis to be conducted on only half the bar, which significantly reduced the computational time.

Table 1. Properties of the bar.

Property	Value
Young's modulus (GPa)	193
Poisson's ratio	0.28
Density (kg/m^3)	8000
Width of bar (mm)	10

Since the finite element method is an approximate method, the structure must be divided into sufficient elements to achieve the required accuracy. The accuracy was checked by refining the mesh until the solution converged. In our case, segments AB and BM were subdivided into 10 and 15 beam elements, respectively, to achieve sufficient accuracy.

4.2. Natural frequencies

An initial linear analysis of the system produced the natural frequencies and mode shapes for each of the symmetric modes. The values of the natural frequencies are given in Table 2. From the results, the location of the vibrational node for the third mode was determined to be approximately 51 mm from point B (see Figure 1).

Table 2. Natural frequencies for the first five symmetric modes.

Mode	Frequency (Hz)
1	27
2	153
3	313
4	432
5	761

4.3. Damping

Damping mechanisms are complex and difficult to accurately define for many systems. There are various mathematical models that are used to represent damping. In this investigation, we used Rayleigh damping, which is a form of viscous damping commonly used in structural analysis [7]. Rayleigh damping assumes that the damping matrix **C** in equation 2 is a linear combination of the stiffness and mass matrices such that

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K} \tag{3}$$

The values of α and β are determined by specifying the damping ratios, $\zeta_{1,2}$, for two different reference frequencies $\Omega_{1,2}$. The damping ratios are given in terms of α and β by

$$\zeta_n = \frac{1}{2}\left(\frac{\alpha}{\Omega_n} + \beta\Omega_n\right) \tag{4}$$

Equation 4 can be solved for α and β. Some care is needed to ensure that the resulting level of damping is realistic, and that important frequencies are not damped out artificially. However, increasing the damping ratios reduced the amount of mode coupling that could be practically measured. Since the objective of this work was to establish if finite element analysis could predict mode coupling, lower but less realistic values were used in this analysis.

4.4. Impulse

The impulse applied to the bar must also be modelled. The shape and duration of the impulse will influence the analysis, and any comparison between the modelled behaviour and experimental results would require careful correlation.

Three shapes were investigated, namely triangular, parabolic and half-sine. All produced nonlinear coupling behaviour to some extent. However, the various shapes and durations of the impulses affected the ability to distinguish the amplitude of the generated coupled mode. The triangular impulse shape was finally chosen because it produced the greatest definition in the peaks of the Fast Fourier Transform spectrums. The impulse length was adjusted based on the analysis time, and this resulted in applying quite rapid excitations.

4.5. Time-stepping

The dynamic equations of motion (equation 2) were solved within Strand7 using the Newmark time-stepping scheme [7]. The nonlinear effects in this analysis occur because the stiffness matrix, **K**, is a function of the deformation and the axial load in each member. Hence, an iterative procedure is required to solve the equations of motion. Strand7 is capable of including both effects on **K** as options in the time-stepping procedure.

The time step used for the calculations must be sufficiently small to ensure that the results are accurate and the impulse is modelled correctly. As discussed below, the results from the finite element analysis were post-processed using a Fast Fourier Transform. Hence, the time step should also be small enough to avoid aliasing in the transform calculations [9].

4.6. Analysis parameters

The behaviour of the kinked bar was determined using an impulse applied at the node of the third mode. Table 3 summarises the parameters that were used for the finite element analysis.

Table 3. Values/inputs of parameters used for the finite element analysis.

Parameter	Value/Input
Impulse Shape	Triangular
Impulse Length (s)	0.015
Analysis Time (s)	0.4
Damping frequencies (Hz)	5, 320
Damping Factors (%)	0.5, 0.5

5. RESULTS

The bar shown in Figure 1 together with the parameters from Tables 1 and 3 was analysed using Strand7. Of particular interest was the amplitude of the harmonically-related second and third modes. The finite element analysis was repeated for a variety of impulse magnitudes. The results of each analysis were evaluated based on the size of the amplitudes of the second and third modes relative to each other. By measuring the variation in the vertical displacement of point M with time, displacement versus time curves were obtained. The results were transformed into the frequency domain using a Fast Fourier Transform with a Hann window [10].

Figure 2 shows the maximum generated amplitude of mode 3 as a function of the initial

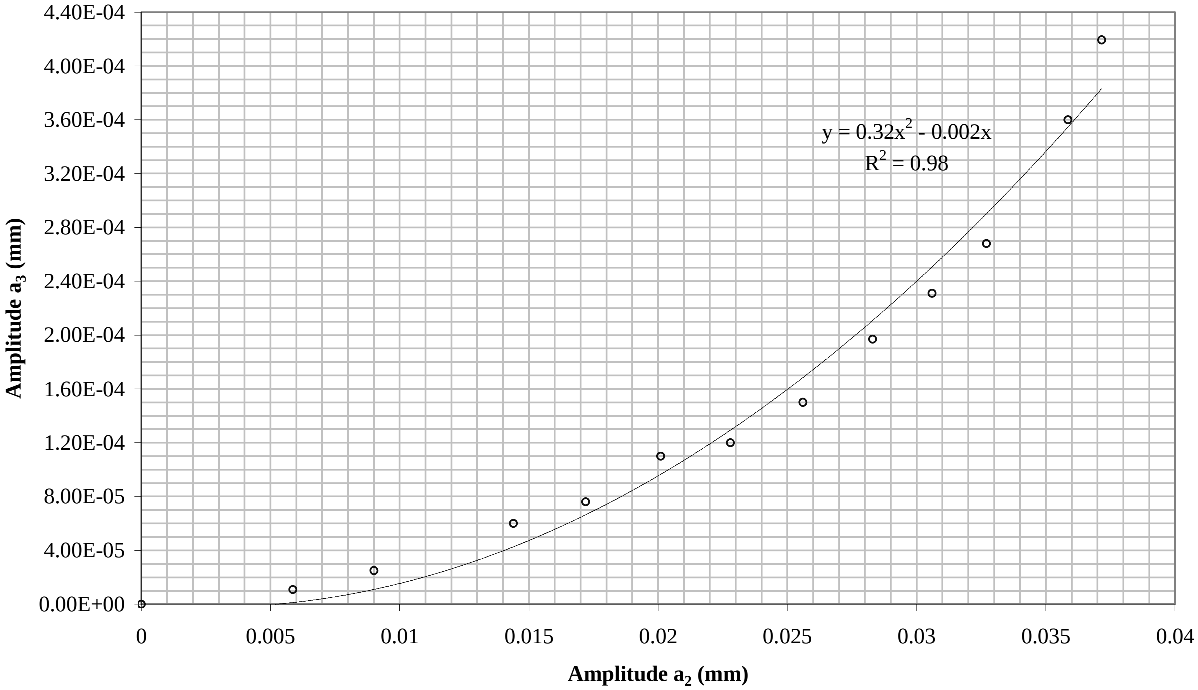

Figure 2. Variation of mode amplitudes.

amplitude of mode 2. The curve of best fit through the results is

$$a_3 = 0.32a_2^2 - 0.002a_2 \tag{5}$$

where a_2 and a_3 denote the amplitudes of the respective modes.

The linear coefficient in equation 5 is quite small compared to the quadratic coefficient, and can be neglected. The resulting quadratic relationship between a_2 and a_3 clearly indicates mode coupling between the second and third modes arising from the amplitude-dependent tension. The finite element results are in qualitative agreement with the results from Legge and Fletcher [6].

6. CONCLUSIONS

Standard linear analysis of simple structures predicts that the natural modes of vibration are independent. However, the inclusion of nonlinear terms in the governing equations for a structure indicates that energy can be transferred between some harmonically-related modes. In this paper, we examined the application of the finite element method for the solution of the nonlinear equations of a kinked bar. The use of the finite element method confirmed the existence of nonlinear coupling between harmonically related modes. Further studies will be undertaken using finite element analysis to model the transient behaviour of more complex musical structures.

REFERENCES

[1] N.H. Fletcher and T.D. Rossing. *The Physics of Musical Instruments*. Springer-Verlag, Berlin, second edition, 1998.

[2] J. Petrolito and K.A. Legge. Benchmarks for nonlinear elastic frame analysis. *Steel Construction, AISC*, 29(1):2–12, 1995.

[3] J. Petrolito and K.A. Legge. Optimal undercuts for the tuning of percussive beams. *Journal of the Acoustical Society of America*, 102(4):2432–2437, 1997.

[4] J. Petrolito and K.A. Legge. Designing musical plates using numerical optimisation. In *Proceedings, 16th National Congress, Australian Institute of Physics [CD-Rom]*, pages 1–4, Canberra, 2005.

[5] K.A. Legge and N.H. Fletcher. Nonlinearity, chaos and the sound of shallow gongs. *Journal of the Acoustical Society of America*, 86(6):2439–2442, 1989.

[6] K.A. Legge and N.H. Fletcher. Nonlinear mode coupling in symmetrically kinked bars. *Journal of Sound and Vibration*, 118(1):23–34, 1987.

[7] O.C. Zienkiewicz and R.L. Taylor. *The Finite Element Method for Solid and Structural Mechanics*, volume 2. Butterworth-Heinemann, Oxford, sixth edition, 2005.

[8] Strand7 Software. *Theoretical Manual*. Strand7 Pty. Ltd., Sydney, 2005.

[9] D.J. Inman. *Engineering Vibrations*. Prentice Hall, New Jersey, 1994.

[10] A.V. Oppenheim and R.W. Schafer. *Digital Signal Processing*. Prentice Hall, New Jersey, 1975.