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GAUGE INVARIANCE APPROACH TO ACOUSTIC FIELDS

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Abstract

We extend the gauge invariance property of the Maxwell's equations to the acoustic field equations. We use the analogy of the sound velocity as equivalent to the magnetic field and the acoustic stress field as equivalent to the electric field. We apply to sound propagation in solids and negative refraction. For sound propagation in solids, we use the equation of motion and the strain-displacement relation and express the velocity field and the stress field in terms of the vector potential and the scalar potential. We solve the inhomogeneous wave equation for these potentials and obtain the velocity field and the stress field in terms of these potentials. This gives a more rigorous solution than that from the Christoffel equation. Gauge invariance has symmetry property. We obtain the gauge form for the acoustic scalar potential and the vector potential and demonstrate their invariant property. We also show that the inhomogeneous wave equation in terms of the vector potential and the scalar potential possesses symmetry property. We also show the symmetry property of the acoustic field equation. We also derive the acoustic Lorentz condition from the equation of continuity. For the application of gauge invariance to negative refraction , we first show that the acoustic field equations also possess left hand and right hand symmetry and this gives rise to negative refraction. This is a different approach from that of Veselago who derives negative refraction from the negative permittivity and negative permeability. Here we extend negative refraction to anisotropic materials. In anisotropic materials, the compliance and stiffness possess rotational symmetry. When rotating in the clockwise direction, it gives rise to lefthanded phenomenon such as negative refraction. When rotating in the anticlockwise direction, it gives rise to the righthanded phenomenon such as positive refraction. Due to symmetry, both the righthanded phenomenon and the lefthanded phenomenon satisfy the acoustic field equations. The stress field, the velocity field and the acoustic Poynting vector together form a righthanded triplet or a lefthanded triplet depending on the direction of energy flow or the direction of the Poynting vector. According to parity conservation, acoustic law at the deepest level, there is no differentiation of righthanded and lefthanded treatment. The performance of an object and that of its mirror image will satisfy the same law of physics. The negative refraction in fact is a mirror image of the positive refraction.

1. INTRODUCTION

We extend the gauge invariance property of the Maxwell's equation for electromagnetic theory to the acoustic field equations. Gauge invariance which includes symmetries is a basic property of field theory which covers strong nuclear forces, electromagnetic force, and gravitational force. In extending gauge invariance approach to acoustic fields, it will be a more sophisticated approach than the vector theory of acoustic fields and will include also the characteristics of the vector theory. We will also show that vector theory is a subset of gauge theory. We address the symmetry properties of the acoustic field equations, the application of gauge invariance to negative refraction and interpretation of the inhomogeneous wave equation in terms of gauge invariance. Gauge invariance has long been applied to electromagnetic wave theory. Due to the similarities between electromagnetic waves and acoustic waves, we feel that the interpretation of acoustic fields in terms of gauge invariance will provide more understanding of the acoustic fields and throw lights on new potential applications. This paper is the first application of gauge invariance to acoustic fields.

2. GAUGE INVAIANCE FORMULATION OF THE INHOMOGENEOUS WAVE EQUATION

We will start with the derivation of the acoustic Lorentz gauge condition. First, we show that for electromagnetic waves, the Lorentz gauge condition gives rise to the equation of continuity. In electromagnetic theory, the Lorentz gauge condition is given as

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{1}$$

where $\vec{A} =$ vector potential and $\phi =$ scalar potential.

The potential theory is used here. By substituting in the retarded potential for $\vec{A}$ and ϕ, we have

$$\nabla \cdot \frac{\mu_0}{4\pi}\int \frac{j(x'_\alpha)}{r(x_\alpha, x'_\alpha)} dV' + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{1}{4\pi\epsilon_0}\right)\int \frac{\rho(x'_\alpha)}{r(x_\alpha, x'_\alpha)} = 0 \tag{2}$$

where $j =$ current density, $\mu =$ permeability, $\epsilon =$ permittivity, $\rho =$ charge density, $V =$ volume, $r =$ radial distance.

This can be simplified to

$$\mu_0 \nabla \cdot j + \frac{1}{c^2}\frac{\partial}{\partial t}\frac{1}{\epsilon_0}\rho(x'_\alpha) = 0 \tag{3}$$

With

$$\mu_0 = \frac{1}{c^2 \epsilon_0}$$

Eqn. (2) becomes

$$\nabla \cdot j + \frac{\partial \rho}{\partial t} = 0 \tag{4}$$

which is the equation of continuity.

On the other hand the acoustic equation of continuity is

$$\nabla(\rho V) + \frac{\partial \rho}{\partial t} = 0 \tag{5}$$

with just replacing j by ρV which is correct with charge density being analogous to the mass density for the acoustic case.

So we can use $\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$ as the acoustic Lorentz gauge condition where $c =$ sound velocity.

The two main acoustic fields are $\vec{T}$ and $\vec{V}$, the stress field and the velocity field respectively.

We are developing an acoustic field theory based on gauge invariance. The acoustic fields will be expressed in terms of potential functions, the vector potential and scalar potential.

From the electromagnetic theory, the electric field $\vec{E}$ and the magnetic field $\vec{B}$ are expressed in terms of potential functions as:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{6}$$

and

$$\vec{B} = \nabla \times \vec{A} \tag{7}$$

where $\phi =$ scalar potential and $\vec{A} =$ vector potential. These potentials are invariant under gauge transformation. We will use the analogy of $\vec{E}$ as $\vec{T}$ the acoustic stress field. This can be verified as follows.

The usual acoustical wave equations (homogeneous and inhomogeneous) are expressed in terms of wave functions which are ϕ or $\vec{A}$ in this case. This gives only partial aspects of the acoustic field $\vec{V}$ and $\vec{T}$. The basic acoustic field equations [1] are given by

$$\nabla \cdot \vec{T} = \frac{\partial \vec{P}}{\partial t} - \vec{F} \tag{8}$$

and

$$\nabla_s \cdot \vec{V} = \frac{\partial \vec{S}}{\partial A} \tag{9}$$

where $\vec{P} = \rho\vec{V} =$ momentum density, $\vec{F} =$ volume density of force, and $\vec{S} =$ stress field.

Eqn. (8) and (9) now completely parallel the Maxwell equations:

$$-\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} \tag{10}$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}_c + \vec{J}_s \tag{11}$$

where $\vec{E} =$ electric field, $\vec{H} =$ magnetic field, $\vec{J}_c =$ conduction current density and $J_s =$ source current density.

The strongest analogy of the acoustic field equations with the electromagnetic equations is obtained by taking $\vec{T}$ as equivalent to $\vec{E}$, $\vec{V}$ as equivalent to $\vec{H}$, $\vec{P}$ as equivalent to $\vec{B}$ and $\vec{S}$ as equivalent to $\vec{D}$.

Also we have

$$S_{ij} = S_{ijkl} T_{kl} \tag{12}$$

and

$$D_i = \epsilon_{ij} E'_j \tag{13}$$

So the compliance constants S_{ijkl} describe the elastic properties of a medium in a manner analogous to the description of its electrical properties by the permittivity matrix elements ϵ_{ij}. So we can write the stress field of the acoustic field as

$$\vec{T} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{14}$$

From the Maxwell's equation

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{15}$$

For the acoustic field

$$\nabla \cdot \vec{T} = 4\pi\rho \tag{16}$$

Substituting (14) into (16), we have

$$\nabla \left(\cdot - \nabla\phi - \frac{\partial \vec{A}}{\partial t} \right) = 4\pi\rho$$

But Lorentz gauge condition:

$$\nabla \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t}$$

$$\text{thus} \quad -\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2 \phi}{\partial^2 t} = 4\pi\rho \tag{17}$$

which is an inhomogeneous wave equation in ϕ and $\rho =$ field density.

We can also express the field in terms of the scalar potential and vector potential as

$$\vec{V} = \nabla\phi + \nabla \times \vec{A} \tag{18}$$

The Christoffel equation for an isotropic medium with plane wave solutions and harmonic time variation [2] can be written as

$$c_{11}\nabla(\nabla \cdot \vec{V}) - c_{44}\nabla \times \nabla \times \vec{V} = \rho\frac{\partial^2 \vec{V}}{\partial t^2} \tag{19}$$

Substitution of (18) into (19) gives

$$\nabla(c_{11}\nabla^2\phi - \rho\frac{\partial^2}{\partial t^2}\phi) - \nabla \times (c_{44}\nabla \times \nabla \times \vec{A} + \rho\frac{\partial^2 \vec{A}}{\partial t^2}) = 0 \tag{20}$$

since $\nabla \cdot \nabla \times \vec{A} = 0$ and $\nabla \times \nabla\phi = 0$.

For the second term, the quantity in brackets is set equal to the gradient of an arbitrary function f. That is,

$$c_{44}\nabla \times \nabla \times \vec{A} + \rho\frac{\partial^2 \vec{A}}{\partial t^2}) = c_{44}\nabla f$$

Application of the identity

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A} \tag{21}$$

convert this to

$$\nabla(\nabla \cdot \vec{A} - f) - \nabla^2\vec{A} + \frac{1}{V_s^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0$$

where $V_s = (c_{44}|\rho)^{1/2}$. Since f is arbitrary, it can always be chosen to cancel $\nabla \cdot \vec{A}$ in the first term on the left. The vector potential $\vec{A}$ can thus be taken as a solution to the vector wave equation

$$\nabla^2\vec{A} - \frac{1}{V_s^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0 \tag{22}$$

The first term in (20) is made zero by simply requiring that the scalar potential ϕ satisfy the scalar wave equation

$$\nabla^2\phi - \frac{1}{V_l^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \tag{23}$$

where $V_l = (c_{11}|\rho)^{1/2}$.

So we have obtained homogeneous wave equation in A and ϕ.

This shows that the acoustic fields $\vec{T}$ and $\vec{V}$ can be expressed in terms of the solutions of the homogeneous wave equations $\vec{A}$ and ϕ. $\vec{A}$ and ϕ in turn are only partial aspects of $\vec{V}$ and $\vec{T}$. This is a more accurate method of obtaining solutions for the acoustic fields.

The Lorentz gauge condition has the advantage of introducing complete symmetry between the scalar and vector potentials i.e. it makes both potentials satisfy the same wave equation as that obeyed by the fields. Equation (22) and (23) are a symmetrical set of equations.

By using the analogy of momentum density $\vec{P}$ as equivalent to $\vec{B}$ and stress field $\vec{T}$ as equivalent to $\vec{E}$, we have

$$\vec{T} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{24}$$

and

$$\vec{P} = \nabla \times \vec{A} \tag{25}$$

By inspection of Eqs. (24) and (25), we find that the resultant stress field and momentum density field are unchanged by transformation of the type [3]

$$A' = A - \nabla\phi \tag{26}$$

$$\phi' = \phi + \frac{\partial \phi}{\partial t} \tag{27}$$

where ϕ is a function of the coordinates and the time. This means that if any physical law involving elastic energy interactions is to be expressed in terms of the general elastic energy potential $\vec{A}$ and ϕ, then such a physical law must be unaffected by a transformation of that given by Eqns. (26) and (27). These transformations are usually known as gauge transformations, and a physical law that is invariant under such a transformation is said to be gauge invariant. The property of gauge invariance ensures that the physical law will not lead to consequences that cannot be expressed in the field formulation of the interaction of elastic properties.

From the above field theory approach, we find that both the vector fields $\vec{V}$ and $\vec{T}$ can be expressed in terms of a scalar potential and a vector potential. In another words, the vector potential is only a part of the field. Hence the vector theory is a subset of the field theory.

3. APPLICATION OF GAUGE INVARIANCE APPROACH OF ACOUSTIC FIELDS TO NEGATIVE DIFFRACTION

Veselago[3] derived the negative refraction theory from the consideration of negative permittivity and negative permeability, and is limited only to isotropic materials and electromagnetic wave. In this paper, we extend his theory to acoustic wave and to anisotropic materials such as piezoelectric materials. We apply gauge invariance to this study. It is well known that an important property of gauge invariance is symmetry.

In an anisotropic material, the compliance and stiffness possess rotational symmetry. Compliance and stiffness together describe the intrinsic elastic properties of the medium. The compliance constant describe the elastic properties of a medium in a manner analogous to the description of its electrical properties by the permittivity matrix elements. If the medium itself is symmetric with respect to a particular transformation of coordinates, then the compliance and stiffness matrices must themselves be unchanged by the same transformation. Symmetries for anisotropic media are much more complicated than for the isotropic case. When rotating in the clockwise direction, it gives rise to lefthanded phenomenon such as the negative refraction. When rotating in the anticlockwise direction, it give rise to the righthanded phenomenon such as positive refraction. Due to rotational symmetry, both the righthanded phenomenon and lefthanded phenomenon satisfy the acoustic field equations. The stress field, the velocity field and the acoustic Poynting vector together form a righthanded triplet or a lefthanded triplet depending on the direction of energy flow or the direction of the Poynting vector. According to parity conservation, acoustic law at the deepest level, there is no differentiation of righthanded and lefthanded treatment. The performance of an object and that of its mirror image will satisfy the same law of physics. The negative refraction in fact is a mirror image of the positive refraction. The lefthanded triplet is a mirror image of the righthanded triplet. The lefthanded triplet gives rise to negative refraction. So from the consideration of parity conservation or mirror symmetry, both negative refraction and positive refraction satisfy the physical laws of acoustics.

$$\vec{T} = c\vec{S} \tag{28}$$

where $c =$ compliance.

Since compliance has rotational symmetry, it follows that $\vec{T}$ will also have rotational symmetry.

The acoustic Poynting vector is given as

$$\vec{P} = \frac{-\vec{V}^* \cdot \vec{T}}{2} \tag{29}$$

It follows that $\vec{P}$ will also have rotational symmetry.

Since both $\vec{T}$ and $\vec{P}$ are two components of the triplet it follows that the triplet has rotational symmetry.

4. CONCLUSION

Gauge invariance or gauge field theory is a more sophisticated treatment of acoustic fields than vector theory. It gives a more accurate solution for the velocity field and the stress field than those obtained from the inhomogeneous wave equations. The symmetry argument also explains the phenomenon of negative refraction for the anisotropic case. It also gives the difference between the strain field and the velocity field which previously all treat them as wave functions. It provides better understanding of the stress field and the velocity field.

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