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FREE VIBRATION OF A CANTILEVER BEAM WITH A PARTIAL SPAN OF DISTRIBUTED MASS

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ABSTRACT
A modified transfer matrix method is developed to obtain exact solution of natural frequencies and mode shapes of a cantilever beam loaded with distributed mass over an intermediate span. As verification, free vibration of a cantilever beam with concentrated mass is studied as a limiting case and the computational natural frequencies agree very well to the data available in the literature. Computational results are presented to demonstrate influence of added mass position and length on vibrating behavior of the mass-loaded cantilever beam.
As the mass position is fixed, the effect of mass length is found to be significant for natural frequencies of higher modes, implying that in this case error will appear if a distributed mass is modeled as a concentrated one. However, when the length of the distributed mass is rather small, say 1/1000 of the beam length, then it can be regarded as a concentrated one with good accuracy.

INTRODUCTION
Free vibration of non-uniform beams has been a subject of numerous investigations because of its relevance to aeronautical, mechanical and structural engineering. A special case of non-uniform beams is the beam loaded with mass. Such type of structure is common in engineering.
Vibration of the beam with concentrated mass or masses has been extensively studied by many authors using analytical or approximate approaches [1-7]. However, the loaded mass is usually distributed in engineering practice. A literature survey shows very few of

works on this vibration problem. Chan and Zhang [8] has studied vibration of a cantilever tube partially filled with liquid, which was modeled as a cantilever beam loaded with uniformly distributed mass.

Chan and Zhang's approach was conceptually a transfer matrix method. In present paper, their approach is extended to obtain exact solutions for a cantilever beam loaded with a partially distributed mass in an intermediate span. By shortening the span length, the partially distributed mass is in effect a concentrated one. As verification, free vibration of a cantilever beam with concentrated mass is studied using the developed approach, and the computational natural frequencies agree very well to the data in the open literature.

Computational results are presented to show some features of natural frequencies and mode shapes of the studied cantilever as function of the position of mass center. As the position of mass center is fixed, the effect of mass length on the natural frequencies is also demonstrated. It is found that such effect is significant for higher modes, implying that error will appear if a distributed mass is still modeled as a concentrated one in this case. Therefore, the present approach can be applied to a more general case of mass-loaded beams with good accuracy.

THEORY DERIVATION

In figure 1 is shown a partially loaded cantilever beam with the coordinate system, the origin located at the left end of the added mass span, which is l_d from the left end of the beam. The distributed mass has a length of l_a and its mass per unit length is m_a. The stiffness of the beam is *EI*, length *L* and mass per unit length *m*, all constant along the beam.

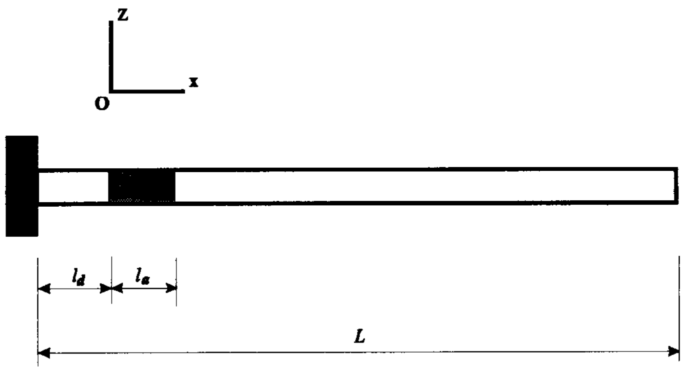

Figure 1. Diagram of a beam loaded with an intermediate section of distributed mass.

According to the Euler-Bernoulli beam theory, the equation of motion for free vibration of the mass-loaded beam can be written as,

$$EI\frac{\partial^4 W}{\partial x^4}+\left\{m+m_a\left[H(x)-H\left(x-l_a\right)\right]\right\}\frac{\partial^2 W}{\partial t^2}=0 \tag{1}$$

where W is transverse displacement of the beam, $H()$ is the Heaviside step function.

Divide the loaded beam as three fields at the two junctions, x=0 and $x=l_a$, each field representing an uniform Euler beam, then the vibrating displacement in each field can be expressed as,

$$W_i(x,t) = \left(A_i \sin k_i x + B_i \cos k_i x + C_i \sinh k_i x + D_i \cosh k_i x\right) e^{j\omega t} \tag{2}$$

where i is the number of field, i=1 for $-l_d \le x < 0$, i=2 for $0 < x < l_a$, and i=3 for $l_a < x \le L - l_d$.

The wave numbers in (2) should read,

$$k_1^4 = k_3^4 = \frac{m\omega^2}{EI} = k^4, \text{ and } k_2^4 = \frac{(m + m_a)\omega^2}{EI} \tag{3}$$

For later use it is convenient to define that,

$$\gamma = \frac{k_2}{k_1},\ \gamma_1 = \frac{1}{2}(1+\gamma^2),\ \gamma_2 = \frac{1}{2}(1-\gamma^2),\ \eta = \frac{1}{\gamma};\ \text{and } u_0 = k_1 L,\ u_1 = k_1 l_d,\ u_2 = k_2 l_a,\ u_3 = k_3 l_a \tag{4}$$

At the junctions, the state vectors can be written as,

$$z_j = \begin{bmatrix} W \\ \varphi \\ M \\ V \end{bmatrix}_j \tag{5}$$

where j is the junction number, j=1 for x=0 and j=2 for $x=l_a$. φ is the slope, M the bending moment, and V the shear force. By Euler-Bernoulli beam theory, $\varphi = {}^{\partial W}\!/_{\partial x}$, $M = EI\, {}^{\partial^2 W}\!/_{\partial^2 x}$, and $V = EI\, {}^{\partial^3 W}\!/_{\partial^3 x}$. Then substitute (2) into the above variables, the state vectors can be expressed as,

$$z_j^L = [T_j] \begin{pmatrix} A_j \\ B_j \\ C_j \\ D_j \end{pmatrix}, \text{ and } z_j^R = [T_{j+1}] \begin{pmatrix} A_{j+1} \\ B_{j+1} \\ C_{j+1} \\ D_{j+1} \end{pmatrix} \tag{6}$$

where z_j^L and z_j^R are the state vectors at the left and the right sides of junction j, respectively.

Due to the continuity and equilibrium conditions at the junctions, one has,

$$z_j^L = z_j^R,\ j\text{=1, 2} \tag{7}$$

written in matrix form as,

$$\begin{pmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{pmatrix} = [\Lambda] \begin{pmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{pmatrix} \text{ and } \begin{pmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{pmatrix} = [T_2]^{-1}[T_3] \begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix} \tag{8}$$

where

$$[\Lambda] = \begin{pmatrix} \gamma\gamma_1 & 0 & \gamma\gamma_2 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 \\ \gamma\gamma_2 & 0 & \gamma\gamma_1 & 0 \\ 0 & \gamma_2 & 0 & \gamma_1 \end{pmatrix}, \text{ and}$$

$$[T_2]=\begin{pmatrix} \sin u_2 & \cos u_2 & \sinh u_2 & \cosh u_2 \\ \cos u_2 & -\sin u_2 & \cosh u_2 & \sinh u_2 \\ -\sin u_2 & -\cos u_2 & \sinh u_2 & \cosh u_2 \\ -\cos u_2 & \sin u_2 & \cosh u_2 & \sinh u_2 \end{pmatrix},\ [T_3]=\begin{pmatrix} \sin u_3 & \cos u_3 & \sinh u_3 & \cosh u_3 \\ \eta\cos u_3 & -\eta\sin u_3 & \eta\cosh u_3 & \eta\sinh u_3 \\ -\eta^2\sin u_3 & -\eta^2\cos u_3 & \eta^2\sinh u_3 & \eta^2\cosh u_3 \\ -\eta^3\cos u_3 & \eta^3\sin u_3 & \eta^3\cosh u_3 & \eta^3\sinh u_3 \end{pmatrix} \quad (9)$$

Then one obtains,

$$\begin{pmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{pmatrix} = [\Lambda][T_2]^{-1}[T_3]\begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix} \quad (10)$$

The cantilever beam has the boundary conditions at both ends, written as,

$$W_1(-l_d,t)=W_1'(-l_d,t)=W_3''(L-l_d,t)=W_3'''(L-l_d,t)=0 \quad (11)$$

which yielding the following matrix equation,

$$[K_1]\begin{pmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{pmatrix}+[K_3]\begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix}=0 \quad (12)$$

where

$$[K_1]=\begin{pmatrix} -\sin u_1 & \cos u_1 & -\sinh u_1 & \cosh u_1 \\ \cos u_1 & \sin u_1 & \cosh u_1 & -\sinh u_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ [K_3]=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sin(u_0-u_1) & -\cos(u_0-u_1) & \sinh(u_0-u_1) & \cosh(u_0-u_1) \\ -\cos(u_0-u_1) & \sin(u_0-u_1) & \cosh(u_0-u_1) & \sinh(u_0-u_1) \end{pmatrix} \quad (13)$$

Substitute (10) into (12) yields,

$$\left\{[K_1][\Lambda][T_2]^{-1}[T_3]+[K_3]\right\}\begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix}=0 \quad (14)$$

This is a eigenvalue problem of free vibration of the partially loaded cantilever beam. Exact natural frequencies can be calculated by finding the roots of the following frequency equation,

$$\left|[K_1][\Lambda][T_2]^{-1}[T_3]+[K_3]\right|=0 \quad (15)$$

The corresponding eigenfactors give the coefficients in (2) which describe the mode shapes.

COMPUTATIONAL RESULTS

In figures 2a-2d are shown the frequency shift curves for the first four modes as function of the center position of the distributed mass. One can observe that as the distributed mass shifts from the clamped end to the free one, the fundamental natural frequency decreases while the natural frequencies of the other modes vary cyclically.

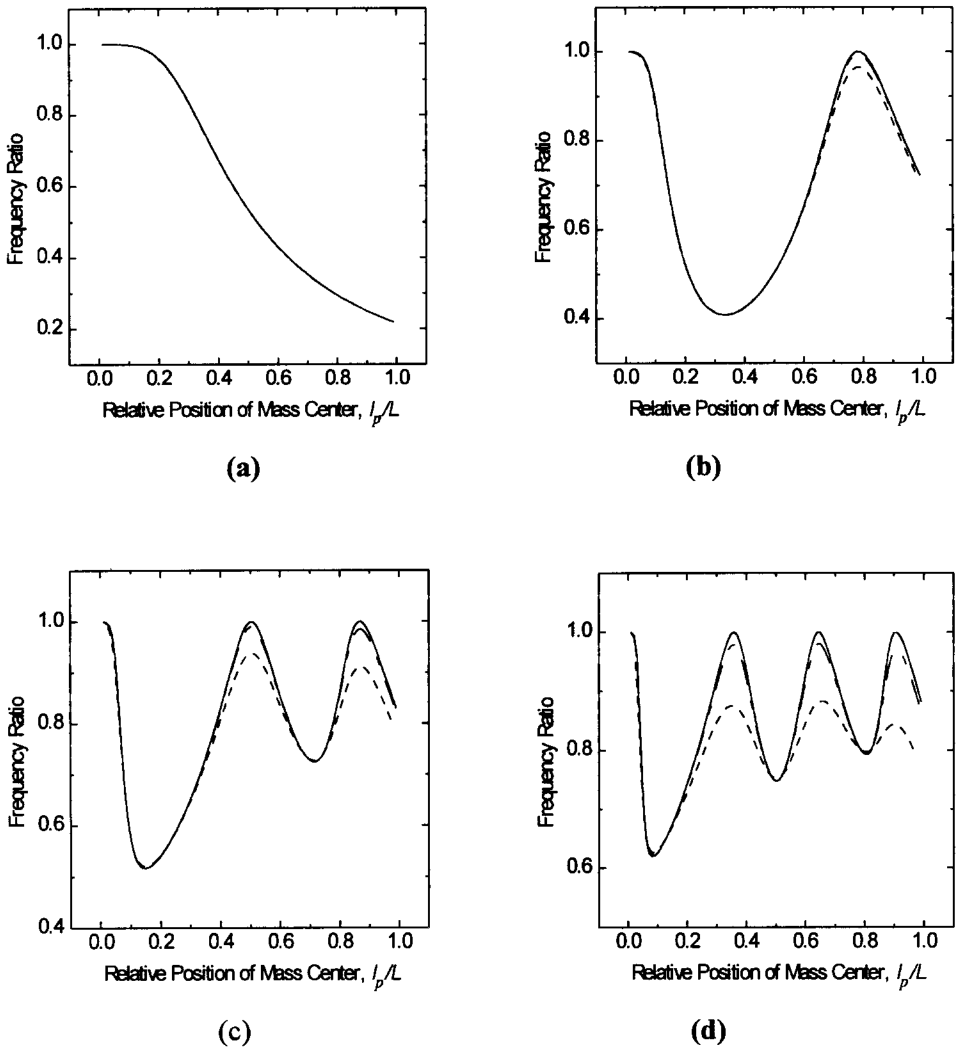

Figure 2. Frequency shift due to position of mass center. The frequency is rated by corresponding one of uniform beam. The ratio of added mass to the beam mass is 5. ———, l_a/L=0.0001; — — — ·, l_a/L=0.001; — · — · ·, l_a/L=0.02; · · · · ·, l_a/L=0.05. (a) Mode 1; (b) Mode 2; (c) Mode 3; (d) Mode 4.

The nodal points of corresponding mode shapes for the fourth mode, shown in figure 3, shift with the distributed mass in a 'swing' manner, and the amplitudes of mode shapes are suppressed at the location of the mass.

From figures 2a-2d, it can also be seen that when the length of the distributed mass is very small compared with the beam length, say no more than $0.001L$, the frequency shift curves are almost same, as shown in figure 2 for the curves of l_a/L=0.0001 and 0.001. However, when the mass length increases further, the frequency shift curves show significant changes.

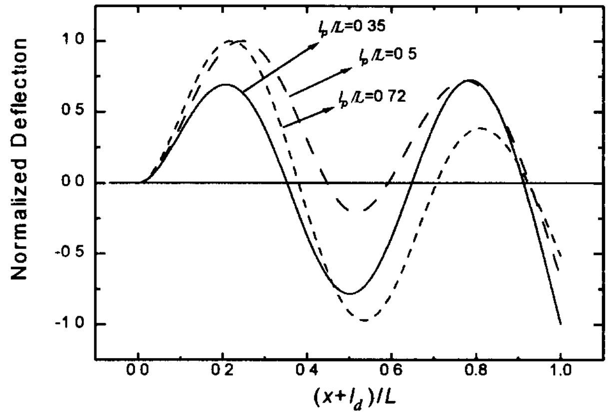

Figure 3. Mode shapes for the fourth mode with the added mass located at different positions. The ratio of added mass to the beam mass is 5.

When the length ratio of the distributed mass to the beam is very small, the mass can be treated as a concentrated one with good accuracy. In figure 4 is shown computational frequency shift curves, compared with the data reproduced from Goel [2]. The curves are computational results using the present method, where the mass length, l_a, is $0.001L$. The parameters used in computation are same as those in [2]. One can observe that the agreement is very good, and the validity of the developed method is therefore proved.

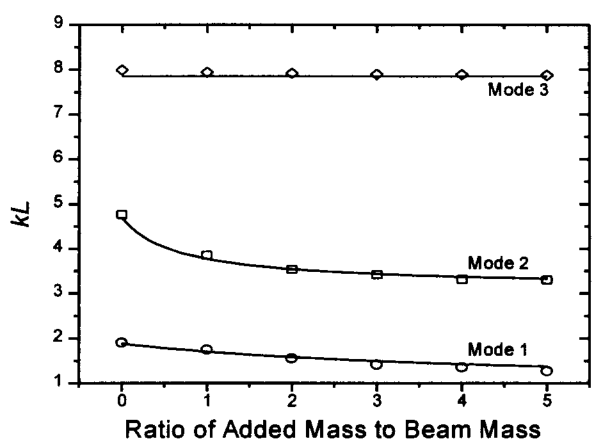

Figure 4. Frequency shift as function of the ratio of added mass to beam mass, compared with data given by Goel [2]. Curves are computational results using the present approach, and symbols are data reproduced from Goel [2].

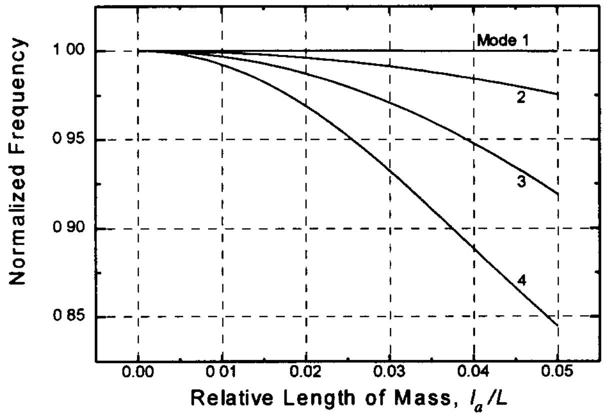

Figure 5. Frequency shift due to mass length when its position is fixed (l_p/L=0.9). The frequency is normalized by that of the mass-loaded cantilever with l_a/L=0.001. The ratio of added mass to the beam mass is 5.

When the mass length is relatively large compared with the beam length, however, errors will appear if a distributed mass is still modeled by a concentrated one, as already shown in figures 2a-2d. To demonstrate such errors more clearly, computational frequency shift curves are shown in figure 5, as function of mass length when the position of the mass center is fixed. One can see that the error is sometimes significant, for example, exceeding 5% for the third mode and 10% for the fourth mode with l_a/L=0.04, and even up to 15% for the fourth mode with l_a/L=0.05. Therefore, the present approach can be applied to more general case of vibration of beam loaded with mass.

CONCLUSIONS

In this paper, the transfer matrix method is modified to calculate exact natural frequencies and mode shapes of a cantilever beam with an intermediate section of distributed mass. The developed method is verified by comparing with the results in the literature. Some computational results are presented to show the effect of the mass position and length on vibrating behavior. The results also show that errors may be up to 10-15% if a distributed mass is always modeled as a concentrated one.

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