# Analysis of acoustic radiation of a ring-stiffened cylindrical shell in underwater based on precise integration transfer matrix method 

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#### Abstract

Based on transfer matrix theory and inhomogeneous precise integration method, precise integration transfer matrix method (PITMM) is developed to study the dynamic response of stiffened cylindrical shell. Firstly, field transfer matrix and point transfer matrix for stiffened cylindrical shell are obtained based on Flügge theory, transfer matrix theory and Helmholtz equation. The effect of generalized acoustic pressure excitation is handled with the inhomogeneous precise integration method and increment-dimensional storage methods. According to the boundary conditions for both ends of cylindrical shell and motion continuity condition on the fluid-solid interaction, the coefficient of acoustic pressure is calculated. Thus, acoustic radiation of the stiffened cylindrical shell is determined. The accuracy of the proposed approach has been demonstrated by comparing the current results with those in the literature and with some experimental results. On this basis, the effects of boundary conditions, loss factors of structures and fluid medium on the acoustic radiation of stiffened cylindrical shell are discussed. It is shown that, the acoustic radiation pressure of structure is largest for free boundary condition, and becomes smaller in that order for simply supported and clamped boundary condition. With an increase of loss factors of structures, the acoustic radiation pressure monotonically decreases. However, the radiation pressure increases with an increase of fluid medium impedance.


Keywords: Non-homogeneous terms; Precision integration transfer matrix; Acoustic radiation. R3

## 1. INTRODUCTION

In the engineering applications, especially in the field of modern military defense, stiffened cylindrical shell is basically a simplified model of many weapons and equipments, such as: torpedoes, missiles, submarines and so on. So the analysis acoustic coupling mechanism of stiffened cylindrical shell, has been a research hotspot among domestic and foreign scholars. In the numerical aspects, FEM,BEM,SEA have been widely application. Theoretically, numerical methods can handle any complex structure. But the accuracy of solution is limited by computing frequency scope. And numerical methods are difficult to analyze the mechanism, All these restrict the development of finite element technique. While the traditional analytical methods can only give a simple structure's analytical solution. Therefore, semi-analytical method is now becoming an effective method in analyzing stiffened cylindrical shell acoustic radiation. For many of the same types of units ordered by the combination of a similar chain of shell structures, the transfer matrix method is relatively simple method to solve dynamic problems.

Initially H. Tottenham and K. Shimizu ${ }^{[1]}$ first put forward the transfer function method for solving the free vibration of cylindrical shell. Later T. Irie ${ }^{[2]}$, who conducted further application,form a more perfect transfer matrix method,the method is successfully applied to free vibration of the structure such as the axial cylindrical shells with discrete spring-loaded, cone shell variable thickness and taper - cylinder shell etc. However, these studies are focused on studying characteristics of structural free

[^0]vibration. None of these studies is related to the response of structural vibration and underwater acoustic radiation problems. CAI Xian xin ${ }^{[3]}$ who absorbed the thought of T.Irie expands the state vector of rotary shell into the form of series in the circumferential direction by using the model of rotary shell structure.He derives first-order vibration differential equation of rotate the shell structure, and gives a kinds of semi-analytical solution for solving the free vibration of rotating shell. However, the accuracy of this method is low.And it requires relatively slight segments. CAO Lei ${ }^{[4]}$ who uses Riccati transfer matrix method to analyze the ring stiffened cylindrical shell's performance of underwater acoustic radiation. But he uses a piecewise interpolation theory and polynomial approximation to approach non-homogeneous terms in calculating the non-homogeneous terms of the first-order non-homogeneous matrix differential equations. the problems such as inaccurate values and loss of accuracy can't be avoided.

In connection with the above shortcomings, this paper proposes the precise integration transfer matrix method. It is mainly used to solve the problem of structural vibration response and underwater acoustic radiation. This paper uses a more accurate approach when deal with the non-homogeneous terms of first-order non-homogeneous matrix differential equations .This approach is academician ZHONG Wanxie's inhomogeneous terms precise integration method ${ }^{[5]}$.It divides fine segment of cylindrical shell into precise integral step. Based on Taylor series expansion and the addition theorem in precise integral step, highly accurate results of inhomogeneous terms in fine segment are obtained through the recycling processing, numerical calculation precision is higher and more accurate. This method will be discussed amply in paragraph 2.3.

## 2. THEORETICAL DERIVATION

### 2.1 Cylindrical shell segment field transfer matrix

Model of this dissertation is finite stiffened cylindrical shell immersed in an infinite flow field medium. The assumption is that both ends of the cylindrical shell are infinite rigid baffles. The model is shown in Fig.1.

For obtaining precise values, we expound the shell deformation by thin shell theory which is based on linear assumptions. In order to obtain precise values, relatively accurate Flügge shell theory ${ }^{[6]}$ is used in this chapter. The force balance equation is obtained by analyzing cylindrical shell micro-element stress. In this paper equations are based on the kinetic theory, so many terms include time items with the purpose of facilitate the writing and derivation. Dynamic response time items $e^{-i \omega t}$ is omitted in writing hereinafter. Cylindrical shell coordinates system $(r, \varphi, x)$ and displacement positive direction are shown in Fig. 2 .


Figure 1. Model of stiffened cylindrical shell


Figure 2. Cylindrical shell coordinates system

On the ground of Flügge shell theory, force balance equation of cylindrical shell is given:

$$
\begin{gathered}
\frac{\partial N_{x}}{\partial x}+\frac{1}{R} \frac{\partial N_{\varphi x}}{\partial \varphi}+\rho h \omega^{2} u=0 \quad, \frac{1}{R} \frac{\partial N_{\phi}}{\partial \varphi}+\frac{\partial N_{x \varphi}}{\partial x}-\frac{Q_{\varphi}}{R}+\rho h \omega^{2} w=0, \frac{N_{\varphi}}{R}+\frac{\partial Q_{x}}{\partial x}+\frac{1}{R} \frac{\partial Q_{\varphi}}{\partial \varphi}-\rho h \omega^{2} w=0 \\
Q_{\varphi}=\frac{1}{R} \frac{\partial M_{\varphi}}{\partial \varphi}+\frac{\partial M_{x \varphi}}{\partial x}, \quad Q_{x}=\frac{\partial M_{x}}{\partial x}+\frac{1}{R} \frac{\partial M_{\varphi x}}{\partial \varphi}
\end{gathered}
$$

Reference[3]. By eliminating eight unknown quantities and retaining eight unknown quantities. All quantities become dimensionless quantities and expand to trigonometric function form along the circumferential direction. Then, first-order matrix differential equation is obtained through complicated simplifying :

$$
\begin{equation*}
\frac{d\{\boldsymbol{Z}(\xi)\}}{d \xi}=\mathbf{U}(\xi)\{\boldsymbol{Z}(\xi)\}+\{\boldsymbol{F}(\xi)\}-\{\boldsymbol{p}(\xi)\} \tag{1}
\end{equation*}
$$

The state vector of cylindrical shell is: $\boldsymbol{Z}(\xi)=\left\{\begin{array}{llllllll}\bar{u} & \bar{v} & \bar{w} & \bar{\psi} & \overline{M_{x}} & \bar{v}_{x} & \bar{S}_{x \varphi} & \overline{N_{x}}\end{array}\right\}^{\mathrm{T}} \cdot(\bar{u}, \bar{v}, \bar{w})$ are dimensionless quantities of axial direction( $x$ direction), circumferential direction( $\phi$ direction) and normal displacement $(\gamma$ direction $) \cdot \bar{\psi}$ is a dimensionless angle, $\overline{N_{x}}$ is a dimensionless film force, $\overline{M_{x}}$ is a dimensionless moment, $\left(\overline{V_{x}}, \quad \overline{S_{x}}\right)$ are dimensionless Kelvin-Kirchhoff shear force and shear force; $\boldsymbol{Z}(\xi)$ is the shell element's state vector and also a function of dimensionless variables $\xi \cdot \mathbf{U}(\xi)$ as the differential coefficient is field transfer matrix of cylindrical shell structure's state vector, and is an eight-order square matrix. There are 22 non-zero elements in $\mathbf{U}(\xi)$, see Appendix A..

### 2.2 The point transfer matrix of ring ribs

Due to the effect of the ring rib, cylindrical shell state vector changes in the ring rib place. Using continuous condition connection cylindrical shell with the ring rib and rib motion control equation (tensile and bending vibrations within the plane, the bending and torsional vibrations out of the plane), it is easy to obtain the ring rib point transfer matrix $\boldsymbol{R}_{k}$.A ring rib locates at cylindrical shells $\xi_{k}$, due to the presence of ring ribs, two faces internal and two faces external forces of stiffening rings left and right end change, and state vector of left and right end satisfy the following equation:

$$
\boldsymbol{Z}\left(\xi_{k}^{R}\right)=\boldsymbol{R}_{k} \boldsymbol{Z}\left(\xi_{k}^{L}\right)
$$

$\boldsymbol{R}_{k}$ is an $8 \times 8$ point transfer matrix whose nonzero matrix elements are:

$$
R_{i i}=1, i=1,2,3, \ldots 8 .
$$

$$
\begin{gathered}
R_{51}=-\frac{R}{K}\left\{\left[\frac{E I_{2}}{R_{b}^{3}} n^{2}+\frac{G J}{R_{b}^{3}} n^{2}\right] h+e h\left[\frac{E I_{2}}{R_{b}^{4}} n^{4}+\frac{G J}{R_{b}^{4}} n^{2}-\rho A \omega^{2}\right]\right\} \\
R_{54}=-\frac{R}{K}\left\{\left[\frac{E I_{2}}{R_{b}^{3}} R_{b}-\frac{E I_{2}}{R_{b}^{3}} e n^{2}-\frac{G J}{R_{b}^{3}} e n^{2}+\frac{G J}{R_{b}^{3}} R_{b} n^{2}-\rho I_{p} \omega^{2}\right] \frac{h}{R}+e \frac{h}{R}\left[-\frac{E I_{2}}{R_{b}^{4}} e n^{4}+\frac{E I_{2}}{R_{b}^{4}} R_{1} n^{2}-\frac{G J}{R_{b}^{4}} e n^{2}+\frac{G J}{R_{b}^{4}} R_{1} n^{2}-\rho A \omega^{2} e\right]\right\} \\
R_{62}=-h \frac{R^{2}}{K}\left[\frac{E I_{1} R_{b}}{R_{b}^{4}} \frac{b}{R} n^{3}+\frac{E A R_{b}}{R_{b}^{2}}\right], \quad R_{63}=-h \frac{R^{2}}{K}\left[\frac{E I_{1}}{R_{b}^{4}} \frac{R_{b}}{R} n^{4}+\frac{E A}{R_{b}^{2}}\left(1+\frac{e}{R} n^{2}\right)-\rho A \omega^{2}\right] \\
R_{72}=-h \frac{R^{2}}{K}\left[\frac{E I_{1}}{R_{b}^{4}} \frac{R_{b}}{R} n^{2}+\frac{E A}{R_{b}^{2}} \frac{R_{b}}{R} n^{2}-\rho A \omega^{2} \frac{R_{b}}{R}\right], R_{73}=-h \frac{R^{2}}{K}\left[\frac{E I_{1}}{R_{b}^{4}} \frac{R_{b}}{R} n^{3}+\frac{E A}{R_{b}^{2}} n \frac{R-e}{R}-\rho A \omega^{2} n \frac{e}{R}\right] \\
R_{81}=-h \frac{R^{2}}{K}\left[\frac{E I_{2}}{R_{b}^{4}} n^{4}+\frac{G J}{R_{b}^{4}} n^{2}-\rho A \omega^{2}\right], R_{84}=\frac{h}{R} \frac{R^{2}}{K}\left[-\frac{E I_{2}}{R_{b}^{4}} e n^{4}+\frac{E I_{2}}{R_{b}^{4}} R_{1} n^{2}-\frac{G J}{R_{b}^{4}} e n^{2}+\frac{G J}{R_{b}^{4}} R_{1} n^{2}-\rho A \omega^{2} e\right]
\end{gathered}
$$

Where $R_{b}, ~ I_{1}, ~ I_{2}, ~ I_{P^{`}} J$ respectively represents the radius of the ring ribs at shaft, the inertia moment about longitudinal symmetry axis, the inertia moment about radial symmetry axis, the polar inertia moment and torsional moment; $A, ~ \rho, ~ G$ respectively is ring rib cross-sectional area, density, shear modulus. $R_{b}=R+e$, For eccentricity $e, e$ take positive with outside rib.

### 2.3 Inhomogeneous terms use of precise integration for solving dynamic response

For non-homogeneous linear differential Eq.(1), the general solution is:

$$
\begin{equation*}
\boldsymbol{Z}(\xi)=e^{\boldsymbol{U} \Delta \xi} \boldsymbol{Z}\left(\xi_{0}\right)+\int_{\xi_{0}}^{\xi} e^{\boldsymbol{U}(\xi-\tau)} \boldsymbol{r}(\tau) d \tau \tag{2}
\end{equation*}
$$

Where $\boldsymbol{r}(\tau)=\{\boldsymbol{F}(\xi)\}-\{\boldsymbol{p}(\xi)\}$, the second term at equation's right is the state response caused by a non-homogeneous terms. Solving the above equation is the first step to solve ring-stiffened cylindrical shell acoustic radiation meanwhile it is a very important step. For Solving $e^{\boldsymbol{U} \Delta \xi}$, readers can refer to academician ZhONG Wanxie index matrix precise integration method[7]. It will not be mentioned here. If the non-homogeneous terms are calculated, state vector $\boldsymbol{Z}(\xi)$ can be calculated.

Helmholtz equation in drain field is expressed as:

$$
\begin{equation*}
\nabla^{2} p+k_{0}^{2} p=0 \tag{3}
\end{equation*}
$$

The acoustic pressure also satisfies the boundary conditions at infinity distance:

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \sqrt{r}\left(\frac{\partial p}{\partial r}+i k_{0} p\right)=0 \quad \text { (4), }\left.\quad \frac{\partial p}{\partial x}\right|_{x=0}=\left.\frac{\partial p}{\partial x}\right|_{x=L}=0 \tag{5}
\end{equation*}
$$

Helmholtz equation can be solved by variable separation approach, considering the Eqs.(4), (5) and other boundary conditions, the acoustic radiation pressure of the surrounding fluid domain can be expressed as:

$$
\begin{equation*}
p=\cos \left(k_{m} x\right) \sin \sum_{\alpha=0}^{1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_{m n} H_{n}^{(1)}\left(k_{r} r\right)\left(n \theta+\frac{\alpha \pi}{2}\right) \tag{6}
\end{equation*}
$$

Where $k_{m}=(m \pi / L), m=0,1, \ldots N, k_{0}=\omega / c_{0} \quad, c_{0}=1500 \mathrm{~m} / \mathrm{s}$ is speed of acoustic in water.
When the argument $x$ is a real number: $H_{n}^{(1)}(x)=J_{n}(x)+i Y_{n}(x)$
When the argument $x$ is imaginary: $H_{n}^{(1)}(i x)=(-i)^{n+1} \frac{2}{\pi} K_{n}(x)$
Corresponding to wavenumber $(m, n)$,the monomial generalized acoustic pressure is expressed as:

$$
r(\tau)=H_{n}^{(1)}\left(k_{r} R\right) \cos k_{n}
$$

It is taken into eq.(2),response terms can be obtained:

$$
\begin{equation*}
\int_{0}^{\Delta \xi} e^{\boldsymbol{U} \tau} \boldsymbol{r}\left(\xi_{k+1}-\tau\right) d \tau=\boldsymbol{e} \cos \left(k_{m} l \xi_{k+1}\right) \int_{0}^{\Delta \xi} e^{\boldsymbol{U} \tau} \cos \left(k_{m} l \tau\right) d \tau+\boldsymbol{e} \sin \left(k_{m} l \xi_{k+1}\right) \int_{0}^{\Delta \xi} e^{\boldsymbol{U} \tau} \sin \left(k_{m} l \tau\right) d \tau \tag{7}
\end{equation*}
$$

Where $\boldsymbol{e}$ is known item, $\boldsymbol{e}=H_{n}^{(1)}\left(k_{r} R\right)\left[\begin{array}{llllllll}0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\end{array}\right]^{T}$, assuming $\alpha=\mathrm{i} k_{m} l$,

$$
\begin{aligned}
& \boldsymbol{p} \boldsymbol{p}=\boldsymbol{e} \cos \left(k_{m} l \xi_{k+1}\right), \boldsymbol{q q}=\boldsymbol{e} \sin \left(k_{m} l \xi_{k+1}\right) \text {, Then: } \\
& \int_{0}^{\Delta \xi} e^{U \tau} \boldsymbol{r}\left(\xi_{k+1}-\tau\right) d \tau=\operatorname{Re}\left(\int_{0}^{\Delta \xi} e^{U \tau} e^{\alpha \tau} d \tau\right) \boldsymbol{p} \boldsymbol{p}+\operatorname{Im}\left(\int_{0}^{\Delta \xi} e^{\boldsymbol{U \tau}} e^{\alpha \tau} d \tau\right) \boldsymbol{q} \boldsymbol{q}
\end{aligned}
$$

There is only one unknown quantity $\int_{0}^{\Delta \xi} e^{U \tau} e^{\alpha \tau} d \tau$ in the equation.
Where $\boldsymbol{E}(\Delta \xi)=\int_{0}^{\Delta \xi} e^{\boldsymbol{U} \tau} e^{\alpha \tau} d \tau, \quad \varphi_{0}(l)=e^{\alpha l}$ By the addition theorem available:

$$
\begin{equation*}
\boldsymbol{E}(2 \Delta \xi)=\int_{0}^{2 \Delta \xi} e^{\boldsymbol{U} \tau} e^{\alpha \tau} d \tau=\int_{0}^{\Delta \xi} e^{\boldsymbol{U} \tau} e^{\alpha \tau} d \tau+\int_{\Delta \xi}^{2 \Delta \xi} e^{\boldsymbol{U} \tau} e^{\alpha \tau} d \tau=\int_{0}^{\Delta \xi} e^{\boldsymbol{U} \tau} e^{\alpha \tau} d \tau+e^{\boldsymbol{U \Delta \xi}} e^{\partial \Delta \xi} \int_{0}^{\Delta \xi} e^{\boldsymbol{U} \tau} e^{\alpha \tau} d \tau \tag{8}
\end{equation*}
$$

The same available:

$$
\Phi_{0}(2 \Delta \xi)=\left(\Phi_{0}(\Delta \xi)\right)^{2}, \quad \varphi_{0}(2 \Delta \xi)=\left(\varphi_{0}(\Delta \xi)\right)^{2}
$$

When the fine segment $\Delta \xi$ is divided into precise integral step $\tau\left(\tau=\Delta \xi / 2^{M}\right), M$ is recommended to take 20. $\boldsymbol{E}(\tau), \Phi_{0}(\tau), \varphi_{0}(\tau)$ can be expanded by Taylor series:

$$
\begin{gather*}
\boldsymbol{E}(\tau)=\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \int_{0}^{\Delta \xi} \frac{\boldsymbol{U}^{k} \alpha^{m} t^{k+m}}{k!m!} d t=\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\boldsymbol{U}^{k} \alpha^{m} \tau^{k+m+1}}{k!m!(k+m+1)}  \tag{9}\\
\boldsymbol{\Phi}_{0}(\tau)=e^{\boldsymbol{U} \tau}=\sum_{k=0}^{\infty} \frac{A^{k} \tau^{k}}{k!} \approx \boldsymbol{I}_{8}+\boldsymbol{\Phi}_{0}^{\prime}(\tau)(10), \quad \varphi_{0}(\tau)=e^{\partial \tau}=\sum_{k=0}^{\infty} \frac{\partial^{k} \tau^{k}}{k!} \approx 1+\varphi_{0}^{\prime}(\tau) \tag{11}
\end{gather*}
$$

If using addition theorem directly based on Eqs.(10),(11), it will lead to loss of precision. On account of that $\varphi_{0}^{\prime}(\tau), ~ \boldsymbol{\Phi}_{0}^{\prime}(\tau)$ is trace relative 1 and $\boldsymbol{I}_{8}$. Mantissa will appear error because of computer rounding errors, then, lead to loss of precision. Therefore, they should be rewritten to increase the micro- increments $\Phi_{0}^{\prime}(\tau)$ and $\varphi_{0}^{\prime}(\tau)$ :

$$
\begin{equation*}
\Phi_{0}^{\prime}(2 \tau)=2 \Phi_{0}^{\prime}(\tau)+\Phi_{0}^{\prime}(\tau) \Phi_{0}^{\prime}(\tau)(12), \quad \varphi_{0}^{\prime}(2 \tau)=2 \varphi_{0}^{\prime}(\tau)+\varphi_{0}^{\prime}(\tau) \varphi_{0}^{\prime}(\tau \tag{13}
\end{equation*}
$$

According to Eq. $(7)$,obtaining: $\boldsymbol{E}(2 \tau)=\boldsymbol{E}(\tau)+\left[1+\varphi_{0}^{\prime}(\tau)\right]\left[\boldsymbol{I}_{8}+\boldsymbol{\Phi}_{0}^{\prime}(\tau)\right] \boldsymbol{E}(\tau)$
Where $\boldsymbol{E}_{1}$ is real component of $\boldsymbol{E}(\tau)$

$$
\begin{equation*}
\boldsymbol{E}_{1}=\boldsymbol{E}_{1}+\operatorname{real}\left[1+\varphi_{0}^{\prime}(\tau)\right]\left[\boldsymbol{I}_{8}+\boldsymbol{\Phi}_{0}^{\prime}(\tau)\right] \boldsymbol{E}_{1}-\operatorname{imag}\left[1+\varphi_{0}^{\prime}(\tau)\right]-\left[\boldsymbol{I}_{8}+\boldsymbol{\Phi}_{0}^{\prime}(\tau)\right] \boldsymbol{E}_{2} \tag{14}
\end{equation*}
$$

Where $\boldsymbol{E}_{2}$ is imaginary part of $\boldsymbol{E}(\tau)$

$$
\begin{equation*}
\boldsymbol{E}_{2}=\boldsymbol{E}_{2}+\operatorname{imag}\left[1+\varphi_{0}^{\prime}(\tau)\right]\left[\boldsymbol{I}_{8}+\boldsymbol{\Phi}_{0}^{\prime}(\tau)\right] \boldsymbol{E}_{1}+\operatorname{real}\left[1+\varphi_{0}^{\prime}(\tau)\right]\left[\boldsymbol{I}_{8}+\boldsymbol{\Phi}_{0}^{\prime}(\tau)\right] \boldsymbol{E}_{2} \tag{15}
\end{equation*}
$$

Then run the program according to the following algorithm:

$$
\text { For (iter }=0 \text {; iter }<\mathrm{M} \text {; iter++) }
$$

Run eqs (12),(13), (14),(15).
Obtaining $\boldsymbol{\Phi}_{0}^{\prime}(\Delta \xi), \varphi_{0}^{\prime}(\Delta \xi), \boldsymbol{E}_{1}, \boldsymbol{E}_{2}$ in the fine segment $\Delta \xi$ after $M$ assignment.

$$
\boldsymbol{\Phi}_{0}(\Delta \xi)=\boldsymbol{I}_{8}+\boldsymbol{\Phi}_{0}^{\prime}(\Delta \xi), \varphi_{0}(\Delta \xi)=1+\varphi_{0}^{\prime}(\Delta \xi), \boldsymbol{E}_{1}=\operatorname{Re}\left(\int_{0}^{\Delta \xi} e^{\boldsymbol{U \tau}} e^{\alpha \tau} d \tau\right), \boldsymbol{E}_{2}=\operatorname{Im}\left(\int_{0}^{\Delta \xi} e^{\boldsymbol{U \tau}} e^{\alpha \tau} d \tau\right)
$$

Inhomogeneous terms $\int_{0}^{\Delta \xi} e^{\boldsymbol{U \tau}} \boldsymbol{r}\left(\xi_{k+1}-\tau\right) d \tau$ can be expressed as:

$$
\int_{0}^{\Delta \xi} e^{\boldsymbol{U} \tau} \boldsymbol{r}\left(\xi_{k+1}-\tau\right) d \tau=\boldsymbol{E}_{1} \boldsymbol{p} \boldsymbol{p}+\boldsymbol{E}_{2} \boldsymbol{q} \boldsymbol{q}
$$

So far, we have obtained precise results of non-homogeneous terms by using the addition theorem in precise integral step. Meanwhile the Eq.(2) under the effect of acoustic pressure can be written in matrix form:

$$
\left[\begin{array}{c}
\boldsymbol{Z}\left(\xi_{k+1}\right)  \tag{16}\\
1
\end{array}\right]=\left[\begin{array}{cc}
\boldsymbol{\Phi}_{0}(\Delta \xi) & \operatorname{Re}(\boldsymbol{E}(\Delta \xi)) \boldsymbol{p}+\operatorname{Im}(\boldsymbol{E}(\Delta \xi)) \boldsymbol{q} \boldsymbol{q} \\
0 & 1
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{Z}\left(\xi_{k}\right) \\
1
\end{array}\right]=\boldsymbol{M}_{k}\left[\begin{array}{c}
\boldsymbol{Z}\left(\xi_{k}\right) \\
1
\end{array}\right]
$$

Concentrated force's impacts on cylindrical shell are similar to ring rib's. It only changes the state vector where concentrated force impacts, the left and right ends of concentrated force satisfy the following equation:

$$
\left[\begin{array}{c}
\boldsymbol{Z}\left(\xi_{k}^{R}\right)  \tag{17}\\
1
\end{array}\right]=\left[\begin{array}{cc}
\boldsymbol{I} & \boldsymbol{F}_{k} \\
0 & 1
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{Z}\left(\xi_{k}^{L}\right) \\
1
\end{array}\right]=\boldsymbol{I} \boldsymbol{F}_{k}\left[\begin{array}{c}
\boldsymbol{Z}\left(\xi_{k}^{L}\right) \\
1
\end{array}\right]
$$

$\boldsymbol{F}_{k}=R K^{-1}\left[\begin{array}{llllllll}0 & 0 & 0 & 0 & 0 & f_{n} & 0 & 0\end{array}\right]^{T}, \quad \boldsymbol{I} \boldsymbol{F}_{k}$ represents concentration point transfer matrix.

### 2.4 The Solution of Stiffened cylindrical shell's acoustic radiation

### 2.4.1 State vector under concentrated force

For the structural segment $\xi_{k} \sim \xi_{k+1}$, the left state vector $\boldsymbol{Z}\left(\xi_{k}\right)$ transmits to $\boldsymbol{Z}\left(\xi_{k+1}\right)$ through field transfer matrix $\Phi_{0}(l)$. The concentration or stiffener is present at a section ${ }_{k+1}$. The left state vector $\boldsymbol{Z}\left(\xi_{k+1}^{L}\right)$ transmits to the right state vector $\boldsymbol{Z}\left(\xi_{k+1}^{R}\right)$ through $\boldsymbol{I} \boldsymbol{F}_{k+1}$ or $\boldsymbol{R}_{k+1}$.Repeating the above process, then the relationship of the left state vector $\boldsymbol{Z}\left(\xi_{0}^{L}\right)$ and the right state vector $\boldsymbol{Z}\left(\xi_{N}^{R}\right)$ at end of ring stiffened cylindrical shell satisfy:

$$
\boldsymbol{Z}\left(\xi_{N}^{R}\right)=\prod_{k=1}^{N} \boldsymbol{C}_{k P} \boldsymbol{C}_{k F} \boldsymbol{Z}\left(\xi_{0}^{L}\right)
$$

According to the boundary conditions at both ends of the cylindrical shell, the state vector of both ends can be calculated. Then substituted into the above equation, the state vector at the nodes of any segment can be obtained. We obtain the radial displacement $w_{f}^{n}(x)$ under each circumferential wave number $n$ and the concentrated force eventually.

### 2.4.2 State vector under concentrated force

For monomial acoustic pressure $H_{n}^{(1)}\left(k_{r} R\right) \cos \left(k_{m} x\right)$ under the wavenumber ( $m, n$ ), the state vector under monomial acoustic pressure can be solved by using PITMM. In this way, field transfer matrix is obtained. Using the above method, the total transfer matrix can be obtained:

$$
\boldsymbol{Z}\left(\xi_{N}^{R}\right)=\prod_{k=1}^{N} \mathbf{C}_{k P} \boldsymbol{M}_{k} \boldsymbol{Z}\left(\xi_{0}^{L}\right)
$$

Similarly, considering the boundary conditions at both ends, we can obtain the radial displacement $w_{m}^{n}(x)$ under the corresponding $(m, n)$ order wave number and the acoustic pressure by using PITMM.

### 2.4.3 The solution of acoustic radiation

According to the linear superposition principle, the radial displacement which corresponds to each circumferential wavenumber $n$ satisfies:

$$
\begin{equation*}
w^{n}(x)=w_{f}^{n}(x)+\sum_{m=0}^{\infty} p_{m n} w_{m}^{n}(x) \tag{18}
\end{equation*}
$$

As $p_{m n}$ is unknown, we must firstly solve the $p_{m n}$ for getting $w^{n}(x)$.

According to the continuity conditions on the contact surface, the radial velocity of the fluid must equal to the radial velocity of structure:

$$
\frac{1}{i \omega \rho} \frac{\partial p}{\partial r}=\left.\frac{\partial w}{\partial t}\right|_{r=R}
$$

$p_{m n}$ is solved by the Moore-Penrose generalized inverse method, detailed process:
Define: $\quad\left[U_{j m}\right]_{n}=\frac{\partial H_{n}^{(1)}\left(k_{r} R\right)}{\partial r} \cos \left(k_{m} x_{j}\right)-\rho_{0} \omega^{2} w_{m}^{n}\left(x_{j}\right), \quad\left\{p_{m 1}\right\}_{n}=p_{m n}, \quad\left\{Q_{j}\right\}_{n}=\rho_{0} \omega^{2} w_{f}^{n}\left(x_{j}\right)$
Because any point on the structure satisfies interface continuity conditions, therefore we can select $M$ points in the longitudinal direction of the structure. $M$ is greater than the axial wave number m , meanwhile it also satisfies $M=2 \pi \lambda^{-1}$. Taking $M$ points into Eq.(19), and then bring Eqs.(6) (18) into Eq.(19). Then it can be transformed into a equation which can solve the acoustic pressure coefficients under each wave number corresponding:

$$
\sum_{m=0}^{\infty} p_{m n}\left(\frac{\partial H_{n}^{(1)}\left(k_{r} R\right)}{\partial r} \cos \left(k_{m} x_{j}\right)-\rho_{0} \omega^{2} w_{m}^{n}\left(x_{j}\right)\right)=\rho_{0} \omega^{2} w_{f}^{n}\left(x_{j}\right), \quad j=1,2,3 . . M
$$

Obviously there is only one unknown column vector $\left\{p_{m 1}\right\}_{n}$, so it can be solved.

$$
\begin{equation*}
[U]_{n}\{p\}_{n}=\{Q\}_{n} \tag{20}
\end{equation*}
$$

Dealing with $[U]$ by the singular value decomposition, we obtain: $[U]=[A][D][V]^{T}$.
Dealing with $[U]$ by Moore-Penrose inverse, we obtain: $[U]^{-1}=[V][D]^{-1}[A]^{T}$.
Substitution it into Eq.(20), we obtain : $\{p\}_{n}=[V][D]^{-1}[A]^{T}{ }_{n}\{Q\}_{n}$
The radiated acoustic pressure in the flow field can be obtained by Substitution coefficient matrix $\{p\}$ into Eq.(6).According to radiated acoustic pressure, We have obtained radiation corresponding SPL: $L_{p}=20 \lg \left(p / p_{0}\right)$.Reference acoustic pressure is: $p_{0}=1 \mu P a$. The radial displacement of each point on the cylindrical shell is:

$$
w(x, \theta)=\sum_{\partial} \sum_{m} \sum_{n} W_{m n} \cos \left(k_{m} x\right) \sin \left(n \theta+\frac{\partial \pi}{2}\right)
$$

The radial quadratic velocity on the surface of structure satisfies:

$$
\dot{w}(x, \theta) \dot{w}^{*}(x, \theta)=\frac{1}{4} \sum_{m} \sum_{n} \dot{W}_{m n} \dot{W}_{m n}^{*} \frac{1}{\varepsilon_{n}}
$$

Velocity level is: $L_{v}=10 \lg \left(\left\langle\dot{w} \dot{w}^{*}\right\rangle / \dot{w}_{o}^{2}\right)$,Velocity level basis: $\dot{w}_{0}=5 \times 10^{-8}(\mathrm{~m} / \mathrm{s})$

## 3. NUMERICAL RESULTS

### 3.1 Introduction about verified model

Compared with data of reference [8], [9], [4],to Verify the accuracy of the method PITMM.the results are shown in Table 1, Table 2.

The parameter in reference[8]:young's modulus of the material $E=2.06 \times 10^{11} \mathrm{~Pa}$, Poisson's ratio $\mu=0.3$, Fluid density $\rho=7800 \mathrm{~kg} / \mathrm{m}^{3}$, Loss factor $\eta=0.01$. Acoustic wave propagation velocity $c_{0}=1500 \mathrm{~m} / s$.Cylindrical shell geometry parameter, Length $L=0.6 \mathrm{~m}$, radius $R=0.2 \mathrm{~m}$, Thickness $h=0.003 \mathrm{~m}$, Inner rib geometry parameter $0.002 \times 0.003 \mathrm{~m}$, the number of inner rib is 9 . Radial excitation force acts on cylindrical shell ( $L / 2,0, R$ ), force amplitude 1 N . Measurement points ( $L / 2, \theta, 2 \mathrm{~m}$ ).


Figure 3.Model test schematic diagram shown in Fig. 3.

The parameter in reference [9]:Parameter of material and fluid is same as reference [8]. Cylindrical
shell geometry parameter, length $L=0.6 \mathrm{~m}$, radius $R=0.175 \mathrm{~m}$, thickness $h=0.002 \mathrm{~m}$, outer rib geometry parameter $0.002 \times 0.025 \mathrm{~m}$, the number of outer rib is 5 . Radial excitation force acts on cylindrical shell ( $L / 2,0, R$ ), force amplitude 1 N, Measurement points ( $L / 2, \theta, 1.175 \mathrm{~m}$ ).

### 3.2 The contrast of calculation results

Table1 simply-simply supported Stiffened cylindrical shell SPL (db)

| angle | $\mathrm{f}=6300 \mathrm{~Hz}, 9$ |  |  | inner ribs |
| :---: | :---: | :---: | :---: | :---: |
|  | $[8]$ |  | $[4]$ | PITMM |
|  | calc | test | calc | calc |
| 0 | 135.63 | 129.23 | 138.56 | 137.02 |
| 20 | 131.26 | 127.05 | 135.46 | 134.81 |
| 40 | 130.06 | 121.28 | 131.97 | 123.90 |
| 60 | 129.43 | 122.91 | 122.40 | 127.20 |
| 80 | 130.28 | 125.34 | 119.61 | 119.77 |
| 100 | 118.50 | 127.67 | 119.85 | 128.85 |
| 120 | 128.10 | 127.89 | 126.68 | 131.88 |
| 140 | 120.27 | 121.79 | 120.16 | 123.85 |
| 160 | 121.11 | 115.19 | 118.37 | 126.35 |
| 180 | 127.63 | 117.41 | 125.08 | 127.33 |

Table 2 simply-simply supported Stiffened cylindrical shell SPL (db)

| angle | $\mathrm{f}=4000 \mathrm{~Hz}, 5$ outer ribs |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | $\left.{ }^{0}{ }^{0}\right)$ | $[9]$ |  | $[4]$ |
|  | calc | test | calc | calc |
| 0 | 129.78 | 129.90 | 125.32 | 127.80 |
| 30 | 127.61 | 127.96 | 121.47 | 124.80 |
| 60 | 129.03 | 119.44 | 123.81 | 123.07 |
| 90 | 116.59 | 124.30 | 117.76 | 123.30 |
| 120 | 128.78 | 130.59 | 126.96 | 127.98 |
| 150 | 126.74 | 129.90 | 125.24 | 128.05 |
| 180 | 128.18 | 133.96 | 128.34 | 128.19 |

Tab. 1 and Tab. 2 show that the result data of this method agrees well with test values and calculated values from the model of the reference [8], [9].Comparing with the data of reference [4],the Result data of this method is closer to the test data. It also further validates the precise integration method of this article is exacter than the precise number of polynomial approximation from the reference [4] in dealing with inhomogeneous terms. In the paper, this method is the biggest advantage compared to the reference [4] method in dealing with inhomogeneous terms.

## 4. RESEARCH ON VIBRATION NOISE CHARACTERISTICS OF STIFFENED CY LINDRICAL SHELL

To analyze some parameters of stiffened cylindrical shell influencing acoustic radiation by using PITMM based on the model parameters of reference [9]

### 4.1 Effects of boundary conditions on the acoustic radiation of stiffened cylindrical shell

Table 3 boundary conditions have effects on the stiffened cylindrical shell SPL(db)
Tab. 3 and Fig. 4 compare the two end boundary conditions on the influence SPL of stiffened

| angle $\left(^{\circ}\right)$ | $\mathrm{f}=4000 \mathrm{hz}$ |  |  |
| :---: | :---: | :---: | :---: |
|  | $\mathrm{F}-\mathrm{F}$ | $\mathrm{S}-\mathrm{S}$ | $\mathrm{C}-\mathrm{C}$ |
| 0 | 131.20 | 127.79 | 126.15 |
| 30 | 129.88 | 124.80 | 123.31 |
| 60 | 128.19 | 123.07 | 119.20 |
| 90 | 124.46 | 123.30 | 118.80 |
| 120 | 130.65 | 127.98 | 124.04 |
| 150 | 131.94 | 128.05 | 123.96 |
| 180 | 131.72 | 128.19 | 124.59 |



Fig 4. Circumferential cylindrical shell SPL(db)
cylindrical shell underwater. From the free ends of the boundary conditions, simply supported, to rigidly fixed, the radiation acoustic pressure of structure decreases with a fixed strengthen, This is because the stronger the structure is fixed at both ends to generate more difficult vibration, acoustic radiation pressure is the smaller.

### 4.2 Effects of structural damping on the acoustic radiation of stiffened cylindrical shell

Table4 Structural damping affects
stiffened cylindrical shell SPL(db).

|  | $\mathrm{f}=4000 \mathrm{hz}$ |  |  |
| :---: | :---: | :---: | :---: |
| angle $\left(^{\circ}\right)$ |  |  |  |
|  | $\eta_{=0.01}$ | $\eta_{=0.05}$ | $\eta_{=0.1}$ |
| 0 | 127.79 | 126.93 | 125.60 |
| 30 | 124.79 | 124.97 | 125.14 |
| 60 | 123.07 | 124.52 | 126.36 |
| 90 | 123.30 | 117.84 | 116.26 |
| 120 | 127.98 | 126.79 | 125.01 |
| 150 | 128.05 | 128.52 | 128.01 |
| 180 | 128.19 | 128.94 | 128.43 |



Fig 5. Circumferential cylindrical shell SPL(db)
Table 4 and figure 5 compares the structural loss factor on the influence SPL of stiffened cylindrical structural radiation acoustic pressure decreases with the increasing of loss factor, but a few sets of data does not meet the law; Near the exciting force ( $\theta=0^{0}$ ), the influences of loss factor on structural radiation acoustic pressure is small, away from the exciting force ( $\theta=180^{\circ}$ ), the influences of loss factor on structural radiation acoustic pressure is obvious.

### 4.3 Effects of the fluid medium on the acoustic radiation of stiffened cylindrical shell

The density of water is $1000 \mathrm{~kg} / \mathrm{m}^{3}$, the velocity of sound is $1500 \mathrm{~m} / \mathrm{s}$ in the water, the impedance of the water is $1.5 \mathrm{Mpa} /(\mathrm{m} \cdot \mathrm{s})$, the density of air is $1.293 \mathrm{~kg} / \mathrm{m}^{3}$, the velocity of sound is $340 \mathrm{~m} / \mathrm{s}$ in air, the impedance of the air is $4.4 \times 10^{-4} \mathrm{Mpa} /(\mathrm{m} \cdot \mathrm{s})$.
Table5 the fluid medium affects the stiffened cylindrical shell SPL(db)

| angle $\left({ }^{\circ}\right)$ | $\mathrm{f}=4000 \mathrm{hz}$ |  |
| :---: | :---: | :---: |
|  | water | air |
| 0 | 127.79 | 103.30 |
| 30 | 124.79 | 106.81 |
| 60 | 123.07 | 106.11 |
| 90 | 123.30 | 105.46 |
| 120 | 127.98 | 95.13 |
| 150 | 128.05 | 94.89 |
| 180 | 128.19 | 97.87 |



Fig 6. Circumferential cylindrical shell SPL(db)

Table 5, Figure 6 compare the fluid medium on the influence SPL to stiffened cylindrical shell underwater. It can be apparently found that radiation acoustic pressure of structure in large fluid medium impedance (water) is smaller than in fluid medium small impedance (air). the reason for this phenomenon is that acoustic waves cause the pressure variation of water medium is greater than air medium as the impedance of water is relatively lager than the air's.

### 4.4 Effects of The thickness on the acoustic radiation of stiffened cylindrical shell

Measurement point ( $L / 2,0,1.175 \mathrm{~m}$ ).


Fig 7. Lv comparison of stiffened cylindrical shell with different thickness


Fig 8. SPL comparison of stiffened cylindrical shell with different thickness

Fig. 7 and Fig. 8 compare the thickness of stiffened cylindrical shell influence on LV and SPL underwater. It can be seen from the figure 7 that LV peak points move backwards and the peak values basically remain stable with the increase of shell thickness. Outside of the peak point segment, LV of stiffened cylindrical shell decrease gradually with the increase of shell thickness; It can be seen from fig. 8 that SPL peak points move backwards while peak values have the trend of gradual increase with the increase of the shell thickness. Outside of the peak point segment, SPL of stiffened cylindrical shell decrease gradually with the increase of shell thickness.

## 5. CONCLUSIONS

Papers of using the transfer matrix method to solve the problem of acoustic radiation cylindrical shell have been published in domestic and foreign. But this is the first time that the non-homogeneous terms precision integration is introduced to the transfer matrix method for solving the problem of acoustic radiation. And the accuracy of the method has been validated in this article. What's more the method is not limited to the boundary conditions at both ends of the cylindrical shell. So, the method can be widely used.

Based on the characteristics of transfer matrix method, this method can also be extended to solve the acoustic radiation problem about variable thickness cylindrical shell, reinforced cone shells and other rotating body structure.

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## APPENDIX A

$\boldsymbol{U}_{12}=\boldsymbol{U}_{75}=-\boldsymbol{U}_{78}=\mu \neq \boldsymbol{U}_{13}=\boldsymbol{U}_{68}=-\mu, U_{18}=\frac{\tilde{h}}{12}, \quad \boldsymbol{U}_{21}=-\boldsymbol{U}_{87}=n, \quad \boldsymbol{U}_{24}=-\frac{n \tilde{h}^{2}}{6}$,
$\boldsymbol{U}_{27}=\frac{\tilde{h}}{6(1-\mu)} \boldsymbol{U}_{34}=\boldsymbol{U}_{56}=1, \quad \boldsymbol{U}_{43}=\boldsymbol{U}_{65}=\mu n^{2}, \quad \boldsymbol{U}_{45}=\frac{1}{h}, \quad \boldsymbol{U}_{54}=2(1-\mu) n^{2} \tilde{h}$,
$\boldsymbol{U}_{62}=-\frac{12\left(1-\mu^{2}\right) n}{h}, \quad \boldsymbol{U}_{63}=-\left(12+n^{4} \tilde{h}^{2}\right) \frac{1-v^{2}}{\tilde{h}}+\frac{12 \lambda^{2}}{\tilde{h}}, \quad \boldsymbol{U}_{72}=\frac{12}{h}\left\{\left(1-v^{2}\right) n^{2}-\lambda^{2}\right\}$,
$\boldsymbol{U}_{73}=\left(12+n^{2} \tilde{h}^{2}\right) \frac{\left(1-v^{2}\right) n}{h}, \quad \boldsymbol{U}_{81}=-\frac{12 \lambda^{2}}{h}, \quad \boldsymbol{U}_{84}=(1-v) n^{2} \tilde{h}$.
$\xi=\frac{x}{l}, \quad \bar{l}=\frac{l}{R}, \quad \bar{h}=\frac{h}{R}, \quad \lambda^{2}=\frac{\rho h R^{2} \omega^{2}}{D}$
$(u, w)=h \sum_{\alpha=0}^{1} \sum_{n}(\tilde{u}, \tilde{w}) \sin \left(n \theta+\frac{\alpha \pi}{2}\right)$
$v=h \sum_{\alpha=0}^{1} \sum_{n} \tilde{v} \cos \left(n \theta+\frac{\alpha \pi}{2}\right)$
$\varphi=\frac{h}{R} \sum_{\alpha=0}^{1} \sum_{n} \tilde{\varphi} \sin \left(n \theta+\frac{\alpha \pi}{2}\right)$
$\left(M_{s}, M_{\theta}\right)=\frac{K}{R} \sum_{\alpha=0}^{1} \sum_{n}\left(\tilde{M}_{s}, \tilde{M}_{\theta}\right) \sin \left(n \theta+\frac{\alpha \pi}{2}\right)$
$\left(M_{s \theta}, M_{\theta s}\right)=\frac{K}{R} \sum_{\alpha=0}^{1} \sum_{n}\left(\tilde{M}_{s \theta}, \tilde{M}_{\theta s}\right) \cos \left(n \theta+\frac{\alpha \pi}{2}\right)$
$\left(N_{s \theta}, N_{\theta s}, Q_{\theta}, S_{s \theta}\right)=\frac{K}{R^{2}} \sum_{\alpha=0}^{1} \sum_{n}\left(\tilde{N}_{s \theta}, \tilde{N}_{\theta s}, \tilde{Q}_{\theta}, \tilde{S}_{s \theta}\right) \cos \left(n \theta+\frac{\alpha \pi}{2}\right)$
$\left(N_{s}, N_{\theta}, Q_{s}, V_{s}\right)=\frac{K}{R^{2}} \sum_{\alpha=0}^{1} \sum_{n}\left(\tilde{N}_{s}, \tilde{N}_{\theta}, \tilde{Q}_{s}, \tilde{V}_{s}\right) \sin \left(n \theta+\frac{\alpha \pi}{2}\right)$


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