Acoustic forcing of flexural waves and acoustic fields for a thin plate in a fluid

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ABSTRACT

Consistency with conservation of energy for coupled acoustic fields and plate flexural waves, discussed in another paper for this conference, is used to derive the amplitude and phase of flexural and acoustic waves for an infinite thin plate – fluid system excited by an incident acoustic plane wave. The acoustic interaction of the plate – fluid system is defined by 1. specula reflection from the plate surface, 2.transmission through the plate material and 3. plate flexural waves taking into account fluid loading. This reproduces the well-known peak in plate flexural wave amplitudes above the coincidence frequency where the trace wavenumber of the incident acoustic plane wave along the plate equals the plate – vacuum flexural wavenumber. This is essentially a resonance with the resonant frequency that varies with the direction of the incident plane wave. The width of the resonance is governed by fluid loading which manifests as radiation damping of the flexural waves. It is found that flexural waves affect the acoustic reflectivity of the plate through coherent interference of the acoustic field from flexural waves with the specula reflected field, but only if there is a nonzero phase shift in specula reflection. Energy conservation considerations predict that a plate becomes acoustically soft close to the resonance condition. A simple formula for the approximate resonance width is also derived.

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1. INTRODUCTION

The theory of vibration of a structure immersed in a fluid is important for understanding acoustic phenomena such as scattering and radiation from ships and submarines, and the effect of structures on sonar signals and sonar self noise. Some useful concepts are found by just considering a plane acoustic wave incident onto a flat plate immersed in a fluid, then use well-known relations for acoustic reflection and transmission at the plate – fluid surface. The latter relations need the densities and sound speeds for the fluid and plate materials, plus plate thickness, Young’s modulus and Poisson ratio. These parameters are sufficient for modeling longitudinal (compression) waves in the fluid and plate, and shear waves in the plate. Consideration of flexural (i.e. bending) waves is often omitted, motivating the theory of this paper enabling flexural wave effects to be included into a convenient but artificial model of plate acoustic reflections.

There are treatments of the problem of plane wave acoustic excitation of infinite thin plate flexural waves in text books (1, 2, 3), but they do not specifically discuss energy densities and energy density fluxes. Usually it can be taken for granted that energy is conserved in the solutions of wave equations, however for artificial models such as acoustically hard plates conservation of energy must be imposed separately. This paper defines a more general artificial plate model, where idealized acoustically hard and soft plates are special cases, and derives constraints on the model parameters from conservation of energy.

This paper applies previously derived energy conservation relations (4) to derive in Section 2.1 the amplitudes and phases of acoustically excited thin plate flexural waves while taking into account fluid loading. Since all waves are travelling waves, fluid loading in this case is just radiation damping of the flexural waves. The effect of flexural waves on the plate acoustic reflectivity and the near acoustic field close to a plate are then derived in Sections 2.2 and 2.4 respectively. Section 2.3

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shows that close to the resonance condition, where the trace phase speed of acoustic waves over the plate equals the phase speed of flexural waves, energy conservation does not allow a plate to be acoustically hard.

2. FORCED VIBRATION OF A PLATE DERIVED BY AN ENERGY CONSERVATION METHOD

2.1 Forced vibration of an infinite thin plate in a fluid by an external acoustic plane wave

In a previous paper (4) the following energy conservation equation is derived:

$$\frac{\partial U(x,t)}{\partial t}+\frac{\partial \Gamma(x,t)}{\partial x}+\left(W_F(x,t)+\frac{\partial U_F(x,t)}{\partial t}\right)-W_{EF}(x,t)=W_{PF}(x,t)-W_{EF}(x,t)=0 \tag{1}$$

where $U(x,t)$ is the sum of kinetic and potential energy densities in the plate, $U_F(x,t)$ is the added energy density of the plate – fluid system from the added mass of the fluid, $\Gamma(x,t)$ is the energy density flux in the plate, $W_F(x,t)$ is the energy density transfer rate between the plate and the fluid, and $W_{EF}(x,t)$ is the energy density transfer rate between the plate – fluid system and an external force. Formulae for these functions in terms of the plate wave amplitude are derived in reference (4). $W_{PF}(x,t)$ is the energy density transfer rate, either energy lost or gained from the plate – fluid system, which is nonzero only when it is forced to conform to a particular waveform that is not a plate – fluid natural mode (4)[2]. The RHS of eqn. (1) is zero because we are considering the equilibrium situation of an energy density transfer rate from an external force exactly balanced by the energy density transfer rate of the plate – fluid system.

Consider the externally applied energy density transfer rate $W_{EF}(x,t)$ needed to force plate – fluid waves to satisfy eqn. (1) (i.e. $W_{EF}(x,t)=W_{PF}(x,t)\neq 0$) supplied by an external acoustic plane wave. Incident and specula reflected acoustic waves bend the plate such that plate acceleration creates a pressure change that partially cancels these two external acoustic pressure sources. This pressure change is the fluid loading effect contained in $W_{PF}(x,t)$. Consequently flexural waves displace the acoustic reflectivity of the plate-fluid system away from non-bending plate-fluid specula reflectivity. What proportion of the energy of an incident acoustic wave that is converted into plate flexural wave energy depends on plate stiffness D, mass per unit area M and how much of the incident acoustic energy is absorbed and reflected by internal plate compression and shear mechanisms. In general the incident acoustic energy gets distributed into specula reflected energy, energy absorbed into plate material longitudinal and shear waves, plate flexural waves that radiate acoustic energy back into the fluid, and plate flexural waves that propagate energy along the plate – fluid surface.

An acoustic plane wave from far field (i.e. large z) and incident onto a thin plate has a real wavenumber vector $k'_x\neq 0, k'_z<0, k''_x=0, k''_z=0$[3]. The specula reflected wave has the same $k'_x\neq 0, k''_x=0, k''_z=0$ but $k'_z>0$. The acoustic wave generated by the flexural wave, equivalent to fluid loading, must also have real wavenumbers equal to that of the incident wave in the x direction and opposite sign in the z direction. The complex incident, specula reflected and flexural wave generated acoustic wave pressures are given by respectively

$$p_I(x,z,t)=p_0^{(-)}\exp\left[ik'_x x-ik'_z z-i\omega t+i\phi_0^{(-)}\right] \tag{2a}$$

$$p_{SR}(x,z,t)=p_0^{(+)}\exp\left[ik'_x x+ik'_z z-i\omega t+i\phi_0^{(+)}\right] \tag{2b}$$

$$p_F(x,z,t)=p_f\exp\left[ik'_x x+ik'_z z-i\omega t+i\phi_f\right] \tag{2c}$$

where pressure amplitudes $p_0^{(\pm)}$, p_f and phases $\phi_0^{(\pm)}$, ϕ_f distinguish the origin and directions of incident, specula reflected and plate flexural wave generated acoustic waves. For simplicity the fluid is only on one side of the plate, and the other side is a vacuum.

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[2] Five natural non-leaky and leaky flexural wave modes were identified in reference (5) for a thin plate in a fluid. The solution to any forced vibration problem implies deviations from these natural modes (see references (6) and (7)).

[3] Any complex number q is written as $q=q'+iq''$ to distinguish the real part q' from the imaginary part q''.

For later brevity we define a specula reflection coefficient $0 \le \mu \le 1$ by

$$\mu = \frac{p_0^{(+)}}{p_0^{(-)}} \tag{3a}$$

and specula reflection phase shift $-\pi \le \delta \le \pi$ by

$$\delta = \phi_0^{(+)} - \phi_0^{(-)} \tag{3b}$$

We assume that the external pressure from incident and specula reflected waves contribute equally to causing plate bending (i.e. flexural) waves, and the internal pressure of waves within the plate material do not directly contribute to plate bending waves. So a proportion μ of the incident acoustic wave pressure contributes to plate bending waves, and specula reflection, also a proportion μ of the incident acoustic wave pressure, also contributes to plate bending waves albeit with a phase change δ. The proportion $1-\mu \ne 0$ of the incident acoustic wave pressure drives compression and shear waves within the plate material leading to complete energy absorption within the material[4]. Commonly used but somewhat artificial cases are acoustic "hard" materials with $\mu = 1$ (hence bending waves are the only energy loss mechanism) and acoustic "soft" materials with $\mu = 0$ (hence no bending waves are excited and all energy is absorbed by the plate material).

The external acoustic pressure $p_D(x,t)$ at $z = 0$ driving the plate flexural wave is

$$\begin{aligned} p_D(x,t) &= \mu p_I^{(-)}(x,0,t) + \varepsilon p_{SR}^{(+)}(x,0,t) = \mu\left(1+\varepsilon e^{i\delta}\right)p_I^{(-)}(x,0,t) \\ &= \mu p_0^{(-)}\left(1+\varepsilon e^{i\delta}\right)\exp\left[ik_x' x - i\omega t + i\phi_0^{(-)}\right] \end{aligned} \tag{4}$$

The parameter ε is used to keep track of the contribution of the specula reflected acoustic wave to driving flexural waves, and their effect on the net reflectivity of the plate and net pressure at the plate surface. It is implicit that $\varepsilon = 1$.

$p_D(x,t)$ can be rewritten in terms of an amplitude p_0 and phase deviation N from $\phi_0^{(-)}$ by

$$p_D(x,t) = p_0 e^{i\mathrm{N}} \exp\left[ik_x' x - i\omega t + i\phi_0^{(-)}\right] \tag{5a}$$

where

$$p_0 = \mu p_0^{(-)}\sqrt{\left(1+\varepsilon\cos(\delta)\right)^2 + \left(\varepsilon\sin(\delta)\right)^2} = \mu p_0^{(-)}\sqrt{1+\varepsilon^2+2\varepsilon\cos(\delta)} \tag{5b}$$

$$e^{i\mathrm{N}} = \frac{\left(1+\varepsilon e^{i\delta}\right)}{\sqrt{1+\varepsilon^2+2\varepsilon\cos(\delta)}}, -\frac{\pi}{2} \le \mathrm{N} \le \frac{\pi}{2} \tag{5c}$$

Another, later useful, form of eqns. (5a, b, c) introduces a variable η related to phase N by

$$\eta = \frac{\mu p_0^{(-)}}{p_0}\left(1+\varepsilon\cos(\delta)\right) = \cos(\mathrm{N}), -\frac{\pi}{2} \le \mathrm{N} \le \frac{\pi}{2} \tag{6a}$$

$$sign(\sin(\delta))\sqrt{1-\eta^2} = \frac{\mu p_0^{(-)}}{p_0}\varepsilon\sin(\delta) = \sin(\mathrm{N}), -\frac{\pi}{2} \le \mathrm{N} \le \frac{\pi}{2} \tag{6b}$$

$\sqrt{a}$ denotes the positive square root of any number a so the sign is given explicitly.

The phase N is nonzero only if $\delta \ne n\pi, n = 0, \pm 1, \ldots$ where $\delta = 0$ for an acoustic hard surface and $\delta = \pm\pi$ for an acoustic soft surface.

The plate flexural wave displacement $\xi(x,t)$ is given by

$$\xi(x,t) = \xi_0 \exp\left[ik_x' x - i\omega t + i\phi_x\right] \tag{7}$$

$\xi(x,t)$ is defined as positive for a plate displacement in the $+z$ direction.

Equations (37a) and (37b) of reference (4) relate p_f and ϕ_f to ξ_0 and ϕ_x by

$$p_f = M_a''\omega^2\xi_0 = \frac{\rho_0\omega^2}{k_z'}\xi_0 = D\xi_0\alpha'' k_f^4 \tag{8a}$$

[4] Energy dissipation within the plate material needs to exist if $1-\mu \ne 0$ since a vacuum is present on the far side of the thin plate.

$$\phi_f = \phi_x - \frac{\pi}{2} \tag{8b}$$

$$k'_z = \sqrt{k_0^2 - k_x'^2} \tag{8c}$$

Here $k_0 = \omega / c_0$ is the acoustic wavenumber in the fluid with phase speed c_0, $k_f = \left(M\omega^2 / D\right)^{\frac{1}{4}}$ is the plate–vacuum flexural wavenumber, M is the plate mass per unit area and D its bending stiffness, and ρ_0 is the fluid density. The plate flexural response to a plane wave acoustic pressure can only be to radiate, so the added mass is only the imaginary part iM''_a, and α'' is the ratio $\alpha'' = M''_a / M$.

It remains to determine ξ_0, $p_0^{(+)}$, ϕ_x and $\phi_0^{(+)}$ from $p_0^{(-)}$ and $\phi_0^{(-)}$ by calculating the work rate $W_{EF}(x,t)$ of the driving pressure wave $p_D(x,t)$ on the plate and equating it to $W_{PF}(x,t)$. Below we use from reference (4) a phase $\psi'(x,t)$ for energy density and flux variations where

$$\psi'(x,t) = 2(k'_x x - \omega t) + 2\phi_x \tag{9}$$

$W_{EF}(x,t)$ is given by

$$W_{EF}(x,t) = -\dot{\xi}'(x,t)\, p'_D(x,t) = -\frac{1}{4}\left(\dot{\xi} + \dot{\xi}^*\right)\left(p_D + p_D^*\right) \tag{10}$$

The minus sign in eqn. (10) arises from the acoustic force of the driving pressure being in the opposite direction $-z$ to plate positive velocity direction $+z$.

From eqns. (4) and (10)

$$W_{EF}(x,t) = W_{EF}^{(-)}(x,t) + W_{EF}^{(+)}(x,t) \tag{11a}$$

$$W_{EF}^{(-)}(x,t) = \frac{1}{2}\omega\xi_0 \mu p_0^{(-)}\left(\cos\left(\pi + \phi_x - \phi_0^{(-)}\right)\sin\left(\psi'(x,t)\right) + \sin\left(\pi + \phi_x - \phi_0^{(-)}\right)\left(1 - \cos\left(\psi'(x,t)\right)\right)\right) \tag{11b}$$

$$W_{EF}^{(+)}(x,t) = \frac{1}{2}\omega\xi_0 \varepsilon\mu p_0^{(-)}\left(\cos\left(\pi + \phi_x - \phi_0^{(+)}\right)\sin\left(\psi'(x,t)\right) + \sin\left(\pi + \phi_x - \phi_0^{(+)}\right)\left(1 - \cos\left(\psi'(x,t)\right)\right)\right) \tag{11c}$$

where $W_{EF}^{(\pm)}(x,t)$ are the incident and specula reflected wave parts of the external acoustic work rate for plate bending. The minus sign in eqn. (10) is absorbed by $\phi_x \rightarrow \pi + \phi_x$ in the phase terms of eqns. (11b, c).

From eqn. (45b) of reference (4), noting that $\alpha' = 0$ by $k''_z = 0$,

$$W_{PF}(x,t) = \frac{D\omega}{2}\xi_0^2\left(\left(k_x'^4 - k_f^4\right)\sin\left(\psi'(x,t)\right) + \alpha'' k_f^4\left(1 - \cos\left(\psi'(x,t)\right)\right)\right) \tag{12}$$

The RHS of eqns. (11a, b, c) and (12) must be equal for all phases $\psi'(x,t)$ leading to

$$D\xi_0\left(k_x'^4 - k_f^4\right) = \mu p_0^{(-)}\left(\cos\left(\pi + \phi_x - \phi_0^{(-)}\right) + \varepsilon\cos\left(\pi + \phi_x - \phi_0^{(+)}\right)\right) \tag{13a}$$

$$D\xi_0 \alpha'' k_f^4 = \mu p_0^{(-)}\left(\sin\left(\pi + \phi_x - \phi_0^{(-)}\right) + \varepsilon\sin\left(\pi + \phi_x - \phi_0^{(+)}\right)\right) \tag{13b}$$

Using eqn. (3 b), $\phi_0^{(+)}$ can be eliminated from eqns. (13a, b) giving

$$D\xi_0\left(k_x'^4 - k_f^4\right) = \mu p_0^{(-)}\left(\left(1 + \varepsilon\cos(\delta)\right)\cos\left(\pi + \phi_x - \phi_0^{(-)}\right) + \varepsilon\sin(\delta)\sin\left(\pi + \phi_x - \phi_0^{(-)}\right)\right) \tag{13c}$$

$$D\xi_0 \alpha'' k_f^4 = \mu p_0^{(-)}\left(\left(1 + \varepsilon\cos(\delta)\right)\sin\left(\pi + \phi_x - \phi_0^{(-)}\right) - \varepsilon\sin(\delta)\cos\left(\pi + \phi_x - \phi_0^{(-)}\right)\right) \tag{13d}$$

When eqns. (13c, d) are solved for $\sin\left(\pi + \phi_x - \phi_0^{(-)}\right)$ and $\cos\left(\pi + \phi_x - \phi_0^{(-)}\right)$ we find

$$\sin\left(\pi + \phi_x - \phi_0^{(-)}\right) = \frac{D\xi_0}{\mu p_0^{(-)}}\frac{\varepsilon\sin(\delta)\left(k_x'^4 - k_f^4\right) + \left(1 + \varepsilon\cos(\delta)\right)\alpha'' k_f^4}{\left(1 + \varepsilon\cos(\delta)\right)^2 + \left(\varepsilon\sin(\delta)\right)^2} \tag{13e}$$

$$\cos\left(\pi + \phi_x - \phi_0^{(-)}\right) = \frac{D\xi_0}{\mu p_0^{(-)}}\frac{\left(1 + \varepsilon\cos(\delta)\right)\left(k_x'^4 - k_f^4\right) - \varepsilon\sin(\delta)\alpha'' k_f^4}{\left(1 + \varepsilon\cos(\delta)\right)^2 + \left(\varepsilon\sin(\delta)\right)^2} \tag{13f}$$

From eqns. (13e, f) and (5b) the plate flexural wave amplitude is

$$\xi_0 = \frac{1}{D\sqrt{\left(k_x'^4 - k_f^4\right)^2 + \left(\alpha'' k_f^4\right)^2}}\, p_0 \tag{14a}$$

and the phase $\pi + \phi_x - \phi_0^{(-)}$ can be determined from

$$\tan\left(\pi + \phi_x - \phi_0^{(-)}\right) = \frac{\left(1 + \varepsilon\cos(\delta)\right)\alpha'' k_f^4 + \varepsilon\sin(\delta)\left(k_x'^4 - k_f^4\right)}{\left(1 + \varepsilon\cos(\delta)\right)\left(k_x'^4 - k_f^4\right) - \varepsilon\sin(\delta)\alpha'' k_f^4} \tag{14b}$$

Equation (14a) shows that ξ_0 reaches a maximum at $k_x' = \pm k_f$ which is a resonance that does not diverge owing to damping by radiation from the plate into the fluid. This resonance is only possible at frequencies above the coincidence frequency defined at the condition $k_0 = k_f$.

For brevity define dimensionless and always positive parameter β , which is equivalent to a phase B , by

$$\beta = \frac{\alpha'' k_f^4}{\sqrt{\left(k_x'^4 - k_f^4\right)^2 + \left(\alpha'' k_f^4\right)^2}} = \sin(\mathrm{B}), 0 \le \mathrm{B} \le \pi \tag{15a}$$

$$sign(k_x'^4 - k_f^4)\sqrt{1-\beta^2} = \frac{k_x'^4 - k_f^4}{\sqrt{\left(k_x'^4 - k_f^4\right)^2 + \left(\alpha'' k_f^4\right)^2}} = \cos(\mathrm{B}), 0 \le \mathrm{B} \le \pi \tag{15b}$$

From eqns. (8a) and (14a)

$$p_f = \beta p_0 = \sin(\mathrm{B}) p_0 \tag{16}$$

From eqns. (6a, b), (14a) and (15a, b) we find that eqns. (13e, f) become

$$\sin\left(\pi + \phi_x - \phi_0^{(-)}\right) = \beta\eta + sign(k_x'^4 - k_f^4) sign(\sin(\delta))\sqrt{1-\beta^2}\sqrt{1-\eta^2} = \sin(\mathrm{B} + \mathrm{N}) \tag{17a}$$

$$\cos\left(\pi + \phi_x - \phi_0^{(-)}\right) = \eta\, sign(k_x'^4 - k_f^4)\sqrt{1-\beta^2} - \beta\, sign(\sin(\delta))\sqrt{1-\eta^2} = \cos(\mathrm{B} + \mathrm{N}) \tag{17b}$$

Eqns. (17a, b) show that the phase difference between plate flexural waves and the incident acoustic wave satisfies

$$\pi + \phi_x - \phi_0^{(-)} = \mathrm{B} + \mathrm{N} + 2n\pi, n = 0, \pm 1, \pm 2, \ldots, 0 \le \mathrm{B} \le \pi, -\frac{\pi}{2} \le \mathrm{N} \le \frac{\pi}{2} \tag{18a}$$

Combining eqns. (18a) with (8b) we obtain the phase difference between acoustic radiation by plate flexural waves and the incident acoustic wave

$$\phi_f - \phi_0^{(-)} = \mathrm{B} + \mathrm{N} - \frac{3\pi}{2} + 2n\pi, n = 0, \pm 1, \pm 2, \ldots, 0 \le \mathrm{B} \le \pi, -\frac{\pi}{2} \le \mathrm{N} \le \frac{\pi}{2} \tag{18b}$$

The phase of the driving pressure is $\phi_0^{(-)} + \mathrm{N}$ so using eqn. (18b) the phase difference between the flexural wave acoustic field and the driving field is

$$\phi_f - \phi_0^{(-)} - \mathrm{N} = \mathrm{B} - \frac{3\pi}{2} + 2n\pi, n = 0, \pm 1, \pm 2, \ldots, 0 \le \mathrm{B} \le \pi, -\frac{\pi}{2} \le \mathrm{N} \le \frac{\pi}{2} \tag{18c}$$

At the resonance $p_f = p_0$ from eqn. (16) so we find the flexural wave generated acoustic field amplitude at the plate surface equals the acoustic wave amplitude driving the plate flexural waves. Also at the resonance by eqn. (15a) $\beta = 1, \mathrm{B} = \pi/2$ and by eqn. (18c) $\phi_f - \phi_0^{(-)} - \mathrm{N} = -\pi$. Hence the flexural wave driving field and acoustic field from the plate have opposite phase and so cancel exactly. There remains at the surface only the component of the acoustic field that drives transmission and absorption by the plate material.

Our results derived from energy transfer rate considerations should agree with results for the acoustic excitation of flexural waves for fluid loaded plates derived by other methods. The flexural wave mobility is a convenient comparison function and this is found to be

$$\frac{-\dot{\xi}(x,t)}{p_I^{(-)}(x,0,t)} = \frac{-i\omega}{D}\frac{\mu\left(1 + \varepsilon e^{i\delta}\right)}{k_x'^4 - k_f^4 - i\alpha'' k_f^4} \tag{19a}$$

Substituting for k_x' , k_f and α'' , eqn. (19a) can be rearranged to

$$\frac{-\dot{\xi}(x,t)}{p_I^{(-)}(x,0,t)} = -\frac{\cos(\theta)}{\rho_0 c_0}\frac{\mu\left(1 + \varepsilon e^{i\delta}\right)}{1 - ik_0 h(\rho_s/\rho_0)\cos(\theta)\left(1 - (\omega/\omega_c)^2\sin^4(\theta)\right)} \tag{19b}$$

where θ is the angle of the incident wavenumber vector to the plate normal, ρ_s is the plate density and h the thickness, and $\omega_c = c_0^2\sqrt{M/D} = 2\pi f_c$ where f_c is the coincidence frequency. For $\omega << \omega_c$ reference (1) eqn. (6.49) agrees with eqn. (19b) for $\mu = 1, \delta = 0$ (acoustically hard plate) and $\varepsilon = 1$ (specula reflection contribution). Reference (1) eqn. (6.47) would lead to eqn. (19b) for any ω except the denominator of eqn. (6.47) has an error $(\omega/\omega_c)^3$ instead of $(\omega/\omega_c)^2$.

2.2 Effect of flexural waves on acoustic absorption and reflection by an infinite thin plate in a fluid

We seek formulae for flexural wave contributions to acoustic energy absorption and reflection. Denoting the energy density transfer rate of the incident external acoustic wave onto the plate as $W_I(x,t)$, and denoting $W_R(x,t)$ as the reflected energy density transfer rate, then $W_E(x,t) = W_I(x,t) - W_R(x,t)$ is the energy density transfer rate injected acoustically into the plate – fluid system. $W_R(x,t)$ includes both an acoustic wave part $W_F(x,t)$ generated by plate flexural waves and a specula reflected acoustic wave part $W_{SR}(x,t)$. Similarly $W_E(x,t)$ consists of a flexural wave part $W_{EF}(x,t)$ and a plate material acoustic absorption part $W_{EP}(x,t)$. The acoustic absorption coefficient γ_E of the plate – fluid system is defined from random phase averaged energy density transfer rates by $\gamma_E = \overline{W}_E/\overline{W}_I = \left(\overline{W}_I - \overline{W}_R\right)/\overline{W}_I$ and the acoustic reflection coefficient is $\gamma_R = \overline{W}_R/\overline{W}_I = 1 - \gamma_E$. We define flexural wave and plate material acoustic absorption coefficients $\gamma_{EF} = \overline{W}_{EF}/\overline{W}_I$ and $\gamma_{EP} = \overline{W}_{EP}/\overline{W}_I$ respectively.

From eqns. (11a, b, c) and (14a) using phase relation eqn. (18a)

$$W_{EF}(x,t) = \frac{1}{2\rho_0 c_0}\frac{k_z'}{k_0}p_f^2\left(1 + \frac{\cos(\mathrm{B})}{\sin(\mathrm{B})}\sin(\psi'(x,t)) - \cos(\psi'(x,t))\right) \tag{20}$$

The acoustic energy density rate incident onto the plate is phase sensitive and depends on $\psi'(x,t)$ and $\phi_x - \phi_0^{(-)}$ giving

$$W_I(x,t) = \frac{1}{2\rho_0 c_0}\frac{k_z'}{k_0}p_0^{(-)2}\left(1 - \sin\left(2\left(\phi_x - \phi_0^{(-)}\right)\right)\sin(\psi'(x,t)) - \cos\left(2\left(\phi_x - \phi_0^{(-)}\right)\right)\cos(\psi'(x,t))\right) \tag{21a}$$

From eqns.(17a, b), $\phi_x - \phi_0^{(-)}$ is related to $\mathrm{B} + \mathrm{N}$ and eqn. (21a) becomes

$$W_I(x,t) = \frac{1}{2\rho_0 c_0}\frac{k_z'}{k_0}p_0^{(-)2}\left(1 - \sin(2\mathrm{B} + 2\mathrm{N})\sin(\psi'(x,t)) - \cos(2\mathrm{B} + 2\mathrm{N})\cos(\psi'(x,t))\right) \tag{21b}$$

Comparing the RHS of eqns. (20) and (21b) we see that $W_{EF}(x,t)$ and $W_I(x,t)$ are not in phase with each other which makes the ratio $W_{EF}(x,t)/W_I(x,t)$ phase sensitive. Consequently random phase averaged energy rates are used to define γ_{EF}. From eqns. (20) and (21a, b)

$$\overline{W}_{EF} = \frac{1}{2\rho_0 c_0}\frac{k_z'}{k_0}\beta^2 p_0^2 = \frac{1}{2\rho_0 c_0}\frac{k_z'}{k_0}p_f^2 \tag{22a}$$

$$\overline{W}_I = \frac{1}{2\rho_0 c_0}\frac{k_z'}{k_0}p_0^{(-)2} \tag{22b}$$

The coefficient γ_{EF} for the average acoustic input energy density transfer rate into flexural waves is

$$\gamma_{EF} = \frac{\overline{W}_{EF}}{\overline{W}_I} = \left(\frac{p_f}{p_0^{(-)}}\right)^2 = \beta^2\left(\frac{p_0}{p_0^{(-)}}\right)^2 = \beta^2\mu^2\left(1 + \varepsilon^2 + 2\varepsilon\cos(\delta)\right) \tag{23}$$

γ_{EF} is the sum of two parts

$$\gamma_{EF} = \gamma_{EF}^{(+)} + \gamma_{EF}^{(-)} \tag{24a}$$

where $\gamma_{EF}^{(-)}$ is an incident acoustic contribution and $\gamma_{EF}^{(+)}$ is a specula reflected wave contribution. These are easily distinguished by the terms that depend on ε and give

$$\gamma_{EF}^{(-)} = \beta^2\mu^2 \tag{24b}$$

$$\gamma_{EF}^{(+)} = \beta^2\mu^2\varepsilon^2 + 2\beta^2\mu^2\varepsilon\cos(\delta) \tag{24c}$$

The first term on the RHS of eqn. (24c) is from the specula reflected generated flexural wave, and the second term is the effect of coherent interference of the incident wave generated and specula reflected generated flexural waves.

To derive γ_R it is first necessary to calculate the coherent sum of specula reflected and plate flexural wave generated wave pressures. From eqns. (2b, c) the pressure for the reflected wave is

$$p_R(x,z,t) = p_{SR}(x,z,t) + p_F(x,z,t) \tag{25a}$$

Using eqns. (25a) and (2b, c) a reflected wave amplitude $p_R^{(+)}$ and phase $\phi_R^{(+)}$ are determined from

$$\begin{aligned} p_R(x,z,t) &= p_0^{(+)}\exp\left[ik'_x x + ik'_z z - i\omega t + i\phi_0^{(+)}\right] + p_f \exp\left[ik'_x x + ik'_z z - i\omega t + i\phi_f\right] \\ &= p_R^{(+)}\exp\left[ik'_x x + ik'_z z - i\omega t + i\phi_R^{(+)}\right] \end{aligned} \tag{25b}$$

$$p_R^{(+)^2} = p_0^{(+)^2} + p_f^2 + 2\cos(\phi_f - \phi_0^{(+)})p_0^{(+)}p_f \tag{25c}$$

Since by eqn. (8b) $\phi_f - \phi_0^{(+)} = \phi_f - \phi_x + \phi_x - \phi_0^{(+)} = \phi_x - \phi_0^{(+)} - \pi/2$

$$p_R^{(+)^2} = p_0^{(+)^2} + p_f^2 + 2\sin(\phi_x - \phi_0^{(+)})p_0^{(+)}p_f = p_0^{(+)^2} + p_f^2 - 2\sin(\pi + \phi_x - \phi_0^{(+)})p_0^{(+)}p_f \tag{25d}$$

From phase relations eqns. (3b) and (18a) we obtain $\sin(\pi + \phi_x - \phi_0^{(+)}) = \sin(\mathrm{B} + \mathrm{N} - \delta)$ which can be used in eqn. (25d) together with eqns. (16), (4) and (3a) to give

$$p_R^{(+)^2} = p_0^{(+)^2} + p_f^2 - 2\beta p_0^{(+)}\left[\mu p_0^{(-)}\sin(\mathrm{B} - \delta) + \varepsilon p_0^{(+)}\sin(\mathrm{B})\right] \tag{25e}$$

This shows that the net reflected wave is determined by specula reflection, the acoustic radiation from flexural waves in the plate, and coherent interference of the specula reflection with the radiation of the flexural wave due to incident and specula reflected acoustic sources. Using eqns. (16) and (5b)

$$p_f^2 = \beta^2\left(\mu^2 p_0^{(-)^2} + 2\varepsilon\mu p_0^{(-)}p_0^{(+)}\cos(\delta) + \varepsilon^2 p_0^{(+)^2}\right) \tag{25f}$$

Then eqn. (25e) becomes

$$\begin{aligned} p_R^{(+)^2} &= p_0^{(+)^2} + \beta^2\mu^2 p_0^{(-)^2} - 2\beta\mu p_0^{(+)}p_0^{(-)}\sin(\mathrm{B} - \delta) + \beta^2\left(\varepsilon p_0^{(+)}\right)^2 - 2\beta\varepsilon p_0^{(+)^2}\sin(\mathrm{B}) \\ &+ 2\beta\mu\varepsilon p_0^{(+)}p_0^{(-)}\cos(\delta)\sin(\mathrm{B}) \end{aligned} \tag{25g}$$

The parameter ε allows easier identification of 6 different contributions to the reflected acoustic wave. The first 3 RHS terms of eqn. (25g) are specula reflection, acoustic radiation from the flexural wave generated by the incident wave and coherent interference of specula reflection with flexural wave acoustic radiation generated by the incident wave. The last 3 RHS terms are from acoustic radiation of the flexural wave generated by the specula reflected wave, coherent interference of flexural wave acoustic radiation generated by the specula reflection with the specula reflection, and coherent interference of specula reflection excited flexural wave acoustic radiation with the incident wave excited flexural wave acoustic radiation.

The acoustic energy density transfer rate reflected from the plate is $W_R(x,t)$ that can be derived in terms of $p_R^{(+)}$, $\phi_R^{(+)}$ and plate phase ϕ_x similar to eqn. (21a) and the result is

$$W_R(x,t) = \frac{1}{2\rho_0 c_0}\frac{k'_z}{k_0}p_R^{(+)2}\left(1 - \sin\left(2\left(\phi_x - \phi_R^{(+)}\right)\right)\sin(\psi'(x,t)) - \cos\left(2\left(\phi_x - \phi_R^{(+)}\right)\right)\cos(\psi'(x,t))\right) \tag{26a}$$

The random phase averaged reflected energy density transfer rate is

$$\overline{W}_R = \frac{1}{2\rho_0 c_0}\frac{k'_z}{k_0}p_R^{(+)2} \tag{26b}$$

Then using eqns. (22b), (26b), (16) and (25g) the reflectivity coefficient is

$$\begin{aligned} \gamma_R &= \frac{\overline{W}_R}{\overline{W}_I} = \frac{p_R^{(+)^2}}{p_0^{(-)^2}} = \mu^2\left(1 + \beta^2(1-\varepsilon)(1 - \varepsilon - 2\cos(\delta)) + 2\beta\, sign(k_x'^4 - k_f^4)\sqrt{1-\beta^2}\sin(\delta)\right) \\ &= \mu^2\left(1 + \sin^2(\mathrm{B})(1-\varepsilon)(1 - \varepsilon - 2\cos(\delta)) + \sin(2\mathrm{B})\sin(\delta)\right) \end{aligned} \tag{27a}$$

Setting $\varepsilon = 1$, eqn. (27a) reduces to

$$\gamma_R = \mu^2\left(1 + 2\beta\, sign(k_x'^4 - k_f^4)\sqrt{1-\beta^2}\sin(\delta)\right) = \mu^2\left(1 + \sin(2\mathrm{B})\sin(\delta)\right) \tag{27b}$$

Comparing eqn. (27b) with eqn. (27a) we see that $\varepsilon = 1$ leads to cancellation of most flexural wave terms leaving only the flexural wave contribution from the interference of the incident wave induced flexural wave acoustic field with the specula reflected field. This flexural wave effect on the reflectivity only exists if $\sin(\delta) \neq 0$ and $\beta \neq 1$ (i.e. not exactly at the resonance condition). Interestingly the idealized acoustically hard and soft plate models with $\sin(\delta) = 0$ have no flexural wave effect on the reflectivity.

2.3 Energy conservation constraints on acoustic excitation of flexural waves of an infinite thin plate in a fluid

By energy conservation, the average rate of work of the incident acoustic wave on plate flexural waves cannot exceed the average energy flux contained in the incident acoustic wave. This requires then $0 \leq \gamma_{EF} \leq 1$, and from eqn. (23) gives

$$0 \leq \beta^2 \mu^2 \left(1 + \varepsilon^2 + 2\varepsilon \cos(\delta)\right) \leq 1 \tag{28}$$

Setting $\varepsilon = 0$ (i.e. no specula reflection contribution to flexural wave excitation) eqn. (28) is always satisfied since $\beta \leq 1$ and $\mu \leq 1$. Setting $\varepsilon = 1$, when $\beta^2 << 1$ there is no problem satisfying eqn. (28) for any $0 \leq \mu \leq 1$ and any $-1 \leq \cos(\delta) \leq 1$. However for $\beta^2 \approx 1$ near the resonance the middle function in eqn. (28) would be greater than 1 for sufficiently large μ and $\cos(\delta)$. This leads to an interesting conclusion for understanding acoustic reflection from a plate when the flexural wave resonance is taken into account. Specula reflection can create a higher pressure amplitude at the plate surface than in the incident wave, so it might seem the rate of work by this higher pressure on plate bending waves could exceed the power available from the incident wave unless the parameters μ and $\cos(\delta)$ are constrained to satisfy eqn. (28). This implies that specula reflection must be significantly modified at the flexural wave resonance condition even if the plate is acoustically hard away from the resonance[5].

Setting $\beta^2 \approx 1$ then eqn. (28) requires

$$-1 \leq \cos(\delta) \leq -1 + \frac{1}{2\mu^2} \tag{29}$$

Equation (29) eliminates the possibility of an acoustic hard plate defined by $\mu^2 \approx 1, \cos(\delta) = 1$. If $\cos(\delta) = 1$ were still possible at the resonance, then μ is constrained by

$$0 \leq \mu \leq \frac{1}{2} \tag{30}$$

that represents a fairly soft acoustic reflection by a significant energy loss mechanism within the plate material. If there is no significant energy loss mechanism within the plate, then $\mu^2 \approx 1$ and eqn. (29) becomes

$$-1 \leq \cos(\delta) \leq -\frac{1}{2} \tag{31}$$

Eqn. (31) is realistic for a thin plate with a vacuum on one side since it approximates a pressure release (i.e. acoustic soft) surface. For the ideal acoustic soft plate defined by $\cos(\delta) = -1$ so that $\gamma_{EF} = 0$ since the incident and specula reflected acoustic fields cancel and do no net work on plate flexural waves.

Consider another constraint $0 \leq \gamma_R \leq 1$ which requires the reflected acoustic energy flux to not exceed the incident acoustic energy flux. This is easily satisfied for weak specula reflections giving $\mu^2 << 1$. However for no significant energy losses in the plate, $\mu^2 \approx 1$ and eqn. (27b) requires

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[5] An acoustically hard surface $\mu = 1, \delta = 0$ makes the flexural wave driving pressure twice the incident wave pressure. Real surfaces that approximate this model over some frequency range are used for sonar signal enhancement at a sensor close to a "signal conditioning" plate. It might seem the average work rate on the sensor can be four times the average incident wave energy rate, but a typical sensor is almost acoustically transparent such that energy conservation is assured. However if the plate is itself a sensor to detect signal induced vibrations, energy conservation limits μ and δ making an acoustically hard surface impossible near the resonance condition.

$$-1 \le \sin(2\mathrm{B})\sin(\delta) \le 0 \tag{32}$$

Using eqn. (15b), eqn. (32) shows that

$$(k_x'^4 - k_f^4)\sin(\delta) \le 0 \tag{33}$$

At the resonance $k_x'^4 - k_f^4 = 0$, $2\mathrm{B} = \pi$, $\sin(2\mathrm{B}) = 0$, so eqns. (32) and (33) do not constrain $\sin(\delta)$. If $\sin(\delta) \ne 0$ then $\sin(\delta)$ must have opposite signs just above and below the resonance, $sign(\sin(\delta)) \ge 0$ above the resonance frequency, and $sign(\sin(\delta)) \le 0$ below the resonance frequency. Also for $\mu^2 \approx 1$ eqn. (31) rules out $\delta = 0$, so there must be a discontinuity in δ at the resonance where $\delta \to -\delta$. This again shows that specula reflection and flexural waves must be linked to get consistency with energy conservation near the resonance condition.

A more detailed analysis of specula reflection from thin plates in a fluid, with a vacuum on one side, using more detailed acoustic and elasticity theory is needed to check the conclusion that a plate must be acoustically soft near the resonance condition. If the width of the resonance from radiation damping is quite narrow, the softness effect may not have been previously noticed experimentally. In many practical cases the frequency and direction of incident acoustic waves are well away from the resonance condition enabling almost any level of acoustic hardness / softness of the plate.

2.4 Acoustic field close to the wavenumber resonance condition

The total acoustic field at the plate surface $z = 0$ is given by (see eqns. (2a, b, c))

$$p_T(x,t) = p_I(x,0,t) + p_{SR}(x,0,t) + p_F(x,0,t) = (1-\mu)p_I(x,0,t) + p_D(x,t) + p_F(x,0,t)$$

$$= \left((1-\mu)p_0^{(-)} + p_0 e^{i\mathrm{N}} \left(1 - \sin(\mathrm{B}) e^{i\left(\mathrm{B} - \frac{\pi}{2}\right)} \right) \right) \exp\left[i k_x' x - i\omega t + i\phi_0^{(-)} \right] \tag{34}$$

At the resonance condition $\mathrm{B} = \pi/2$, eqn. (34) shows that the acoustic field $p_D(x,t)$ from the incident and specula reflected waves driving the flexural waves is exactly cancelled by the acoustic field produced by the flexural waves, leaving only the acoustic field component $(1-\mu)p_0^{(-)}$ absorbed as compression and shear waves in the plate material.

Consider the frequency and wave directions for the resonance condition. Equating the incident wave trace wavenumber on the plate surface to the plate-vacuum flexural wavenumber $k_x' = \pm k_f$ we find the resonance condition for angle θ is $\sin(\theta) = \sqrt{\omega_c / \omega}$. Hence the resonance condition is only physically possible for $\omega \ge \omega_c$, although the frequency width of the function β from radiation damping (see (15a)) causes some effect at lower frequencies. For $k_x' \approx \pm k_f$, β can be approximated by

$$\beta \approx \frac{\alpha'' k_f / 4}{\sqrt{(k_x' - k_f)^2 + (\alpha'' k_f / 4)^2}} \tag{35a}$$

showing the approximate resonance wavenumber width is

$$\Delta k_x' \approx \alpha'' k_f / 4 \tag{35b}$$

Using the previously derived formula for α'' (see Section 2.1) this width at the resonance is

$$\Delta k_x' \approx \frac{\rho_0}{4\rho_s h} \tan(\theta) \tag{35c}$$

3. SUMMARY

This paper has analysed theoretically acoustic wave excitation of thin plate flexural waves taking into account fluid loading causing radiation damping. An acoustic plane wave excites a flexural wave in an infinite thin plate with a frequency and direction dependent amplitude and phase given by eqns. (14a, b). The acoustic field generated by the flexural wave has a phase that tends to cancel the incident and specula reflected acoustic fields driving the flexural wave. The maximum cancellation effect occurs at a resonance condition where the trace acoustic phase speed along the plate equals the phase speed of plate-vacuum flexural waves. Near the resonance condition, energy conservation requires that the plate specula reflection be acoustically “soft” ruling out the possibility of an ideal “hard” plate in this case. This example shows that solutions of the wave

equation can be unphysical for artificial acoustic material assumptions. More detailed acoustic analysis using realistic plate material properties should be able to relate the parameters μ and δ and demonstrate their consistency with energy conservation eqns. (28) and (33).

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